Math 210, Exam 1, Spring 2001
Problem 1 Solution

1. Given two vectors $\overrightarrow{\mathbf{a}} = \langle -3, 2, 2\rangle$, $\overrightarrow{\mathbf{b}} = \langle 4, 3, -1\rangle$.

   (a) Find a unit vector in the same direction as $\overrightarrow{\mathbf{a}}$.

   (b) Find the angle between $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$.

Solution:

(a) A unit vector in the same direction as $\overrightarrow{\mathbf{a}}$ is:

$$\hat{\mathbf{e}}_{\mathbf{a}} = \frac{1}{||\overrightarrow{\mathbf{a}}||}\,\overrightarrow{\mathbf{a}}$$

$$\hat{\mathbf{e}}_{\mathbf{a}} = \frac{1}{\sqrt{(-3)^2 + 2^2 + 2^2}}\,\langle -3, 2, 2\rangle$$

$$\hat{\mathbf{e}}_{\mathbf{a}} = \frac{1}{\sqrt{17}}\,\langle -3, 2, 2\rangle$$

$$\boxed{\hat{\mathbf{e}}_{\mathbf{a}} = \left\langle -\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}\right\rangle}$$

(b) We use the dot product to find the angle between $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$.

$$\cos\theta = \frac{\overrightarrow{\mathbf{a}} \cdot \overrightarrow{\mathbf{b}}}{||\overrightarrow{\mathbf{a}}||\;\left|\left|\overrightarrow{\mathbf{b}}\right|\right|}$$

$$\cos\theta = \frac{(-3)(4) + (2)(3) + (2)(-1)}{\sqrt{(-3)^2 + 2^2 + 2^2}\,\sqrt{4^2 + 3^2 + (-1)^2}}$$

$$\cos\theta = \frac{-8}{\sqrt{17}\,\sqrt{26}}$$

$$\boxed{\theta = \cos^{-1}\left(-\frac{8}{\sqrt{17}\,\sqrt{26}}\right)}$$

Math 210, Exam 1, Spring 2001
Problem 2 Solution

2. Find the equation of the plane determined by the three points $(0,0,0)$, $(1,0,0)$, and $(2,3,4)$.

Solution: The equation of the plane will be of the form $ax+by+cz=d$. Knowing that the point $(0,0,0)$ is in the plane tells us the value of d.

$$a(0)+b(0)+c(0)=d \quad \Longleftrightarrow \quad d=0$$

We now use the fact that the point $(1,0,0)$ is in the plane to find a.

$$a(1)+b(0)+c(0)=0 \quad \Longleftrightarrow \quad a=0$$

Finally, we use the fact that the point $(2,3,4)$ is in the plane to get a relationship between b and c.

$$0(2)+b(3)+c(4)=0 \quad \Longleftrightarrow \quad b=-\frac{4}{3}c$$

Thus, the equation of the plane is:

$$\begin{aligned} ax+by+cz&=d \\ 0x-\frac{4}{3}cy+cz&=0 \\ c\left(-\frac{4}{3}y+z\right)&=0 \\ -\frac{4}{3}y+z&=0 \\ \boxed{-4y+3z=0} \end{aligned}$$

Math 210, Exam 1, Spring 2001
Problem 3 Solution

3. The position vector of a moving particle is given by

$$\vec{\mathbf{r}}(t) = \left\langle 3t - 4, 3t^2, 2t^2 + t \right\rangle .$$

(a) Find the velocity $\vec{\mathbf{v}}(t)$.

(b) Find the speed.

(c) Find the acceleration $\vec{\mathbf{a}}(t)$.

(d) Find the curvature $\kappa(t)$.

(e) Write the integral which gives the arclength from the point where $t = 0$ to the point where $t = 5$, do not evaluate the integral.

Solution:

(a) The velocity is the derivative of position.

$$\boxed{\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle 3, 6t, 4t + 1\rangle}$$

(b) The speed is the magnitude of velocity.

$$v(t) = \left|\left|\vec{\mathbf{v}}(t)\right|\right|$$
$$v(t) = \sqrt{3^2 + (6t)^2 + (4t + 1)^2}$$
$$v(t) = \sqrt{9 + 36t^2 + 16t^2 + 8t + 1}$$
$$\boxed{v(t) = \sqrt{52t^2 + 8t + 10}}$$

(c) The acceleration is the derivative of velocity.

$$\boxed{\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \langle 0, 6, 4\rangle}$$

(d) We use the following definition of curvature:

$$\kappa(t) = \frac{\left|\left|\vec{\mathbf{a}}(t) \times \vec{\mathbf{v}}(t)\right|\right|}{\left|\left|\vec{\mathbf{v}}(t)\right|\right|^3}$$

The cross product $\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t)$ is:

$$\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 6 & 4 \\ 3 & 6t & 4t+1 \end{vmatrix}$$
$$\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t) = \hat{\mathbf{i}} \begin{vmatrix} 6 & 4 \\ 6t & 4t+1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 0 & 4 \\ 3 & 4t+1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 0 & 6 \\ 3 & 6t \end{vmatrix}$$
$$\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t) = \hat{\mathbf{i}}\,[(6)(4t+1)-(6t)(4)] - \hat{\mathbf{j}}\,[(0)(4t+1)-(3)(4)] + \hat{\mathbf{k}}\,[(0)(6t)-(3)(6)]$$
$$\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t) = 6\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 18\,\hat{\mathbf{k}}$$
$$\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t) = \langle 6, 12, -18\rangle$$

The curvature function is then:

$$\kappa(t) = \frac{\left|\left|\overrightarrow{\mathbf{a}}(t) \times \overrightarrow{\mathbf{v}}(t)\right|\right|}{\left|\left|\overrightarrow{\mathbf{v}}(t)\right|\right|^3}$$
$$\kappa(t) = \frac{||\langle 6, 12, -18\rangle||}{v(t)^3}$$
$$\kappa(t) = \frac{\sqrt{6^2 + 12^2 + (-18)^2}}{(\sqrt{52t^2 + 8t + 10})^3}$$
$$\boxed{\kappa(t) = \frac{6\sqrt{14}}{(52t^2 + 8t + 10)^{3/2}}}$$

(e) The arclength integral is:

$$L = \int_0^5 \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|\,dt$$
$$\boxed{L = \int_0^5 \sqrt{52t^2 + 8t + 10}\,dt}$$

Math 210, Exam 1, Spring 2001
Problem 4 Solution

4. Let $f(x, y) = ye^{(x^2+y^2)}$.

(a) Find f_x and f_y.

(b) Find $f_{x,y}$.

Solution:

(a) The first partial derivatives f_x and f_y are:

$$f_x = \frac{\partial}{\partial x} ye^{(x^2+y^2)}$$
$$f_x = y \frac{\partial}{\partial x} e^{(x^2+y^2)}$$
$$f_x = ye^{(x^2+y^2)} \cdot \frac{\partial}{\partial x} \left(x^2 + y^2\right)$$
$$f_x = ye^{(x^2+y^2)} \cdot 2x$$

$$f_y = \frac{\partial}{\partial y} ye^{(x^2+y^2)}$$
$$f_y = y \cdot \frac{\partial}{\partial y} e^{(x^2+y^2)} + e^{(x^2+y^2)} \cdot \frac{\partial}{\partial y} y$$
$$f_y = ye^{(x^2+y^2)} \cdot \frac{\partial}{\partial y} \left(x^2 + y^2\right) + e^{(x^2+y^2)} \cdot 1$$
$$f_y = ye^{(x^2+y^2)} \cdot 2y + e^{(x^2+y^2)}$$

(b) The mixed derivative $f_{x,y}$ is:

$$f_{x,y} = (f_x)_y$$
$$f_{x,y} = \frac{\partial}{\partial y} ye^{(x^2+y^2)} \cdot 2x$$
$$f_{x,y} = \frac{\partial}{\partial y} 2xye^{(x^2+y^2)}$$
$$f_{x,y} = 2xy \cdot \frac{\partial}{\partial y} e^{(x^2+y^2)} + e^{(x^2+y^2)} \cdot \frac{\partial}{\partial y} 2xy$$
$$f_{x,y} = 2xye^{(x^2+y^2)} \cdot \frac{\partial}{\partial y} \left(x^2 + y^2\right) + e^{(x^2+y^2)} \cdot 2x \frac{\partial}{\partial y} y$$
$$f_{x,y} = 2xye^{(x^2+y^2)} \cdot 2y + e^{(x^2+y^2)} \cdot 2x$$

Math 210, Exam 1, Spring 2001
Problem 5 Solution

5. Let $f(x,y) = \dfrac{2x}{x-y}$.

(a) Find the domain of f.

(b) Sketch the level curves $f(x,y) = k$ for $k = 0, 1, 2$ and label them.

Solution:

(a) The domain of the function is the set of all pairs (x,y) such that x and y are real numbers satisfying $x - y \neq 0$.

(b) First, when $k = 0$ we have $f(x,y) = 0$ which gives us the level curve:

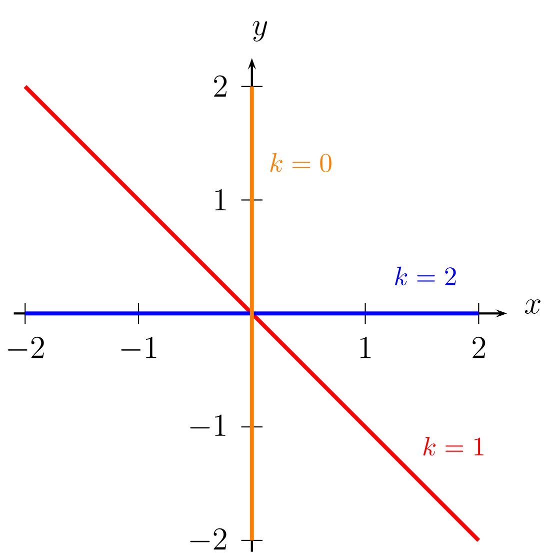
$$\frac{2x}{x-y} = 0 \quad \Longleftrightarrow \quad x = 0$$

Next, when $k = 1$ we have $f(x,y) = 1$ which gives us the level curve:

$$\frac{2x}{x-y} = 1 \quad \Longleftrightarrow \quad y = -x$$

Finally, when $k = 2$ we have $f(x,y) = 2$ which gives us the level curve:

$$\frac{2x}{x-y} = 2 \quad \Longleftrightarrow \quad y = 0$$



Math 210, Exam 1, Spring 2001
Problem 6 Solution

6. (a) Find an equation of the tangent plane to the surface

$$z = x + \ln(2x + y)$$

at the point $(-1, 3, -1)$.

(b) Find the differential of the function $f(x, y) = x + \ln(2x + y)$.

Solution:

(a) We will use the following formula for the plane tangent to $z = f(x, y) = x + \ln(2x + y)$ at the point $(-1, 3, -1)$:

$$z = f(-1, 3) + f_x(-1, 3)(x - (-1)) + f_y(-1, 3)(y - 3)$$

The first partial derivatives of f are:

$$f_x = \frac{\partial}{\partial x}[x + \ln(2x + y)] \qquad f_y = \frac{\partial}{\partial y}[x + \ln(2x + y)]$$

$$f_x = 1 + \frac{1}{2x + y} \cdot \frac{\partial}{\partial x}(2x + y) \qquad f_y = \frac{1}{2x + y} \cdot \frac{\partial}{\partial y}(2x + y)$$

$$f_x = 1 + \frac{2}{2x + y} \qquad f_y = \frac{1}{2x + y}$$

At the point $(-1, 3)$ we have:

$$f_x(-1, 3) = 1 + \frac{2}{2(-1) + 3} \qquad f_y(-1, 3) = \frac{1}{2(-1) + 3}$$

$$f_x(-1, 3) = 3 \qquad f_y(-1, 3) = 1$$

Thus, an equation for the tangent plane is:

$$\boxed{z = -1 + 3(x + 1) + (y - 3)}$$

(b) The differential df of the function $f(x, y) = x + \ln(2x + y)$ is:

$$df = f_x\,dx + f_y\,dy$$

$$\boxed{df = \left(1 + \frac{2}{2x + y}\right) dx + \left(\frac{1}{2x + y}\right) dy}$$

where the partial derivatives f_x and f_y were calculated in part (a).