Math 210, Exam 1, Spring 2008
Problem 1 Solution

1. Complete each of the following:

(a) Find a set of parametric equations for the line through $(1, -1, 2)$ and $(0, -2, -3)$.

(b) Consider the vectors $\vec{\mathbf{v}} = \langle -1, 1, 2\rangle$ and $\vec{\mathbf{w}} = \langle 2, -2, 4\rangle$. Are the vectors perpendicular, parallel, or neither? Explain.

Solution:

(a) In order to find a set of parametric equations for the line, we need a point P_0 on the line and a vector $\vec{\mathbf{v}}$ parallel to it. A vector parallel to the line is the vector from the first point to the second point. This vector is:

$$\vec{\mathbf{v}} = \langle 1-0, -1-(-2), 2-(-3)\rangle = \langle 1, 1, 5\rangle$$

Using the point $(1, -1, 2)$ as a point P_0 on the line, we have:

$$\boxed{x = 1 + t,\ y = -1 + t,\ z = 2 + 5t}$$

(b) The vectors are not parallel because they are not scalar multiples of one another. The vectors are not perpendicular because the dot product:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = (-1)(2) + (1)(-2) + (2)(4) = 4$$

is nonzero. Therefore, the vectors are **neither** parallel nor perpendicular.

Math 210, Exam 1, Spring 2008
Problem 2 Solution

2. Find the equation for the plane containing the point $(1, 2, 3)$ and the line whose vector equation is $\overrightarrow{\mathbf{r}}(t) = \langle 1 - t, t, 2 + 4t \rangle$.

Solution: To find the equation for the plane, we must find a vector $\overrightarrow{\mathbf{n}}$ perpendicular to it. To do this, we take the cross product of two vectors in the plane. One such vector is $\overrightarrow{\mathbf{v}} = \langle -1, 1, 4 \rangle$ whose components are the coefficients of t in $\overrightarrow{\mathbf{r}}(t)$. This vector is parallel to the line and lies in the plane. The other vector can be obtained by constructing a vector $\overrightarrow{\mathbf{w}}$ from the point $(1, 2, 3)$ and to a point on the line. A point on the line is $(1, 0, 2)$, whose coordinates we identify as the constants in $\overrightarrow{\mathbf{r}}(t)$. Therefore, the vector $\overrightarrow{\mathbf{w}}$ is:

$$\overrightarrow{\mathbf{w}} = \langle 1 - 1, 2 - 0, 3 - 2 \rangle = \langle 0, 2, 1 \rangle$$

The vector $\overrightarrow{\mathbf{n}}$ is:

$$\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} = \langle -7, 1, -2 \rangle$$

Thus, the equation for the plane is:

$$\boxed{-7(x - 1) + (y - 2) - 2(z - 3) = 0}$$

Math 210, Exam 1, Spring 2008
Problem 3 Solution

3. Consider the plane whose equation is $x - y + 2z = 1$ and the point $A = (0, -1, 1)$.

(a) Find a point in the plane. Call it P.

(b) Find a vector $\vec{\mathbf{v}}$ perpendicular to the plane and compute $\hat{\mathbf{e}}_{\mathbf{v}}$, the unit vector in the direction of $\vec{\mathbf{v}}$.

(c) Compute $\text{proj}_{\hat{\mathbf{e}}_{\mathbf{v}}} \overrightarrow{PA}$, the vector projection of $\overrightarrow{PA}$ onto $\hat{\mathbf{e}}_{\mathbf{v}}$.

(d) Compute $\left|\left|\text{proj}_{\hat{\mathbf{e}}_{\mathbf{v}}} \overrightarrow{PA}\right|\right|$. What is the geometric meaning of this quantity?

Solution:

(a) Let $P = (1, 0, 0)$. This works since $x - y + 2z = 1 - 0 + 2(0) = 1$.

(b) The components of $\vec{\mathbf{v}}$ are the coefficients of x, y, and z in the plane equation:

$$\vec{\mathbf{v}} = \langle 1, -1, 2 \rangle$$

The unit vector in the direction of $\vec{\mathbf{v}}$ is:

$$\hat{\mathbf{e}}_{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{v}}||} = \frac{\langle 1, -1, 2 \rangle}{\sqrt{1^2 + (-1)^2 + 2^2}} = \boxed{\left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$$

(c) The vector $\overrightarrow{PA}$ is:

$$\overrightarrow{PA} = \langle 0 - 1, -1 - 0, 1 - 0 \rangle = \langle -1, -1, 1 \rangle$$

The projection of $\overrightarrow{PA}$ onto $\hat{\mathbf{e}}_{\mathbf{v}}$ is:

$$\begin{aligned} \text{proj}_{\hat{\mathbf{e}}_{\mathbf{v}}} \overrightarrow{PA} &= \left(\overrightarrow{PA} \cdot \hat{\mathbf{e}}_{\mathbf{v}}\right) \hat{\mathbf{e}}_{\mathbf{v}} \\ &= \left[(-1)\left(\frac{1}{\sqrt{6}}\right) + (-1)\left(-\frac{1}{\sqrt{6}}\right) + (1)\left(\frac{2}{\sqrt{6}}\right)\right] \\ &= \frac{2}{\sqrt{6}} \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle \\ &= \boxed{\left\langle \frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle} \end{aligned}$$

(d) The magnitude of the projection is:

$$\left|\left|\text{proj}_{\hat{\mathbf{e}}_{\mathbf{v}}} \overrightarrow{PA}\right|\right| = \boxed{\frac{2}{\sqrt{6}}}$$

This quantity represents the distance between the point A and the plane.

Math 210, Exam 1, Spring 2008
Problem 4 Solution

4. Consider the vector-valued function $\vec{\mathbf{r}}(t) = \langle e^{3t}, \sin(2t), t^2 \rangle$.

(a) Compute $\vec{\mathbf{r}}'(t)$.

(b) Compute the unit tangent vector, $\vec{\mathbf{T}}(t)$. Do not attempt to simplify your answer.

(c) Given the function $g(t) = 4t + 1$, compute $\frac{d}{dt}\vec{\mathbf{r}}(g(t))$ using the Chain Rule.

Solution:

(a) The derivative $\vec{\mathbf{r}}'(t)$ is:

$$\boxed{\vec{\mathbf{r}}'(t) = \left\langle 3e^{3t}, 2\cos(2t), 2t \right\rangle}$$

(b) The unit tangent vector is:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{||\vec{\mathbf{r}}'(t)||}$$

$$\vec{\mathbf{T}}(t) = \frac{\langle 3e^{3t}, 2\cos(2t), 2t \rangle}{\sqrt{(3e^{3t})^2 + [2\cos(2t)]^2 + (2t)^2}}$$

$$\boxed{\vec{\mathbf{T}}(t) = \frac{\langle 3e^{3t}, 2\cos(2t), 2t \rangle}{\sqrt{9e^{6t} + 4\cos^2(2t) + 4t^2}}}$$

(c) Using the Chain Rule, we have:

$$\frac{d}{dt}\vec{\mathbf{r}}(g(t)) = g'(t)\,\vec{\mathbf{r}}'(g(t))$$

$$\frac{d}{dt}\vec{\mathbf{r}}(g(t)) = \frac{d}{dt}(4t+1)\,\vec{\mathbf{r}}'(4t+1)$$

$$\boxed{\frac{d}{dt}\vec{\mathbf{r}}(g(t)) = 4\left\langle 3e^{3(4t+1)}, 2\cos[2(4t+1)], 2(4t+1) \right\rangle}$$

Math 210, Exam 1, Spring 2008
Problem 5 Solution

5. Find the length of the curve $\vec{\mathbf{r}}(t) = \left\langle \sqrt{2}t, \ln t, \frac{1}{2}t^2 \right\rangle$ for $1 \leq t \leq 2$. (Hint: In the integrand, the expression under the square root is a perfect square.)

Solution: The equation for arc length is:

$$L = \int_a^b \left|\left|\vec{\mathbf{r}}'(t)\right|\right| \, dt$$

The derivative $\vec{\mathbf{r}}'(t)$ and its magnitude $\left|\left|\vec{\mathbf{r}}'(t)\right|\right|$ are:

$$\vec{\mathbf{r}}'(t) = \left\langle \sqrt{2}, \frac{1}{t}, t \right\rangle$$

$$\left|\left|\vec{\mathbf{r}}'(t)\right|\right| = \sqrt{2 + \frac{1}{t^2} + t^2}$$

The arc length of the curve is then:

$$\begin{aligned} L &= \int_1^2 \sqrt{2 + \frac{1}{t^2} + t^2} \, dt \\ &= \int_1^2 \sqrt{\left(\frac{1}{t} + t\right)^2} \, dt \\ &= \int_1^2 \left(\frac{1}{t} + t\right) \, dt \\ &= \left[\ln|t| + \frac{1}{2}t^2\right]_1^2 \\ &= \left[(\ln 2 + 2) - \left(\ln 1 + \frac{1}{2}\right)\right] \\ &= \boxed{\ln 2 + \frac{3}{2}} \end{aligned}$$