Math 210, Exam 2, Fall 2008
Problem 1 Solution

1. Let $f(x,y,z) = \sin(xy-8) - \ln(z+1) + \frac{2x}{y-z}$.

   (a) Compute the gradient $\overrightarrow{\nabla} f$ as a function of x, y, and z.

   (b) Find the equation of the tangent plane to the surface $f(x,y,z) = 4$ at $(4,2,0)$.

   (c) Compute the directional derivative $D_{\hat{\mathbf{u}}} f(4,2,0)$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\langle -2, 1, 0 \rangle$.

Solution:

(a) By definition, the gradient of $f(x,y,z)$ is:

$$\overrightarrow{\nabla} f = \langle f_x, f_y, f_z \rangle$$

The partial derivatives of f are:

$$\begin{aligned} f_x &= y\cos(xy-8) + \frac{2}{y-z} \\ f_y &= x\cos(xy-8) - \frac{2x}{(y-z)^2} \\ f_z &= -\frac{1}{z+1} + \frac{2x}{(y-z)^2} \end{aligned}$$

Thus, the gradient is:

$$\boxed{\overrightarrow{\nabla} f = \left\langle y\cos(xy-8) + \frac{2}{y-z}, x\cos(xy-8) - \frac{2x}{(y-z)^2}, -\frac{1}{z+1} + \frac{2x}{(y-z)^2} \right\rangle}$$

(b) We use the following formula for the equation for the tangent plane:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

because the surface equation is given in **implicit** form. Note that $\overrightarrow{\mathbf{n}} = \overrightarrow{\nabla} f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle$ is a vector normal to the surface $f(x,y,z) = C$ and, thus, to the tangent plane at the point (a,b,c) on the surface.

The partial derivatives evaluated at $(4,2,0)$ are:

$$\begin{aligned} f_x(4,2,0) &= 2\cos((4)(2)-8) + \frac{2}{2-0} = 3 \\ f_y(4,2,0) &= 4\cos((4)(2)-8) - \frac{2(4)}{(2-0)^2} = 2 \\ f_z(4,2,0) &= -\frac{1}{0+1} + \frac{2(4)}{(2-0)^2} = 1 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{3(x-4)+2(y-2)+(z-0)=0}$$

(c) By definition, the directional derivative of $f(x, y, z)$ at $(4, 2, 0)$ in the direction of $\hat{\mathbf{u}}$ is:

$$D_{\hat{\mathbf{u}}}f(4,2,0) = \overrightarrow{\nabla} f(4,2,0) \cdot \hat{\mathbf{u}}$$

From part (b), we have $\overrightarrow{\nabla} f(4,2,0) = \langle 3,2,1\rangle$. Recalling that $\hat{\mathbf{u}}$ must be a unit vector, we multiply $\langle -2,1,0\rangle$ by the reciprocal of its magnitude.

$$\hat{\mathbf{u}} = \frac{1}{|\langle -2,1,0\rangle|}\langle -2,1,0\rangle = \frac{1}{\sqrt{5}}\langle -2,1,0\rangle$$

Therefore, the directional derivative is:

$$D_{\hat{\mathbf{u}}}f(4,2,0) = \overrightarrow{\nabla} f(4,2,0) \cdot \hat{\mathbf{u}}$$

$$D_{\hat{\mathbf{u}}}f(4,2,0) = \langle 3,2,1\rangle \cdot \frac{1}{\sqrt{5}}\langle -2,1,0\rangle$$

$$D_{\hat{\mathbf{u}}}f(4,2,0) = \frac{1}{\sqrt{5}}[(3)(-2)+(2)(1)+(1)(0)]$$

$$\boxed{D_{\hat{\mathbf{u}}}f(4,2,0) = -\frac{4}{\sqrt{5}}}$$

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Problem 2 Solution

2. Let $f(x, y) = x^2 + y^2 - y$, and let $\mathcal{D}$ be the bounded region defined by the inequalities $y \geq 0$ and $y \leq 1 - x^2$.

(a) Find and classify the critical points of $f(x, y)$.

(b) Sketch the region $\mathcal{D}$.

(c) Find the absolute maximum and minimum values of f on the region $\mathcal{D}$, and list the points where these values occur.

Solution: First we note that the domain of $f(x, y)$ is bounded and closed, i.e. compact, and that $f(x, y)$ is continuous on the domain. Thus, we are guaranteed to have absolute extrema.

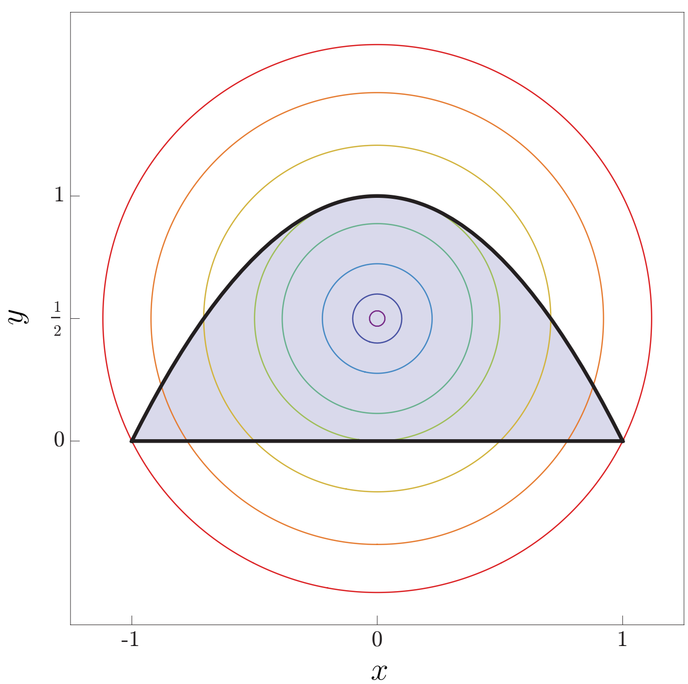
(a) The partial derivatives of f are $f_x = 2x$ and $f_y = 2y - 1$. The critical points of f are all solutions to the system of equations:

$$f_x = 2x = 0$$
$$f_y = 2y - 1 = 0$$

The only solution is $x = 0$ and $y = \frac{1}{2}$, which is an interior point of $\mathcal{D}$. The function value at the critical point is:

$$\boxed{f(0, \tfrac{1}{2}) = -\tfrac{1}{4}}$$

(b) The region $\mathcal{D}$ (shaded) is plotted below along with level curves of $f(x, y)$.



(c) We must now determine the minimum and maximum values of f on the boundary of $\mathcal{D}$. To do this, we must consider each part of the boundary separately:

Part I : Let this part be the line segment between $(-1, 0)$ and $(1, 0)$. On this part we have $y = 0$ and $-1 \leq x \leq 1$. We now use the fact that $y = 0$ to rewrite $f(x, y)$ as a function of one variable that we call $g_I(x)$.

$$\begin{aligned} f(x,y) &= x^2 + y^2 - y \\ g_I(x) &= x^2 + 0^2 - 0 \\ g_I(x) &= x^2 \end{aligned}$$

The critical points of $g_I(x)$ are:

$$\begin{aligned} g_I'(x) &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

Evaluating $g_I(x)$ at the critical point $x = 0$ and at the endpoints of the interval $-1 \leq x \leq 1$, we find that:

$$g_I(0) = 0, \quad g_I(-1) = 1, \quad g_I(1) = 1$$

Note that these correspond to the function values:

$$\boxed{f(0,0) = 0, \quad f(-1,0) = 1, \quad f(1,0) = 1}$$

Part II : Let this part be the parabola $y = 1 - x^2$ on the interval $-1 \leq x \leq 1$. We now use the fact that $y = 1 - x^2$ to rewrite $f(x, y)$ as a function of one variable that we call $g_{II}(x)$.

$$\begin{aligned} f(x,y) &= x^2 + y^2 - y \\ g_{II}(x) &= x^2 + \left(1 - x^2\right)^2 - \left(1 - x^2\right) \\ g_{II}(x) &= x^2 + 1 - 2x^2 + x^4 - 1 + x^2 \\ g_{II}(x) &= x^4 \end{aligned}$$

The critical points of $g_{II}(x)$ are:

$$\begin{aligned} g_{II}'(x) &= 0 \\ 4x^3 &= 0 \\ x &= 0 \end{aligned}$$

Evaluating $g_{II}(x)$ at the critical point $x = 0$ and at the endpoints of the interval $-1 \leq x \leq 1$, we find that:

$$g_{II}(0) = 0, \quad g_{II}(-1) = 1, \quad g_{II}(1) = 1$$

Note that these correspond to the function values:

$$\boxed{f(0,1) = 0, \quad f(-1,0) = 1, \quad f(1,0) = 1}$$

Finally, after comparing these values of f we find that the **absolute maximum** of f is 1 at the points $(-1, 0)$ and $(1, 0)$ and that the **absolute minimum** of f is $-\frac{1}{4}$ at the point $(0, \frac{1}{2})$.

Note: In the figure from part (b) we see that the level curves of f are circles centered at $(0, \frac{1}{2})$. It is clear that the absolute minimum of f occurs at $(0, \frac{1}{2})$ and that the absolute maximum of f occurs at $(-1, 0)$ and $(1, 0)$, which are points on the largest circle centered at $(0, \frac{1}{2})$ that contains points in $\mathcal{D}$.

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Problem 3 Solution

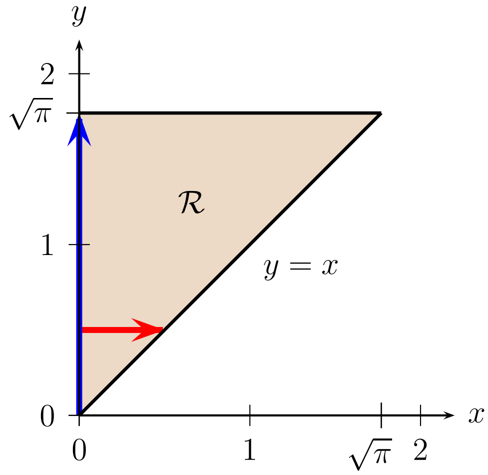
3. Consider the iterated integral $\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos\left(y^2\right)\, dy\, dx$.

(a) Sketch the region of integration.

(b) Compute the integral. (Hint: First reverse the order of integration.)

Solution:

(a) The region of integration $\mathcal{R}$ is sketched below:



(b) The region $\mathcal{R}$ can be described as follows:

$$\mathcal{R} = \left\{(x, y) \,:\, 0 \leq x \leq y,\ 0 \leq y \leq \sqrt{\pi}\right\}$$

Therefore, the value of the integral is:

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \cos\left(y^2\right)\, dy\, dx &= \int_0^{\sqrt{\pi}} \int_0^{y} \cos\left(y^2\right)\, dx\, dy \\
&= \int_0^{\sqrt{\pi}} \cos\left(y^2\right) \Big[x\Big]_0^y \\
&= \int_0^{\sqrt{\pi}} y\cos\left(y^2\right)\, dy \\
&= \left[\frac{1}{2}\sin\left(y^2\right)\right]_0^{\sqrt{\pi}} \\
&= \frac{1}{2}\sin\left(\sqrt{\pi}^2\right) - \frac{1}{2}\sin\left(0^2\right) \\
&= \boxed{0}
\end{aligned}$$

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Problem 4 Solution

4. Let $\mathcal{Q}$ be the part of the unit disk that lies in the second quadrant, i.e.

$$\mathcal{Q} = \{(x, y) \,|\, x \leq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\}$$

(a) Write an iterated integral *in polar coordinates* that represents the area of $\mathcal{Q}$ and compute this area.

(b) Compute $\iint_{\mathcal{Q}} \left(3x^2 + 3y^2\right) dA$.

(c) Compute the average value of $f(x, y) = x^2 + y^2$ over $\mathcal{Q}$.

Solution:

(a) The region $\mathcal{Q}$ can be described in polar coordinates as:

$$\mathcal{Q} = \left\{(r, \theta) \,|\, 0 \leq r \leq 1,\ \frac{\pi}{2} \leq \theta \leq \pi\right\}$$

Using the fact that $dA = r\,dr\,d\theta$ in polar coordinates, the area of $\mathcal{Q}$ is:

$$\begin{aligned} \text{Area}(\mathcal{Q}) &= \iint_{\mathcal{Q}} 1\,dA \\ &= \int_{\pi/2}^{\pi} \int_0^1 r\,dr\,d\theta \\ &= \int_{\pi/2}^{\pi} \left[\frac{1}{2}r^2\right]_0^1 d\theta \\ &= \int_{\pi/2}^{\pi} \frac{1}{2}\,d\theta \\ &= \left[\frac{1}{2}\theta\right]_{\pi/2}^{\pi} \\ &= \boxed{\frac{\pi}{4}} \end{aligned}$$

(b) The function $f(x, y) = 3x^2 + y^2$ can be written in polar coordinates as:

$$f(r, \theta) = 3r^2$$

The integral of $f(r,\theta)$ over the region $\mathcal{Q}$ is then:

$$\begin{aligned}
\text{Area}(\mathcal{Q}) &= \iint_{\mathcal{Q}} f(r,\theta)\,dA \\
&= \int_{\pi/2}^{\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta \\
&= \int_{\pi/2}^{\pi}\left[\frac{3}{4}r^4\right]_0^1 d\theta \\
&= \int_{\pi/2}^{\pi}\frac{3}{4}\,d\theta \\
&= \left[\frac{3}{4}\theta\right]_{\pi/2}^{\pi} \\
&= \boxed{\frac{3\pi}{8}}
\end{aligned}$$

(c) We use the following formula to compute the average value of f:

$$\bar{f} = \frac{\iint_A f(x,y)\,dA}{\iint_A 1\,dA}$$

The function $f(x,y) = x^2 + y^2$ written in polar coordinates is:

$$f(r,\theta) = r^2$$

The integral of $f(r,\theta)$ over the region $\mathcal{Q}$ is then:

$$\begin{aligned}
\iint_{\mathcal{Q}} f(x,y)\,dA &= \int_{\pi/2}^{\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta \\
&= \frac{1}{3}\int_{\pi/2}^{\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta \\
&= \frac{1}{3}\cdot\frac{3\pi}{8} \\
&= \frac{\pi}{8}
\end{aligned}$$

where we used the result from part (b). The integral $\iint_A 1\,dA = \frac{\pi}{4}$ was computed in part (a). Thus, the average value of f is:

$$\bar{f} = \frac{\frac{\pi}{8}}{\frac{\pi}{4}} = \boxed{\frac{1}{2}}$$