Math 210, Exam 2, Fall 2005
Problem 1 Solution

1. Let $F(x, y, z) = 4x^2 - y^2 + 3z^2$. Find the equation of the plane tangent to the level surface $F(x, y, z) = 7$ at the point $(1, -3, 2)$.

Solution: We use the following formula for the equation for the tangent plane:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

because the equation for the surface is given in **implicit** form. Note that $\vec{\mathbf{n}} = \vec{\nabla} F(a, b, c) = \langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c) \rangle$ is a vector normal to the surface $F(x, y, z) = C$ and, thus, to the tangent plane at the point (a, b, c) on the surface.

The partial derivatives of $F(x, y, z) = 4x^2 - y^2 + 3z^2$ are:

$$F_x = 8x, \quad F_y = -2y, \quad F_z = 6z$$

Evaluating these derivatives at $(4, 2, 0)$ we get:

$$F_x(1, -3, 2) = 8(1) = 8$$
$$F_y(1, -3, 2) = -2(-3) = 6$$
$$F_z(1, -3, 2) = 6(2) = 12$$

Thus, the tangent plane equation is:

$$\boxed{8(x - 1) + 6(y + 3) + 12(z - 2) = 0}$$

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Problem 2 Solution

2. Let $f(x, y, z) = x^2 - xz + xyz$.

(a) Find the rate of change of f at the point $(1, 1, 1)$ in the direction of the unit vector $\vec{\mathbf{v}} = \frac{1}{\sqrt{6}}\langle 2, -1, 1\rangle$.

(b) Find the direction in which f increases most rapidly at the point $(1, 1, 1)$, and find the maximum rate of change of f at that point.

(c) Suppose that the function f gives the temperature at each point in space. A bug is flying around, with position function $\vec{\mathbf{p}}(t) = \langle t, t^2, t^3\rangle$, carrying a thermometer in his pocket. Use the chain rule to find the rate of change of his temperature *with respect to time* at the moment when his position is $(1, 1, 1)$.

Solution:

(a) Since $\vec{\mathbf{v}}$ is a unit vector $\left(\left|\vec{\mathbf{v}}\right| = 1\right)$, the rate of change of f at $(1, 1, 1)$ in the direction of $\vec{\mathbf{v}}$ is the directional derivative:

$$D_{\vec{\mathbf{v}}} f(1, 1, 1) = \vec{\nabla} f(1, 1, 1) \cdot \vec{\mathbf{v}}$$

The gradient of f is:

$$\begin{aligned}
\vec{\nabla} f(x, y, z) &= \langle f_x, f_y, f_z\rangle \\
\vec{\nabla} f(x, y, z) &= \langle 2x - z + yz, xz, -x + xy\rangle
\end{aligned}$$

Evaluating at the point $(1, 1, 1)$ we get:

$$\begin{aligned}
\vec{\nabla} f(1, 1, 1) &= \langle 2(1) - 1 + (1)(1), (1)(1), -1 + (1)(1)\rangle \\
\vec{\nabla} f(1, 1, 1) &= \langle 2, 1, 0\rangle
\end{aligned}$$

Therefore, the directional derivative is:

$$\begin{aligned}
D_{\vec{\mathbf{v}}} f(1, 1, 1) &= \vec{\nabla} f(1, 1, 1) \cdot \vec{\mathbf{v}} \\
D_{\vec{\mathbf{v}}} f(1, 1, 1) &= \langle 2, 1, 0\rangle \cdot \frac{1}{\sqrt{6}}\langle 2, -1, 1\rangle \\
D_{\vec{\mathbf{v}}} f(1, 1, 1) &= \frac{1}{\sqrt{6}}\left[(2)(2) + (1)(-1) + (0)(1)\right]
\end{aligned}$$

$$\boxed{D_{\vec{\mathbf{v}}} f(1, 1, 1) = \frac{3}{\sqrt{6}}}$$

(b) The direction of most rapid increase is the direction of **steepest ascent**:

$$\hat{\mathbf{u}} = \frac{1}{\left|\vec{\nabla} f(1,1,1)\right|}\,\vec{\nabla} f(1,1,1)$$

$$\hat{\mathbf{u}} = \frac{1}{|\langle 2,1,0\rangle|}\,\langle 2,1,0\rangle$$

$$\boxed{\hat{\mathbf{u}} = \frac{1}{\sqrt{5}}\,\langle 2,1,0\rangle}$$

(c) We use the Chain Rule for Paths formula:

$$\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right) = \vec{\nabla} f \cdot \vec{\mathbf{p}}'(t)$$

where the gradient of f was computed in part (a) and the derivative $\vec{\mathbf{p}}'(t)$ is:

$$\vec{\mathbf{p}}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle$$

Taking the dot product of these vectors gives us the derivative of $f(\vec{\mathbf{p}}(t))$.

$$\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right) = \vec{\nabla} f \cdot \vec{\mathbf{p}}'(t)$$

$$\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right) = \langle 2x - z + yz, xz, -x + xy\rangle \cdot \left\langle 1, 2t, 3t^2 \right\rangle$$

$$\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right) = (2x - z + yz)(1) + (xz)(2t) + (-x + xy)\left(3t^2\right)$$

We recognize that $t = 1$ when the bug's position is $(1,1,1)$ because $\vec{\mathbf{p}}(1) = \langle 1,1,1\rangle$. Therefore, plugging $t = 1$, $x = 1$, $y = 1$, and $z = 1$ into the derivative we find that:

$$\left.\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right)\right|_{t=1} = (2(1) - 1 + (1)(1))(1) + ((1)(1))(2(1)) + (-1 + (1)(1))\left(3(1)^2\right)$$

$$\boxed{\left.\frac{d}{dt} f\left(\vec{\mathbf{p}}(t)\right)\right|_{t=1} = 4}$$

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Problem 3 Solution

3. Find the critical points of the function $f(x, y) = x^4 + y^4 + 4xy - 1$ and classify them as maximum, minimum or saddle points.

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = x^4 + y^4 + 4xy - 1$ are $f_x = 4x^3 + 4y$ and $f_y = 4y^3 + 4x$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$4x^3 + 4y = 0 \quad (1)$$
$$4y^3 + 4x = 0 \quad (2)$$

Solving Equation (1) for y we get:

$$y = -x^3 \quad (3)$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned}
4y^3 + 4x &= 0 \\
4\left(-x^3\right)^3 + 4x &= 0 \\
-4x^9 + 4x &= 0 \\
-4x\left(x^8 - 1\right) &= 0 \\
-4x(x^4 - 1)(x^4 + 1) &= 0 \\
-4x(x^2 - 1)(x^2 + 1)(x^4 + 1) &= 0 \\
-4x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) &= 0
\end{aligned}$$

This equation has a total of 9 solutions but only 3 are real, those being $x = 0, 1, -1$. We find the corresponding y-values using Equation (3): $y = -x^3$.

- If $x = 0$, then $y = -(0)^3 = 0$.
- If $x = 1$, then $y = -(1)^3 = -1$.
- If $x = -1$, then $y = -(-1)^3 = 1$.

Thus, the critical points are $\boxed{(0, 0)}$, $\boxed{(1, -1)}$, and $\boxed{(-1, 1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = 4$$

The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = \left(12x^2\right)\left(12y^2\right) - (4)^2$$
$$D(x, y) = 144x^2y^2 - 16$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(0, 0)$	-16	0	Saddle Point
$(1, -1)$	128	12	Local Minimum
$(-1, 1)$	128	12	Local Minimum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local minimum of f if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

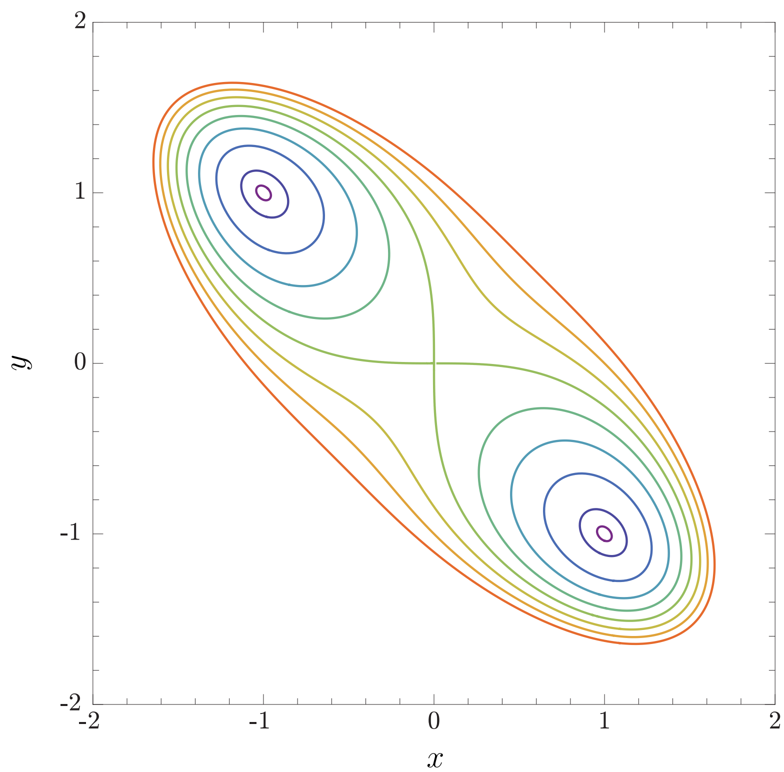

Figure 1: Picture above are level curves of $f(x, y)$. Darker colors correspond to smaller values of $f(x, y)$. It is apparent that $(0, 0)$ is a saddle point and both $(1, -1)$ and $(-1, 1)$ correspond to local minima.

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Problem 4 Solution

4. Let $f(x, y, z) = 1+x^3+y^2-z^3$. Suppose you were using the method of Lagrange multipliers to find the maximum value of the function f on the ellipsoid $x^2 + 3y^2 + 2z^2 = 3$.

(a) Write down the system of 4 algebraic equations in 4 unknowns that you would need to solve. **Do not try to solve these equations**.

(b) State how you would find the maximum value, given the list of solutions to the equations in part (a).

Solution: Let $g(x, y, z) = x^2 + 3y^2 + 2z^2 = 3$.

(a) Using the method of Lagrange multipliers, look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = 1$$

which, when applied to our functions f and g, give us:

$$3x^2 = \lambda(2x) \tag{1}$$
$$2y = \lambda(6y) \tag{2}$$
$$-3z^2 = \lambda(4z) \tag{3}$$
$$x^2 + 3y^2 + 2z^2 = 3 \tag{4}$$

(b) The ellipsoid is compact and f is continuous at all points on the ellipsoid. Therefore, we are guaranteed to find absolute extrema of f. If we had all solutions to the above system of equations, we would then plug all solutions into $f(x, y, z)$. The largest value of f would be the absolute maximum.

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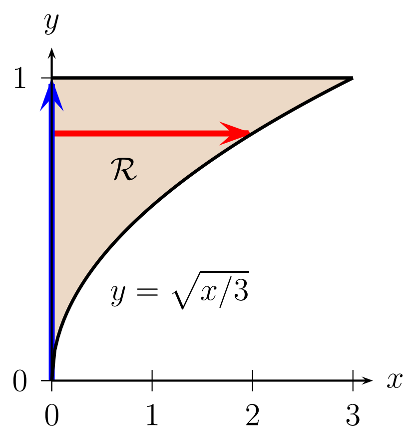
Problem 5 Solution

5. Change the order of integration to compute the iterated integral

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$$

Solution: The region of integration is sketched below:



The region $\mathcal{R}$ can be described as follows:

$$\mathcal{R} = \left\{(x, y) \,:\, 0 \le x \le 3y^2,\ 0 \le y \le 1\right\}$$

where $x = 0$ is the left curve and $x = 3y^2$ is the right curve, obtained by solving the equation $y = \sqrt{x/3}$ for x in terms of y. The projection of $\mathcal{R}$ onto the y-axis is the interval $0 \le y \le 1$. Therefore, the value of the integral is:

$$\begin{aligned}
\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx &= \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy \\
&= \int_0^1 e^{y^3} \Big[x\Big]_0^{3y^2} dy \\
&= \int_0^1 3y^2 e^{y^3}\, dy \\
&= \Big[e^{y^3}\Big]_0^1 \\
&= e^{1^3} - e^{0^3} \\
&= \boxed{e - 1}
\end{aligned}$$

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Problem 6 Solution

6. Find the surface area of the part of the paraboloid $z = -x^2 - y^2$ that lies above the plane $z = -20$.

Solution: The formula for surface area we will use is:

$$S = \iint_{\mathcal{S}} dS = \iint_{\mathcal{R}} \left| \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v \right| dA$$

where the function $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ with domain $\mathcal{R}$ is a parameterization of the surface $\mathcal{S}$ and the vectors $\vec{\mathbf{t}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u}$ and $\vec{\mathbf{t}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v}$ are the tangent vectors.

We begin by finding a parameterization of the paraboloid. Let $x = u\cos(v)$ and $y = u\sin(v)$, where we define u to be nonnegative. Then,

$$\begin{aligned} z &= -x^2 - y^2 \\ z &= -(u\cos(v))^2 - (u\sin(v))^2 \\ z &= -u^2\cos^2(v) - u^2\sin^2(v) \\ z &= -u^2 \end{aligned}$$

Thus, we have $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), -u^2 \rangle$. To find the domain $\mathcal{R}$, we must determine the curve of intersection of the paraboloid and the plane $z = -20$. We do this by plugging $z = -20$ into the equation for the paraboloid to get:

$$\begin{aligned} -x^2 - y^2 &= z \\ -x^2 - y^2 &= -20 \\ x^2 + y^2 &= 20 \end{aligned}$$

which describes a circle of radius $\sqrt{20}$. Thus, the domain $\mathcal{R}$ is the set of all points (x, y) satisfying $x^2 + y^2 \leq 20$. Using the fact that $x = u\cos(v)$ and $y = u\sin(v)$, this inequality becomes:

$$\begin{aligned} x^2 + y^2 &\leq 20 \\ (u\cos(v))^2 + (u\sin(v))^2 &\leq 20 \\ u^2 &\leq 20 \\ 0 \leq u &\leq \sqrt{20} \end{aligned}$$

noting that, by definition, u must be nonnegative. The range of v-values is $0 \leq v \leq 2\pi$. Therefore, a parameterization of $\mathcal{S}$ is:

$$\begin{aligned} \vec{\mathbf{r}}(u, v) &= \left\langle u\cos(v), u\sin(v), -u^2 \right\rangle, \\ \mathcal{R} &= \left\{ (u, v) \,\middle|\, 0 \leq u \leq \sqrt{20},\ 0 \leq v \leq 2\pi \right\} \end{aligned}$$

The tangent vectors $\overrightarrow{\mathbf{t}}_u$ and $\overrightarrow{\mathbf{t}}_v$ are then:

$$\overrightarrow{\mathbf{t}}_u = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial u} = \langle \cos(v), \sin(v), -2u \rangle$$
$$\overrightarrow{\mathbf{t}}_v = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial v} = \langle -u\sin(v), u\cos(v), 0 \rangle$$

The cross product of these vectors is:

$$\begin{aligned}
\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= 2u^2\cos(v)\,\hat{\mathbf{i}} + 2u^2\sin(v)\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}} \\
&= \left\langle 2u^2\cos(v), 2u^2\sin(v), u \right\rangle
\end{aligned}$$

The magnitude of the cross product is:

$$\begin{aligned}
\left|\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v\right| &= \sqrt{\left(2u^2\cos(v)\right)^2 + \left(2u^2\sin(v)\right)^2 + u^2} \\
&= \sqrt{4u^4\cos^2(v) + 4u^4\sin^2(v) + u^2} \\
&= \sqrt{4u^4 + u^2} \\
&= u\sqrt{4u^2+1}
\end{aligned}$$

We can now compute the surface area.

$$\begin{aligned}
S &= \iint_{\mathcal{R}} \left|\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v\right| \, dA \\
&= \int_0^{\sqrt{20}} \int_0^{2\pi} u\sqrt{4u^2+1}\, dv\, du \\
&= \int_0^{\sqrt{20}} u\sqrt{4u^2+1} \Big[v\Big]_0^{2\pi} du \\
&= \int_0^{\sqrt{20}} u\sqrt{4u^2+1} \Big[2\pi - 0\Big] du \\
&= \int_0^{\sqrt{20}} 2\pi u\sqrt{4u^2+1}\, du \\
&= \left[\frac{\pi}{6}\left(4u^2+1\right)^{3/2}\right]_0^{\sqrt{20}} \\
&= \left[\frac{\pi}{6}\left(4\left(\sqrt{20}\right)^2+1\right)^{3/2}\right] - \left[\frac{\pi}{6}\left(4(0)^2+1\right)^{3/2}\right] \\
&= \frac{\pi}{6}(81)^{3/2} - \frac{\pi}{6}(1)^{3/2} \\
&= \boxed{\frac{364\pi}{3}}
\end{aligned}$$