Math 210, Exam 1, Spring 2010
Problem 1a Solution

1a. Consider the curve $\overrightarrow{\mathbf{r}}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$.

(1) Find the arc length of $\overrightarrow{\mathbf{r}}(t)$ from $t = 0$ to $t = 1$.

(2) Find the curvature at $t = 1$.

Solution:

(1) The derivative of $\overrightarrow{\mathbf{r}}(t)$ is $\overrightarrow{\mathbf{r}}'(t) = \langle 1, 2t, 2t^2\rangle$. The magnitude of $\overrightarrow{\mathbf{r}}'(t)$ is computed and simplified as follows:

$$\begin{aligned}
||\overrightarrow{\mathbf{r}}'(t)|| &= \sqrt{1^2 + (2t)^2 + (2t^2)^2} \\
||\overrightarrow{\mathbf{r}}'(t)|| &= \sqrt{1 + 4t^2 + 4t^4} \\
||\overrightarrow{\mathbf{r}}'(t)|| &= \sqrt{(1 + 2t^2)^2} \\
||\overrightarrow{\mathbf{r}}'(t)|| &= 1 + 2t^2
\end{aligned}$$

We can now compute the arc length from $t = 0$ to $t = 1$.

$$\begin{aligned}
L &= \int_0^1 ||\overrightarrow{\mathbf{r}}'(t)||\, dt \\
L &= \int_0^1 \left(1 + 2t^2\right) dt \\
L &= \left[t + \frac{2}{3}t^3\right]_0^1 \\
L &= \left[1 + \frac{2}{3}(1)^3\right] - \left[0 + \frac{2}{3}(0)^3\right] \\
&\boxed{L = \frac{5}{3}}
\end{aligned}$$

(2) The curvature formula we will use is:

$$\kappa(1) = \frac{||\overrightarrow{\mathbf{r}}'(1) \times \overrightarrow{\mathbf{r}}''(1)||}{||\overrightarrow{\mathbf{r}}'(1)||^3}$$

The first two derivatives of $\overrightarrow{\mathbf{r}}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ are:

$$\begin{aligned}
\overrightarrow{\mathbf{r}}'(t) &= \left\langle 1, 2t, 2t^2\right\rangle \\
\overrightarrow{\mathbf{r}}''(t) &= \langle 0, 2, 4t\rangle
\end{aligned}$$

We now evaluate the derivatives at $t = 1$.

$$\vec{\mathbf{r}}'(1) = \langle 1, 2, 2 \rangle$$
$$\vec{\mathbf{r}}''(1) = \langle 0, 2, 4 \rangle$$

The cross product of these vectors is:

$$\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix}$$
$$\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1) = \hat{\mathbf{i}} \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$
$$\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1) = \hat{\mathbf{i}}\,[(2)(4) - (2)(2)] - \hat{\mathbf{j}}\,[(1)(4) - (0)(2)] + \hat{\mathbf{k}}\,[(1)(2) - (0)(2)]$$
$$\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1) = 4\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1) = \langle 4, -4, 2 \rangle$$

We can now compute the curvature.

$$\kappa(1) = \frac{\left|\left|\vec{\mathbf{r}}'(1) \times \vec{\mathbf{r}}''(1)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(1)\right|\right|^3}$$
$$\kappa(1) = \frac{||\langle 4, -4, 2 \rangle||}{||\langle 1, 2, 2 \rangle||^3}$$
$$\kappa(1) = \frac{\sqrt{4^2 + (-4)^2 + 2^2}}{(\sqrt{1^2 + 2^2 + 2^2})^3}$$
$$\kappa(1) = \frac{\sqrt{36}}{(\sqrt{9})^3}$$
$$\boxed{\kappa(1) = \frac{2}{9}}$$

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Problem 1b Solution

1b. A particle moves along the space curve $\vec{\mathbf{r}}(t) = (t\cos(t), t\sin(t), t)$.

(1) Find the velocity, acceleration, and speed as functions of time.

(2) Find the unit tangent and unit normal vectors at $t = 0$.

Solution:

(1) The velocity and acceleration functions are:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle \cos(t) - t\sin(t), \sin(t) + t\cos(t), 1\rangle$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = \langle -2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t), 0\rangle$$

The speed is computed and simplified as follows:

$$||\vec{\mathbf{v}}(t)|| = \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 + 1^2}$$
$$||\vec{\mathbf{v}}(t)|| = \sqrt{\cos^2(t) - 2t\sin(t)\cos(t) + t^2\sin^2(t) + \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t) + 1}$$
$$||\vec{\mathbf{v}}(t)|| = \sqrt{(\cos^2(t) + \sin^2(t)) + t^2(\sin^2(t) + \cos^2(t)) + 1}$$
$$||\vec{\mathbf{v}}(t)|| = \sqrt{1 + t^2(1) + 1}$$
$$||\vec{\mathbf{v}}(t)|| = \sqrt{t^2 + 2}$$

(2) At $t = 0$ the velocity vector is:

$$\vec{\mathbf{v}}(0) = \langle \cos(0) - 0\cdot\sin(0), \sin(0) + 0\cdot\cos(0), 1\rangle = \langle 1, 0, 1\rangle$$

Thus, the unit tangent vector at $t = 0$ is:

$$\vec{\mathbf{T}}(0) = \frac{1}{||\vec{\mathbf{v}}(0)||}\vec{\mathbf{v}}(0)$$
$$\vec{\mathbf{T}}(0) = \frac{1}{||\langle 1, 0, 1\rangle||}\langle 1, 0, 1\rangle$$
$$\vec{\mathbf{T}}(0) = \frac{1}{\sqrt{1^2 + 0^2 + 1^2}}\langle 1, 0, 1\rangle$$
$$\boxed{\vec{\mathbf{T}}(0) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle}$$

To compute the unit normal vector at $t = 0$ we will use the formula:

$$\vec{\mathbf{a}} = a_T\,\vec{\mathbf{T}} + a_N\,\vec{\mathbf{N}}$$

instead of

$$\vec{\mathbf{N}} = \frac{1}{\left|\left|\vec{\mathbf{T}}'\right|\right|}\,\vec{\mathbf{T}}'$$

due to the complicated derivative $\vec{\mathbf{T}}'$. To start, we evaluate $\vec{\mathbf{a}}(0)$ and a_T as follows:

$$\begin{aligned}
\vec{\mathbf{a}}(0) &= \langle -2\sin(0) - 0\cdot\cos(0), 2\cos(0) - 0\cdot\sin(0), 0\rangle = \langle 0, 2, 0\rangle \\
a_T &= \frac{\left\langle \vec{\mathbf{a}}(0)\cdot\vec{\mathbf{v}}(0)\right\rangle}{\left|\left|\vec{\mathbf{v}}(0)\right|\right|} = \frac{\langle 0,2,0\rangle\cdot\langle 1,0,1\rangle}{||\langle 1,0,1\rangle||} = 0
\end{aligned}$$

Therefore, we know that the tangential component of the acceleration is zero at $t = 0$ and:

$$\begin{aligned}
\vec{\mathbf{a}} &= a_T\,\vec{\mathbf{T}} + a_N\,\vec{\mathbf{N}} \\
\vec{\mathbf{a}} &= a_N\,\vec{\mathbf{N}}
\end{aligned}$$

Taking the magnitude of both sides and solving for a_N we get:

$$\begin{aligned}
\left|\left|\vec{\mathbf{a}}\right|\right| &= \left|\left|a_N\,\vec{\mathbf{N}}\right|\right| \\
\left|\left|\vec{\mathbf{a}}\right|\right| &= a_N\,\left|\left|\vec{\mathbf{N}}\right|\right| \\
\left|\left|\vec{\mathbf{a}}\right|\right| &= a_N\,(1) \\
a_N &= \left|\left|\vec{\mathbf{a}}\right|\right| \\
a_N &= ||\langle 0,2,0\rangle|| \\
a_N &= 2
\end{aligned}$$

Thus, the unit normal vector at $t = 0$ is:

$$\begin{aligned}
\vec{\mathbf{N}}(0) &= \frac{1}{a_N}\,\vec{\mathbf{a}}(0) \\
\vec{\mathbf{N}}(0) &= \frac{1}{2}\,\langle 0,2,0\rangle \\
&\boxed{\vec{\mathbf{N}}(0) = \langle 0,1,0\rangle}
\end{aligned}$$

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Problem 2a Solution

2a. Let $f(x, y) = x\sqrt{y} + y$.

(1) Find f_x and f_y at $(2, 4)$.

(2) Write the equation of the tangent plane to the graph of f at the point $(2, 4)$.

Solution:

(1) The first derivatives f_x and f_y are

$$f_x = \sqrt{y}$$
$$f_y = \frac{x}{2\sqrt{y}} + 1$$

At the point $(2, 4)$, the first derivatives are:

$$f_x(2, 4) = \sqrt{4} = \boxed{2}$$
$$f_y(2, 4) = \frac{2}{2\sqrt{4}} + 1 = \boxed{\frac{3}{2}}$$

(2) An equation for the tangent plane at $(2, 4)$ is:

$$z = f(2, 4) + f_x(2, 4)(x - 2) + f_y(2, 4)(y - 4)$$
$$\boxed{z = 8 + 2(x - 2) + \frac{3}{2}(y - 4)}$$

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Problem 2b Solution

2b. Let $f(x,y) = \sqrt{x^2 + 3y^2}$.

(1) Find f_x and f_y at $(1,4)$.

(2) Write the equation of the tangent plane to the graph of f at the point $(1,4)$.

Solution:

(1) The first derivatives f_x and f_y are

$$f_x = \frac{x}{\sqrt{x^2+3y^2}}$$
$$f_y = \frac{3y}{\sqrt{x^2+3y^2}}$$

At the point $(1,4)$, the first derivatives are:

$$f_x(1,4) = \frac{1}{\sqrt{1^2+3(4)^2}} = \boxed{\frac{1}{7}}$$
$$f_y(1,4) = \frac{3(4)}{\sqrt{1^2+3(4)^2}} = \boxed{\frac{12}{7}}$$

(2) An equation for the tangent plane at $(1,4)$ is:

$$z = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$$
$$\boxed{z = 7 + \frac{1}{7}(x-1) + \frac{12}{7}(y-4)}$$

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Problem 3a Solution

3a. Compute $\frac{dw}{dt}$ for $w = e^{-x}\sin(x+y)$, where $x = t^2$ and $y = 1-t$.

Solution: Using the Chain Rule, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$
$$\frac{dw}{dt} = \left[-e^{-x}\sin(x+y) + e^{-x}\cos(x+y)\right](2t) + \left[e^{-x}\cos(x+y)\right](-1)$$
$$\frac{dw}{dt} = 2te^{-x}\left[\cos(x+y) - \sin(x+y)\right] - e^{-x}\cos(x+y)$$
$$\boxed{\frac{dw}{dt} = 2te^{-t^2}\left[\cos\left(t^2+1-t\right) - \sin\left(t^2+1-t\right)\right] - e^{-t^2}\cos\left(t^2+1-t\right)}$$

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Problem 3b Solution

3b. Compute $\frac{dw}{dt}$ for $w = e^{y-x}\sin(y)$, where $x = t^2$ and $y = 1 - t$.

Solution: Using the Chain Rule, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{dw}{dt} = \left[-e^{y-x}\sin(y)\right](2t) + \left[e^{y-x}\sin(y) + e^{y-x}\cos(y)\right](-1)$$

$$\frac{dw}{dt} = -2te^{y-x}\sin(y) - e^{y-x}\left[\sin(y) + \cos(y)\right]$$

$$\boxed{\frac{dw}{dt} = -2te^{1-t-t^2}\sin(1-t) - e^{1-t-t^2}\left[\sin(1-t) + \cos(1-t)\right]}$$

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Problem 4a Solution

4a. Let $f(x, y, z) = x^2 + yz$.

(1) Compute the gradient of f.

(2) Find the derivative of f at $(1, 1, -3)$ in the direction $\vec{\mathbf{u}} = \frac{1}{3}\left(2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}\right)$.

(3) In what direction of f increasing most rapidly at $(1, 1, -3)$.

Solution:

(1) By definition, the gradient of $f(x, y, z)$ is $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$. For the function $f(x, y, z) = x^2 + yz$, we have

$$\boxed{\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle 2x, z, y \rangle}$$

(2) The gradient of f evaluated at $(1, 1, -3)$ is

$$\vec{\nabla} f(1, 1, -3) = \langle 2(1), -3, 1 \rangle = \langle 2, -3, 1 \rangle$$

Thus, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ at $(1, 1, -3)$ is

$$D_{\mathbf{u}} f(1, 1, -3) = \vec{\nabla} f(1, 1, -3) \cdot \vec{\mathbf{u}}$$
$$D_{\mathbf{u}} f(1, 1, -3) = \langle 2, -3, 1 \rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$
$$D_{\mathbf{u}} f(1, 1, -3) = (2)\left(\frac{2}{3}\right) + (-3)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)$$
$$\boxed{D_{\mathbf{u}} f(1, 1, -3) = 1}$$

(3) The direction in which f is increasing most rapidly is

$$\hat{\mathbf{u}} = \frac{1}{\left|\left|\vec{\nabla} f\right|\right|} \vec{\nabla} f$$

At the point $(1, 1, -3)$ we have

$$\hat{\mathbf{u}} = \frac{1}{||\langle 2, -3, 1 \rangle||} \langle 2, -3, 1 \rangle$$
$$\hat{\mathbf{u}} = \frac{1}{\sqrt{14}} \langle 2, -3, 1 \rangle$$
$$\boxed{\hat{\mathbf{u}} = \left\langle \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle}$$

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Problem 4b Solution

4b. Let $f(x, y) = x^2y$.

(1) Compute the gradient of f.

(2) Find the derivative of f at $(1, 2)$ in the direction $\vec{\mathbf{u}} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$.

(3) In what direction of f increasing most rapidly at $(1, 2)$.

Solution:

(1) By definition, the gradient of $f(x, y)$ is $\vec{\nabla} f = \langle f_x, f_y \rangle$. For the function $f(x, y) = x^2y$, we have

$$\boxed{\vec{\nabla} f = \langle f_x, f_y \rangle = \left\langle 2xy, x^2 \right\rangle}$$

(2) The gradient of f evaluated at $(1, 2)$ is

$$\vec{\nabla} f(1, 2) = \left\langle 2(1)(2), 1^2 \right\rangle = \langle 4, 1 \rangle$$

Thus, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ at $(1, 2)$ is

$$D_{\mathbf{u}} f(1, 2) = \vec{\nabla} f(1, 2) \cdot \vec{\mathbf{u}}$$
$$D_{\mathbf{u}} f(1, 2) = \langle 4, 1 \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$
$$D_{\mathbf{u}} f(1, 2) = (4)\left(\frac{4}{5}\right) + (1)\left(-\frac{3}{5}\right)$$
$$\boxed{D_{\mathbf{u}} f(1, 2) = \frac{13}{5}}$$

(3) The direction in which f is increasing most rapidly is

$$\hat{\mathbf{u}} = \frac{1}{\left|\left|\vec{\nabla} f\right|\right|} \vec{\nabla} f$$

At the point $(1, 2)$ we have

$$\hat{\mathbf{u}} = \frac{1}{||\langle 4, 1 \rangle||} \langle 4, 1 \rangle$$
$$\hat{\mathbf{u}} = \frac{1}{\sqrt{17}} \langle 4, 1 \rangle$$
$$\boxed{\hat{\mathbf{u}} = \left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle}$$

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Problem 5a Solution

5a. Consider the three points $A = (0, 0, 0)$, $B = (3, 1, -1)$, $C = (1, 1, 1)$.

(1) Find the equation for the plane which contains the points A, B, C.

(2) What is the area of the triangle ABC?

Solution:

(1) A vector perpendicular to the plane is the cross product of $\overrightarrow{AB} = \langle 3, 1, -1 \rangle$ and $\overrightarrow{BC} = \langle -2, 0, 2 \rangle$ which both lie in the plane.

$$\vec{\mathbf{n}} = \overrightarrow{AB} \times \overrightarrow{BC}$$
$$\vec{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & -1 \\ -2 & 0 & 2 \end{vmatrix}$$
$$\vec{\mathbf{n}} = \hat{\mathbf{i}} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix}$$
$$\vec{\mathbf{n}} = \hat{\mathbf{i}}\,[(1)(2) - (0)(-1)] - \hat{\mathbf{j}}\,[(3)(2) - (-2)(-1)] + \hat{\mathbf{k}}\,[(3)(0) - (-2)(1)]$$
$$\vec{\mathbf{n}} = 2\,\hat{\mathbf{i}} - 8\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{n}} = \langle 2, -8, 2 \rangle$$

Using $A = (0, 0, 0)$ as a point on the plane, we have:

$$\boxed{2(x - 0) - 8(y - 0) + 2(z - 0) = 0}$$

(2) The area of the triangle is half the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, which represents the area of the parallelogram spanned by the two vectors:

$$A = \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|\right|$$
$$A = \frac{1}{2}\sqrt{2^2 + (-8)^2 + 2^2}$$
$$A = \frac{1}{2}\sqrt{72}$$
$$\boxed{A = 3\sqrt{2}}$$

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Problem 5b Solution

5b. Consider the plane $z - 2y + z = 7$ and the point $P = (0, 2, 3)$.

(1) Find the line through P that is perpendicular to the plane.

(2) At what point does the line from part (a) intersect the plane?

(3) Find the distance between P and the plane.

Solution: First, we note that there is a typo in the problem. The equation for the plane should be $x - 2y + z = 7$.

(1) To find the line, we need a vector parallel to the line. This vector is also perpendicular to the plane. From the plane equation, we identify this vector as the coefficients of x, y, and z:

$$\vec{\mathbf{v}} = \langle 1, -2, 1 \rangle$$

Then, using $P = (0, 2, 3)$ as a point on the line we have the following parametric equations for the line:

$$\boxed{x = t, \quad y = 2 - 2t, \quad z = 3 + t}$$

(2) To find the point of intersection, we plug the parametric equations for the line into the plane equation and solve for t:

$$\begin{aligned} x - 2y + z &= 7 \\ t - 2(2 - 2t) + (3 + t) &= 7 \\ 6t &= 8 \\ t &= \frac{4}{3} \end{aligned}$$

Now plug this back into the parametric equations to get the coordinates of the point of intersection:

$$\boxed{x = \frac{4}{3}, \quad y = -\frac{2}{3}, \quad z = \frac{13}{3}}$$

(3) The distance between P and the plane is the distance between the points $P = (0, 2, 3)$ and $Q = \left(\frac{4}{3}, -\frac{2}{3}, \frac{13}{3}\right)$:

$$|PQ| = \sqrt{\left(0 - \frac{4}{3}\right)^2 + \left(2 + \frac{2}{3}\right)^2 + \left(3 - \frac{13}{3}\right)^2} = \sqrt{\frac{96}{9}} = \boxed{\frac{4\sqrt{6}}{3}}$$