Math 210, Exam 1, Spring 2012
Problem 1 Solution

1. (a) Find an equation for the plane containing the points $(1, 0, 3)$, $(0, 4, 2)$, and $(1, 1, 1)$.

(b) Find the distance from the point $(1, 0, 1)$ to the plane from part (a).

Solution:

(a) In order to find an equation for the plane we must know a vector $\overrightarrow{\mathbf{n}}$ perpendicular to plane. Using the fact that the cross product of two vectors is perpendicular to each of the vectors, we let $\overrightarrow{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{QR}$ where $P = (1, 0, 3)$, $Q = (0, 4, 2)$, and $R = (1, 1, 1)$.

$$\overrightarrow{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{QR},$$
$$\overrightarrow{\mathbf{n}} = \langle -1, 4, -1 \rangle \times \langle 1, -3, -1 \rangle,$$
$$\overrightarrow{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 4 & -1 \\ 1 & -3 & -1 \end{vmatrix},$$
$$\overrightarrow{\mathbf{n}} = \hat{\mathbf{i}}[(4)(-1) - (-1)(-3)] - \hat{\mathbf{j}}[(-1)(-1) - (-1)(1)] + \hat{\mathbf{k}}[(-1)(-3) - (4)(1)],$$
$$\overrightarrow{\mathbf{n}} = -7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}},$$
$$\overrightarrow{\mathbf{n}} = \langle -7, -2, -1 \rangle.$$

If (x_0, y_0, z_0) is a point on the plane and $\overrightarrow{\mathbf{n}} = \langle a, b, c \rangle$ is perpendicular to the plane, then we know that an equation for the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Using $P = (1, 0, 3)$ as a point on the plane we have

$$-7(x - 1) - 2y - (z - 3) = 0$$

(b) Let $S = (x, y, z)$ be the point on the plane closest to the point $(1, 0, 1)$. The line that connects (x, y, z) and $(1, 0, 1)$ is perpendicular to the plane. Thus, the vector $\overrightarrow{\mathbf{n}} = \langle -7, -2, -1 \rangle$ is parallel to the line. If (x_0, y_0, z_0) is a point on a line and the vector $\langle a, b, c \rangle$ is parallel to it, then we know that the line is described by the parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Using $(1, 0, 1)$ as a point on the line we have

$$x = 1 - 7t, \quad y = -2t, \quad z = 1 - t$$

S is the point of intersection of the line and the plane. We determine the coordinates of this point by substituting the parametric equations for the line into the plane equation, solving for t, and then plugging this value of t back into the parametric equations.

The value of t is then

$$\begin{aligned}
-7(x-1) - 2y - (z-3) &= 0, \\
7x + 2y + z &= 10, \\
7(1-7t) + 2(-2t) + (1-t) &= 10, \\
7 - 49t - 4t + 1 - t &= 10, \\
-54t &= 2, \\
t &= -\frac{1}{27}.
\end{aligned}$$

Plugging this into the parametric equations for the line gives us

$$x = \frac{34}{27}, \quad y = \frac{2}{27}, \quad z = \frac{28}{27}$$

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Problem 2 Solution

2. If $\overrightarrow{\mathbf{r}}(t) = \langle t^2, t^3 \rangle$, then find the tangential component of the acceleration vector $\overrightarrow{\mathbf{r}}''(t)$ at $t = 1$.

Solution: The velocity and acceleration vectors for the given position vector are

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{r}}'(t) = \left\langle 2t, 3t^2 \right\rangle, \quad \overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{r}}''(t) = \langle 2, 6t \rangle .$$

When $t = 1$ we get

$$\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}(1) = \langle 2, 3 \rangle , \quad \overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{a}}(1) = \langle 2, 6 \rangle .$$

The tangential component of $\overrightarrow{\mathbf{a}}$ is computed as follows

$$\begin{aligned} a_T &= \frac{\overrightarrow{\mathbf{a}} \bullet \overrightarrow{\mathbf{v}}}{||\overrightarrow{\mathbf{v}}||}, \\ a_T &= \frac{\langle 2, 6 \rangle \bullet \langle 2, 3 \rangle}{||\langle 2, 3 \rangle||}, \\ a_T &= \frac{2 \cdot 2 + 6 \cdot 3}{\sqrt{2^2 + 3^2}}, \\ a_T &= \frac{22}{\sqrt{13}}. \end{aligned}$$

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Problem 3 Solution

3. Let ℓ be the line passing through the origin $(0, 0, 0)$ and the point $(2, 0, 0)$. Find the point of intersection between ℓ and the plane $x - 2y + z = 1$.

Solution: A set of parametric equation that describes ℓ is

$$x = 2t, \quad y = 0, \quad z = 0.$$

The point of intersection between ℓ and the plane is found by plugging the parametric equations into the equation of the plane, solving for t, and then plugging this value of t back into the parametric equations.

The value of t is then

$$\begin{aligned} x - 2y + z &= 1, \\ 2t - 2(0) + 0 &= 1, \\ t &= \frac{1}{2}. \end{aligned}$$

Plugging this into the parametric equations for the line gives us

$$x = 1, \quad y = 0, \quad z = 0.$$

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Problem 4 Solution

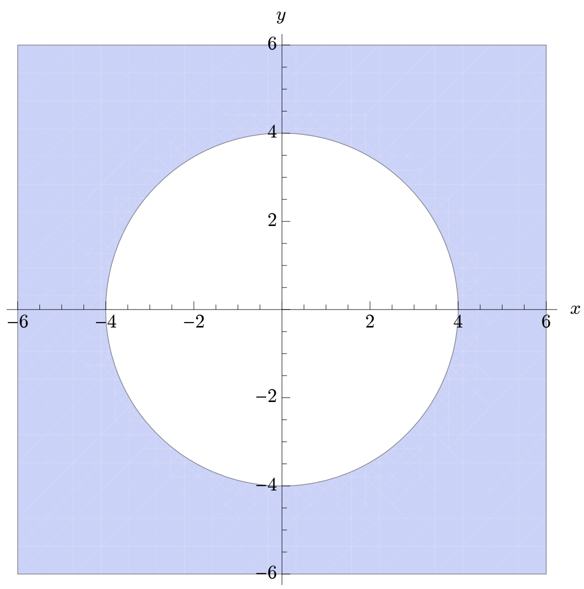
4. Let $f(x,y) = \ln(x^2 + y^2 - 16)$.

(a) Sketch the domain of f.

(b) Compute the partial derivatives f_x, f_y, and f_{xx}.

Solution:

(a) The domain of f is $\{(x,y) : x^2 + y^2 > 16\}$. That is, the domain is the set of all points that lie outside the disk of radius 4 centered at the origin.



(b) The first partial derivatives f_x and f_y are found using the Chain Rule.

$$f_x = \frac{1}{x^2+y^2-16} \cdot \frac{\partial}{\partial x}\left(x^2+y^2-16\right) \qquad f_y = \frac{1}{x^2+y^2-16} \cdot \frac{\partial}{\partial y}\left(x^2+y^2-16\right),$$

$$f_x = \frac{1}{x^2+y^2-16} \cdot 2x \qquad f_y = \frac{1}{x^2+y^2-16} \cdot 2y.$$

The second partial derivative f_{xx} is found using the Quotient Rule.

$$f_{xx} = \frac{\partial}{\partial x} \frac{2x}{x^2+y^2-16},$$

$$f_{xx} = \frac{(x^2+y^2-16)(2) - (2x)(2x)}{(x^2+y^2-16)^2},$$

$$f_{xx} = \frac{-2x^2+2y^2-32}{(x^2+y^2-16)^2}.$$

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Problem 5 Solution

5. Find the arc length of the curve parameterized by $\vec{\mathbf{r}}(t) = \left\langle \ln(t), \frac{t^4}{4}, \frac{t^2}{\sqrt{2}} \right\rangle$ between $t_0 = 1$ and $t_1 = 2$.

Solution: The formula we will use for computing arc length is

$$L = \int_{t_0}^{t_1} \left|\left|\vec{\mathbf{r}}'(t)\right|\right| \, dt.$$

The derivative $\vec{\mathbf{r}}'(t)$ and its magnitude, after some simplification, are

$$\vec{\mathbf{r}}'(t) = \left\langle \frac{1}{t}, t^3, \sqrt{2}t \right\rangle,$$
$$\left|\left|\vec{\mathbf{r}}'(t)\right|\right| = \sqrt{\frac{1}{t^2} + t^6 + 2t^2}.$$

In order to find an antiderivative of $\left|\left|\vec{\mathbf{r}}'(t)\right|\right|$ we first recognize that the term under the square root is a perfect square. In fact,

$$\frac{1}{t^2} + t^6 + 2t^2 = \left(\frac{1}{t} + t^3\right)^2$$

which can be verified by expanding the right hand side. Therefore, the arc length is

$$L = \int_1^2 \sqrt{\left(\frac{1}{t} + t^3\right)^2} \, dt,$$
$$L = \int_1^2 \left(\frac{1}{t} + t^3\right) \, dt,$$
$$L = \left[\ln(t) + \frac{t^4}{4}\right]_1^2,$$
$$L = \left[\ln(2) + \frac{2^4}{4}\right] - \left[\ln(1) + \frac{1^4}{4}\right],$$
$$L = \ln(2) + \frac{15}{4}.$$

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Problem 6 Solution

6. Consider the curve parameterized by

$$\mathbf{r}(t) = \left\langle \cos(t), \sqrt{2}\sin(t), \cos(t) \right\rangle.$$

Find the unit tangent vector $\mathbf{T}(t)$, the principal unit normal $\mathbf{N}(t)$, and the curvature $\kappa(t)$.

Solution: The derivative $\mathbf{r}'(t)$ is

$$\mathbf{r}'(t) = \left\langle -\sin(t), \sqrt{2}\cos(t), -\sin(t) \right\rangle.$$

By definition, the unit tangent vector $\mathbf{T}(t)$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$

where the magnitude of $\mathbf{r}'(t)$ is

$$\begin{aligned}
||\mathbf{r}'(t)|| &= \sqrt{(-\sin(t))^2 + (\sqrt{2}\cos(t))^2 + (-\sin(t))^2}, \\
||\mathbf{r}'(t)|| &= \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)}, \\
||\mathbf{r}'(t)|| &= \sqrt{2\sin^2(t) + 2\cos^2(t)}, \\
||\mathbf{r}'(t)|| &= \sqrt{2}.
\end{aligned}$$

Thus,

$$\mathbf{T}(t) = \frac{\left\langle -\sin(t), \sqrt{2}\cos(t), -\sin(t) \right\rangle}{\sqrt{2}} = \left\langle -\frac{1}{\sqrt{2}}\sin(t), \cos(t), -\frac{1}{\sqrt{2}}\sin(t) \right\rangle.$$

By definition, the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$

where

$$\mathbf{T}'(t) = \left\langle -\frac{1}{\sqrt{2}}\cos(t), -\sin(t), -\frac{1}{\sqrt{2}}\cos(t) \right\rangle$$

and

$$||\mathbf{T}'(t)|| = \sqrt{\frac{1}{2}\cos^2(t) + \sin^2(t) + \frac{1}{2}\cos^2(t)} = 1.$$

Therefore,

$$\mathbf{N}(t) = \left\langle -\frac{1}{\sqrt{2}}\cos(t), -\sin(t), -\frac{1}{\sqrt{2}}\cos(t) \right\rangle.$$

Lastly, the curvature $\kappa(t)$ can be computed via the formula

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{1}{\sqrt{2}}$$