Math 210, Exam 1, Practice Fall 2009
Problem 1 Solution

1. Let $A = (1, -1, 2)$, $B = (0, -1, 1)$, $C = (2, 1, 1)$.

   (a) Find the vector equation of the plane through A, B, C.

   (b) Find the area of the triangle with these three vertices.

Solution:

(a) In order to find the vector equation of the plane we need a point that lies in the plane and a vector $\overrightarrow{\mathbf{n}}$ perpendicular to it. We let $\overrightarrow{\mathbf{n}}$ be the cross product of $\overrightarrow{AB} = \langle -1, 0, -1\rangle$ and $\overrightarrow{BC} = \langle 2, 2, 0\rangle$ because these vectors lie in the plane.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{AB} \times \overrightarrow{BC} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & -1 \\ 2 & 2 & 0 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(0)(0) - (-1)(2)] - \hat{\mathbf{j}}\,[(-1)(0) - (-1)(2)] + \hat{\mathbf{k}}\,[(-1)(2) - (0)(2)] \\
\overrightarrow{\mathbf{n}} &= 2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle 2, -2, -2\rangle
\end{aligned}$$

Using $A = (1, -1, 2)$ as a point in the plane, we have:

$$\boxed{\langle x - 1, y + 1, z - 2\rangle \cdot \langle 2, -2, -2\rangle = 0}$$

as the vector equation for the plane containing A, B, C.

(b) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, which represents the area of the parallelogram spanned by the two vectors:

$$\begin{aligned}
A &= \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|\right| \\
A &= \frac{1}{2}\sqrt{2^2 + (-2)^2 + (-2)^2} \\
A &= \frac{1}{2}\sqrt{12} \\
&\boxed{A = \sqrt{3}}
\end{aligned}$$

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Problem 2 Solution

2. Find the vector of length one in the direction of $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ where $\vec{\mathbf{v}} = \langle 7, 5, 3\rangle$ and $\vec{\mathbf{u}} = \langle 4, 5, 7\rangle$.

Solution: First, the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ is:

$$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \langle 7, 5, 3\rangle - \langle 4, 5, 7\rangle = \langle 3, 0, -4\rangle$$

Next, we convert this vector into a unit vector by multiplying by the reciprocal of its magnitude.

$$\hat{\mathbf{e}} = \frac{1}{||\vec{\mathbf{v}} - \vec{\mathbf{u}}||}\left(\vec{\mathbf{v}} - \vec{\mathbf{u}}\right)$$
$$\hat{\mathbf{e}} = \frac{1}{\sqrt{3^2 + 0^2 + (-4)^2}}\langle 3, 0, -4\rangle$$
$$\hat{\mathbf{e}} = \frac{1}{5}\langle 3, 0, -4\rangle$$
$$\boxed{\hat{\mathbf{e}} = \left\langle \frac{3}{5}, 0, -\frac{4}{5}\right\rangle}$$

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Problem 3 Solution

3. Let $\overrightarrow{\mathbf{r}}(t) = \langle 3t - 1, e^t, \cos(t) \rangle$.

(a) Find the unit tangent vector $\overrightarrow{\mathbf{T}}$ to the path $\overrightarrow{\mathbf{r}}(t)$ at $t = 0$.

(b) Find the speed, $||\overrightarrow{\mathbf{r}}'(t)||$ at $t = 0$.

Solution:

(a) The derivative of $\overrightarrow{\mathbf{r}}(t)$ is $\overrightarrow{\mathbf{r}}'(t) = \langle 3, e^t, -\sin t \rangle$. At $t = 0$ we have $\overrightarrow{\mathbf{r}}'(0) = \langle 3, 1, 0 \rangle$. The unit tangent vector at $t = 0$ is then:

$$\overrightarrow{\mathbf{T}}(0) = \frac{1}{||\overrightarrow{\mathbf{r}}'(0)||}\,\overrightarrow{\mathbf{r}}'(0)$$

$$\overrightarrow{\mathbf{T}}(0) = \frac{1}{\sqrt{3^2 + 1^2 + 0^2}}\,\langle 3, 1, 0 \rangle$$

$$\overrightarrow{\mathbf{T}}(0) = \frac{1}{\sqrt{10}}\,\langle 3, 1, 0 \rangle$$

$$\boxed{\overrightarrow{\mathbf{T}}(0) = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \right\rangle}$$

(b) The speed at $t = 0$ is $||\overrightarrow{\mathbf{r}}'(0)|| = \sqrt{10}$.

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Problem 4 Solution

4. Given a point $P = (0, 1, 2)$ and the vectors $\overrightarrow{\mathbf{u}} = \langle 1, 0, 1\rangle$ and $\overrightarrow{\mathbf{v}} = \langle 2, 3, 0\rangle$, find

(a) an equation for the plane that contains P and whose normal vector is perpendicular to the two vectors $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$,

(b) a set of parametric equations of the line through P and in the direction of $\overrightarrow{\mathbf{v}}$.

Solution:

(a) In order to find an equation for the plane we need a point that lies in the plane and a vector $\overrightarrow{\mathbf{n}}$ perpendicular to it. We let $\overrightarrow{\mathbf{n}}$ be the cross product of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(0)(0) - (3)(1)] - \hat{\mathbf{j}}\,[(1)(0) - (1)(2)] + \hat{\mathbf{k}}\,[(1)(3) - (2)(0)] \\
\overrightarrow{\mathbf{n}} &= -3\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle -3, 2, 3\rangle
\end{aligned}$$

Using $P = (0, 1, 2)$ as a point in the plane, we have:

$$\boxed{-3(x - 0) + 2(y - 1) + 3(z - 2) = 0}$$

as the equation for the plane.

(b) A set of parametric equations of the line through $P = (0, 1, 2)$ and in the direction of $\overrightarrow{\mathbf{v}} = \langle 2, 3, 0\rangle$ is:

$$\boxed{x = 0 + 2t, \quad y = 1 + 3t, \quad z = 2 + 0t}$$

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Problem 5 Solution

5. Find the speed and arclength of the path $\vec{\mathbf{r}}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle$ where $0 \le t \le 2$.

Solution: The derivative of $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}'(t) = \langle -3\sin t, -4\sin t, 5\cos t\rangle$. Speed is the magnitude of $\vec{\mathbf{r}}'(t)$.

$$||\vec{\mathbf{r}}'(t)|| = \sqrt{(-3\sin t)^2 + (-4\sin t)^2 + (5\cos t)^2}$$
$$||\vec{\mathbf{r}}'(t)|| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}$$
$$||\vec{\mathbf{r}}'(t)|| = \sqrt{25\sin^2 t + 25\cos^2 t}$$
$$||\vec{\mathbf{r}}'(t)|| = \sqrt{25}$$
$$\boxed{||\vec{\mathbf{r}}'(t)|| = 5}$$

The arclength of the path is then:

$$L = \int_0^2 ||\vec{\mathbf{r}}'(t)||\ dt$$
$$L = \int_0^2 5\,dt$$
$$L = 5t\Big|_0^2$$
$$\boxed{L = 10}$$

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Problem 6 Solution

6. Find the curvature at $t = 0$ for the curve $\vec{\mathbf{r}}(t) = e^t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$.

Solution: The curvature formula we will use is:

$$\kappa(0) = \frac{\left|\left|\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(0)\right|\right|^3}$$

The first two derivatives of $\vec{\mathbf{r}}(t) = \langle e^t, t^2, t\rangle$ are:

$$\vec{\mathbf{r}}'(t) = \left\langle e^t, 2t, 1\right\rangle$$
$$\vec{\mathbf{r}}''(t) = \left\langle e^t, 2, 0\right\rangle$$

We now evaluate the derivatives at $t = 0$.

$$\vec{\mathbf{r}}'(0) = \left\langle e^0, 2(0), 1\right\rangle = \langle 1, 0, 1\rangle$$
$$\vec{\mathbf{r}}''(0) = \left\langle e^0, 2, 0\right\rangle = \langle 1, 2, 0\rangle$$

The cross product of these vectors is:

$$\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$
$$\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0) = \hat{\mathbf{i}} \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0) = \hat{\mathbf{i}}\,[(0)(0) - (1)(2)] - \hat{\mathbf{j}}\,[(1)(0) - (1)(1)] + \hat{\mathbf{k}}\,[(1)(2) - (1)(0)]$$
$$\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0) = -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0) = \langle -2, 1, 2\rangle$$

We can now compute the curvature.

$$\kappa(0) = \frac{\left|\left|\vec{\mathbf{r}}'(0) \times \vec{\mathbf{r}}''(0)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(0)\right|\right|^3}$$
$$\kappa(0) = \frac{||\langle -2, 1, 2\rangle||}{||\langle 1, 0, 1\rangle||^3}$$
$$\kappa(0) = \frac{\sqrt{(-2)^2 + 1^2 + 2^2}}{(\sqrt{1^2 + 0^2 + 1^2})^3}$$
$$\kappa(0) = \frac{\sqrt{9}}{(\sqrt{2})^3}$$
$$\boxed{\kappa(0) = \frac{3}{2\sqrt{2}}}$$

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Problem 7 Solution

7. Let $\vec{\mathbf{r}}(t) = \langle t, \cos t, \sin t\rangle$.

(a) Find the velocity vector, $\vec{\mathbf{r}}'(t)$.

(b) Find the acceleration vector, $\vec{\mathbf{r}}''(t)$.

(c) Find the component of acceleration in the direction of the velocity when $t = 0$.

Solution:

(a) The velocity vector is $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle 1, -\sin t, \cos t\rangle$.

(b) The acceleration vector is $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = \langle 0, -\cos t, -\sin t\rangle$.

(c) At $t = 0$, the velocity and acceleration vectors are:

$$\vec{\mathbf{v}}(0) = \langle 1, -\sin 0, \cos 0\rangle = \langle 1, 0, 1\rangle$$
$$\vec{\mathbf{a}}(0) = \langle 0, -\cos 0, -\sin 0\rangle = \langle 0, -1, 0\rangle$$

The acceleration can be decomposed into tangential and normal components.

$$\vec{\mathbf{a}} = a_T\,\vec{\mathbf{T}} + a_N\,\vec{\mathbf{N}}$$

By definition, the component of acceleration in the direction of the velocity is a_T. The formula and subsequent computation are shown below.

$$a_T = \frac{\vec{\mathbf{a}}(0) \cdot \vec{\mathbf{v}}(0)}{||\vec{\mathbf{v}}(0)||}$$
$$a_T = \frac{\langle 0, -1, 0\rangle \cdot \langle 1, 0, 1\rangle}{||\langle 1, 0, 1\rangle||}$$
$$a_T = \frac{(0)(1) + (-1)(0) + (0)(1)}{\sqrt{1^2 + 0^2 + 1^2}}$$
$$\boxed{a_T = 0}$$

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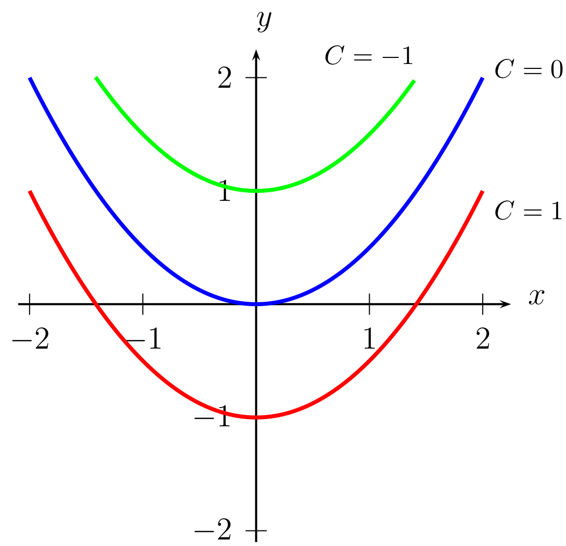
Problem 8 Solution

8. Let $f(x,y) = \frac{1}{2}x^2 - y$. Sketch the three level curves on which $f(x,y) = -1$ or 0 or 1 in the square $-2 \le x \le 2$, $-2 \le y \le 2$.

Solution: The level curves of $f(x,y) = \dfrac{1}{2}x^2 - y$ are the curves obtained by setting $f(x,y)$ to a constant C.

$$C = \frac{1}{2}x^2 - y \iff y = \frac{1}{2}x^2 - C$$

These curves are parabolas with vertices at $(0,-C)$ and are sketched below for the values $C = -1, 0, 1$.



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Problem 9 Solution

9. Find the partial derivatives

$$\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

for the function $f(x, y) = 2x + 3xy - 5y^2$.

Solution: The first partial derivatives of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = 2 + 3y$$
$$\frac{\partial f}{\partial y} = 3x - 10y$$

The second mixed partial derivative $\frac{\partial^2 f}{\partial x \partial y}$ is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x - 10y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 3$$

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Problem 10 Solution

10. Find the partial derivatives

$$\frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}$$

for the function $f(x,y) = e^{2x}\cos(2y)$.

Solution: The first partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = 2e^{2x}\cos(2y)$$
$$\frac{\partial f}{\partial y} = -2e^{2x}\sin(2y)$$

The second partial derivative $\dfrac{\partial^2 f}{\partial x^2}$ is

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2e^{2x}\cos(2y)\right)$$
$$\frac{\partial^2 f}{\partial x^2} = 4e^{2x}\cos(2y)$$

The second partial derivative $\dfrac{\partial^2 f}{\partial y^2}$ is

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-2e^{2x}\sin(2y)\right)$$
$$\frac{\partial^2 f}{\partial y^2} = -4e^{2x}\cos(2y)$$