Math 210, Exam 1, Spring 2013
Problem 1 Solution

1. Let $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle 3, 2, 1\rangle$, and $\mathbf{w} = \langle 2, 1, 3\rangle$. Compute each of the following quantities:

   (a) $\mathbf{u} \cdot \mathbf{v}$

   (b) $\mathbf{u} \times \mathbf{v}$

   (c) $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$

   (d) $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Solution:

(a) The dot product is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(2) + (3)(1)$$

ANSWER $\boxed{\mathbf{u} \cdot \mathbf{v} = 10}$

(b) The cross product is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \hat{\mathbf{k}}$$

$$\mathbf{u} \times \mathbf{v} = [(2)(1) - (3)(2)]\hat{\mathbf{i}} - [(1)(1) - (3)(3)]\hat{\mathbf{j}} + [(1)(2) - (2)(3)]\hat{\mathbf{k}}$$

$$\mathbf{u} \times \mathbf{v} = -4\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

ANSWER $\boxed{\mathbf{u} \times \mathbf{v} = \langle -4, 8, -4\rangle}$

(c) The cross product $\mathbf{v} \times \mathbf{w}$ must be computed first.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{k}}$$

$$\mathbf{v} \times \mathbf{w} = [(2)(3) - (1)(1)]\hat{\mathbf{i}} - [(3)(3) - (1)(2)]\hat{\mathbf{j}} + [(3)(1) - (2)(2)]\hat{\mathbf{k}}$$

$$\mathbf{v} \times \mathbf{w} = 5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\mathbf{v} \times \mathbf{w} = \langle 5, -7, -1\rangle$$

The dot product of $\mathbf{u}$ with this vector gives us the result:

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \langle 1, 2, 3 \rangle \cdot \langle 5, -7, -1 \rangle$$
$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = (1)(5) + (2)(-7) + (3)(-1)$$

ANSWER $\boxed{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = -12}$

(d) The value of $\cos\theta$ may be computed using the definition

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

The dot product was computed to be 10 in part (a). The magnitudes of the vectors $\mathbf{u}$ and $\mathbf{v}$ are

$$||\mathbf{u}|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
$$||\mathbf{v}|| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

Therefore, the value of $\cos\theta$ is

ANSWER $\boxed{\cos\theta = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{5}{7}}$

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Problem 2 Solution

2. Consider the curve C with parametrization $\mathbf{r}(t) = \left\langle t, \frac{t^2}{4} - \frac{\ln(t)}{2}, 6 \right\rangle$, $1 \le t \le 3$.

(a) Compute the unit tangent vector to C at $t = 1$.

(b) Compute the arc length of C.

Solution:

(a) By definition, the unit tangent vector evaluated at $t = 1$ is

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{||\mathbf{r}'(1)||}$$

The derivative is $\mathbf{r}'(t) = \left\langle 1, \frac{t}{2} - \frac{1}{2t}, 0 \right\rangle$. The value of $\mathbf{r}'(1)$ and its magnitude are

$$\mathbf{r}'(1) = \langle 1, 0, 0 \rangle$$
$$||\mathbf{r}'(1)|| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Therefore, the value of $\mathbf{T}'(1)$ is

ANSWER $\boxed{\mathbf{T}'(1) = \langle 1, 0, 0 \rangle}$

(b) The arc length formula is

$$L = \int_a^b ||\mathbf{r}'(t)|| \, dt$$

The magnitude of $\mathbf{r}'(t)$ is computed and simplified as follows:

$$||\mathbf{r}'(t)|| = \sqrt{1^2 + \left(\frac{t}{2} - \frac{1}{2t}\right)^2}$$
$$||\mathbf{r}'(t)|| = \sqrt{1 + \frac{t^2}{4} - \frac{1}{2} + \frac{1}{4t^2}}$$
$$||\mathbf{r}'(t)|| = \sqrt{\frac{t^2}{4} + \frac{1}{2} + \frac{1}{4t^2}}$$
$$||\mathbf{r}'(t)|| = \sqrt{\left(\frac{t}{2} + \frac{1}{2t}\right)^2}$$
$$||\mathbf{r}'(t)|| = \frac{t}{2} + \frac{1}{2t}$$

Thus, the arc length is

$$
\begin{aligned}
L &= \int_1^3 \left(\frac{t}{2} + \frac{1}{2t}\right)\, dt \\
L &= \left[\frac{t^2}{4} + \frac{1}{2}\ln(t)\right]_1^3 \\
L &= \left[\frac{3^2}{4} + \frac{1}{2}\ln(3)\right] - \left[\frac{1^2}{4} + \frac{1}{2}\ln(1)\right]
\end{aligned}
$$

ANSWER $\boxed{L = 2 + \frac{1}{2}\ln(3)}$

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Problem 3 Solution

3. Find the value of the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ or show that it does not exist.

Solution: We use the Two-Path Test to show that the limit does not exist.

- **Path 1** Let $y=0$ and take $x \to 0^+$. The value of the limit is then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = \lim_{x\to 0^+} \frac{x^2 \cdot 0}{x^4+0^2} = \boxed{0}$$

- **Path 2** A path that results in a non-zero limit is $y = x^2$ as $x \to 0^+$. The value of the limit along this path is

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = \lim_{x\to 0^+} \frac{x^2 \cdot (x^2)}{x^4+(x^2)^2} = \lim_{x\to 0^+} \frac{x^4}{x^4+x^4} = \boxed{\frac{1}{2}}$$

Thus, since the value of the limit is different along two different paths, we know that the limit does not exist.

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Problem 4 Solution

4. Consider the plane P with equation $3x + 2y - z + 6 = 0$ and the point $Q(-1, 0, 4)$.

(a) Find the equation of the plane parallel to P and containing the point Q.

(b) Find a set of parametric equations for the line orthogonal to P containing the point Q.

Solution:

(a) From the equation of the given plane P we identify the normal vector as being $\mathbf{n} = \langle 3, 2, -1 \rangle$. We then use the fact that parallel planes have parallel normal vectors to say that this vector is also normal to the plane in question. Using the point $Q(-1, 0, 4)$ as a point on the plane in question we have the following equation that describes it:

ANSWER $\boxed{3(x + 1) + 2(y - 0) - (z - 4) = 0}$

(b) To find the equation for a line we need a vector parallel to the line and a point on the line. The point is given to be Q. The vector we will use is $\mathbf{v} = \langle 3, 2, -1 \rangle$ since we know this vector to be normal to the plane P. Thus, a set of parametric equations that describes the line is:

ANSWER $\boxed{x = -1 + 3t, \quad y = 0 + 2t, \quad z = 4 - t}$

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Problem 5 Solution

5. Compute the first partial derivatives of the function $f(x,y) = \dfrac{y}{3x^2+4y^2}$.

Solution: The partial derivative f_x is

$$f_x = \frac{\partial}{\partial x}\left(\frac{y}{3x^2+4y^2}\right)$$
$$f_x = y\frac{\partial}{\partial x}(3x^2+4y^2)^{-1}$$
$$f_x = y\left[-\frac{1}{(3x^2+y^2)^2}\right]\cdot\frac{\partial}{\partial x}(3x^2+4y^2)$$
$$f_x = y\left[-\frac{1}{(3x^2+y^2)^2}\right]\cdot(6x)$$

ANSWER $\boxed{f_x = -\dfrac{6xy}{(3x^2+4y^2)^2}}$

The partial derivative f_y is

$$f_y = \frac{\partial}{\partial y}\left(\frac{y}{3x^2+4y^2}\right)$$
$$f_y = \frac{(3x^2+4y^2)\frac{\partial}{\partial y}y - y\frac{\partial}{\partial y}(3x^2+4y^2)}{(3x^2+4y^2)^2}$$
$$f_y = \frac{(3x^2+4y^2)-y(8y)}{(3x^2+4y^2)^2}$$

ANSWER $\boxed{f_y = \dfrac{3x^2-4y^2}{(3x^2+4y^2)^2}}$