Math 210, Exam 2, Practice Fall 2009
Problem 1 Solution

1. Let $f(x,y) = 3x^2 + xy + 2y^2$. Find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, at $(1,1)$, and find the best linear approximation of f at $(1,1)$ and use it to estimate $f(1.1, 1.2)$.

Solution: The linearization of $f(x,y) = 3x^2 + xy + 2y^2$ about $(1,1)$ has the form:

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

The first partial derivatives of $f(x,y)$ are:

$$\begin{aligned} f_x &= 6x + y \\ f_y &= x + 4y \end{aligned}$$

At the point $(1,1)$ we have:

$$\begin{aligned} f(1,1) &= 3(1)^2 + (1)(1) + 2(1)^2 = 6 \\ f_x(1,1) &= 6(1) + 1 = 7 \\ f_y(1,1) &= 1 + 4(1) = 5 \end{aligned}$$

Thus, the linearization is:

$$L(x,y) = 6 + 7(x-1) + 5(y-1)$$

The value of $f(1.1, 1.2)$ is estimated to be the value of $L(1.1, 1.2)$:

$$\begin{aligned} &f(1.1, 1.2) \approx L(1.1, 1.2) \\ &f(1.1, 1.2) \approx 6 + 7(1.1-1) + 5(1.2-1) \\ &\boxed{f(1.1, 1.2) \approx 7.7} \end{aligned}$$

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Problem 2 Solution

2. Find and classify the critical points of the function

$$f(x,y) = x^3 - 3xy + y^3.$$

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = x^3 - 3xy + y^3$ are $f_x = 3x^2 - 3y$ and $f_y = -3x + 3y^2$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 3x^2 - 3y = 0 \quad (1)$$

$$f_y = -3x + 3y^2 = 0 \quad (2)$$

Solving Equation (1) for y we get:

$$y = x^2 \quad (3)$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned} -3x + 3y^2 &= 0 \\ -3x + 3\left(x^2\right)^2 &= 0 \\ -3x + 3x^4 &= 0 \\ 3x(x^3 - 1) &= 0 \end{aligned}$$

We observe that the above equation is satisfied if either $x = 0$ or $x^3 - 1 = 0 \iff x = 1$. We find the corresponding y-values using Equation (3): $y = x^2$.

- If $x = 0$, then $y = 0^2 = 0$.
- If $x = 1$, then $y = 1^2 = 1$.

Thus, the critical points are $\boxed{(0,0)}$ and $\boxed{(1,1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3$$

The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = (6x)(6y) - (-3)^2$$
$$D(x, y) = 36xy - 9$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(0, 0)$	-9	0	Saddle Point
$(1, 1)$	27	6	Local Minimum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local minimum of f if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

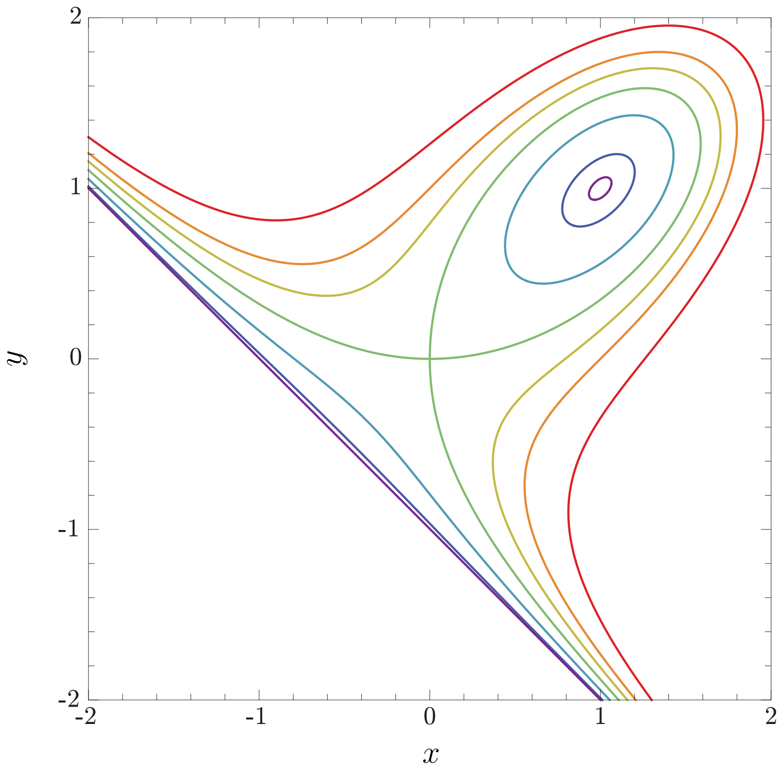

Figure 1: Picture above are level curves of $f(x, y)$. Darker colors correspond to smaller values of $f(x, y)$. It is apparent that $(0, 0)$ is a saddle point and $(1, 1)$ corresponds to a local minimum.

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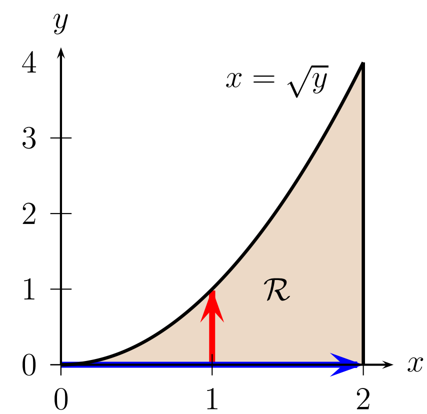
Problem 3 Solution

3. Sketch the region of integration for the integral $\int_0^4 \int_{\sqrt{y}}^2 \sin\left(x^3\right)\, dx\, dy$. Compute the integral.

Solution: The region of integration $\mathcal{R}$ is sketched below:



First, we recognize that $\sin\left(x^3\right)$ has no simple antiderivative. Therefore, we must change the order of integration to evaluate the integral. The region $\mathcal{R}$ can be described as follows:

$$\mathcal{R} = \left\{(x, y) \,:\, 0 \le y \le x^2,\ 0 \le x \le 2\right\}$$

where $y = 0$ is the bottom curve and $y = x^2$ is the top curve, obtained by solving the equation $x = \sqrt{y}$ for y in terms of x. Therefore, the value of the integral is:

$$\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 \sin\left(x^3\right)\, dx\, dy &= \int_0^2 \int_0^{x^2} \sin\left(x^3\right)\, dy\, dx \\
&= \int_0^2 \sin\left(x^3\right) \Big[y\Big]_0^{x^2}\, dx \\
&= \int_0^2 x^2 \sin\left(x^3\right)\, dx \\
&= \left[-\frac{1}{3}\cos\left(x^3\right)\right]_0^2 \\
&= \left[-\frac{1}{3}\cos\left(2^3\right)\right] - \left[-\frac{1}{3}\cos\left(0^3\right)\right] \\
&= \boxed{-\frac{1}{3}\cos(8) + \frac{1}{3}}
\end{aligned}$$

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Problem 4 Solution

4. Find the minimum and maximum of the function $f(x, y, z) = x + y - z$ on the ellipsoid

$$R = \left\{ (x, y, z) \;\middle|\; \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \right\}$$

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that R is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = 1$$

which, when applied to our functions f and g, give us:

$$1 = \lambda \left(\frac{2x}{4} \right) \tag{1}$$

$$1 = \lambda \left(\frac{2y}{9} \right) \tag{2}$$

$$-1 = \lambda (2z) \tag{3}$$

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \tag{4}$$

To solve the system of equations, we first solve Equations (1)-(3) for the variables x, y, and z in terms of λ to get:

$$x = \frac{4}{2\lambda}, \quad y = \frac{9}{2\lambda}, \quad z = -\frac{1}{2\lambda} \tag{5}$$

We then plug Equations (5) into Equation (4) and simplify.

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

$$\frac{(\frac{4}{2\lambda})^2}{4} + \frac{(\frac{9}{2\lambda})^2}{9} + \left(-\frac{1}{2\lambda} \right)^2 = 1$$

$$\frac{\frac{16}{4\lambda^2}}{4} + \frac{\frac{81}{4\lambda^2}}{9} + \frac{1}{4\lambda^2} = 1$$

At this point we multiply both sides of the equation by $4\lambda^2$ to get:

$$4\lambda^2\left(\frac{\frac{16}{4\lambda^2}}{4}+\frac{\frac{81}{4\lambda^2}}{9}+\frac{1}{4\lambda^2}\right)=4\lambda^2(1)$$
$$\frac{16}{4}+\frac{81}{9}+1=4\lambda^2$$
$$4+9+1=4\lambda^2$$
$$\lambda^2=\frac{7}{2}$$
$$\lambda=\pm\sqrt{\frac{7}{2}}$$
$$\lambda=\pm\frac{\sqrt{14}}{2}$$

- When $\lambda=\frac{\sqrt{14}}{2}$, Equations (5) give us the first candidate for the location of an extreme value:
$$x=\frac{4\sqrt{14}}{14},\quad y=\frac{9\sqrt{14}}{14},\quad z=-\frac{\sqrt{14}}{14}$$
- When $\lambda=-\frac{\sqrt{14}}{2}$, Equations (5) give us the first candidate for the location of an extreme value:
$$x=-\frac{4\sqrt{14}}{14},\quad y=-\frac{9\sqrt{14}}{14},\quad z=\frac{\sqrt{14}}{14}$$

Evaluating $f(x,y,z)$ at these points we find that:

$$f\left(\frac{4\sqrt{14}}{14},\frac{9\sqrt{14}}{14},-\frac{\sqrt{14}}{14}\right)=\sqrt{14}$$
$$f\left(-\frac{4\sqrt{14}}{14},-\frac{9\sqrt{14}}{14},\frac{\sqrt{14}}{14}\right)=-\sqrt{14}$$

Therefore, the absolute maximum value of f on R is $\sqrt{14}$ and the absolute minimum of f on R is $-\sqrt{14}$.

Note: The level surfaces $f(x,y,z)=\sqrt{14}$ and $f(x,y,z)=-\sqrt{14}$ are planes tangent to the ellipsoid at the critical points.

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Problem 5 Solution

5. Find the tangent plane to the surface:

$$S = \left\{(x, y, z) \,:\, x^2 + y^3 - 2z = 1\right\}$$

at the point $(1, 2, 4)$.

Solution: Let $F(x, y, z) = x^2 + y^3 - 2z = 1$ be the equation for the surface. We use the following formula for the equation for the tangent plane:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

because the equation for the surface is given in **implicit** form. Note that $\overrightarrow{\mathbf{n}} = \overrightarrow{\nabla} F(a, b, c) = \langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c)\rangle$ is a vector normal to the surface $F(x, y, z) = C$ and, thus, to the tangent plane at the point (a, b, c) on the surface.

The partial derivatives of $F(x, y, z) = x^2 + y^3 - 2z$ are:

$$F_x = 2x, \quad F_y = 3y^2, \quad F_z = -2$$

Evaluating these derivatives at $(1, 2, 4)$ we get:

$$\begin{aligned} F_x(1, 2, 4) &= 2(1) = 2 \\ F_y(1, 2, 4) &= 3(2)^2 = 12 \\ F_z(1, 2, 4) &= -2 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{2(x - 1) + 12(y - 2) - 2(z - 4) = 0}$$

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Problem 6 Solution

6. Let $F(x, y, z) = 3x^2 + y^2 - 4z^2$. Find the equation of the tangent plane to the level surface $F(x, y, z) = 1$ at the point $(1, -4, 3)$.

Solution: We use the following formula for the equation for the tangent plane:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

because the equation for the surface is given in **implicit** form. Note that $\vec{\mathbf{n}} = \vec{\nabla} F(a, b, c) = \langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c)\rangle$ is a vector normal to the surface $F(x, y, z) = C$ and, thus, to the tangent plane at the point (a, b, c) on the surface.

The partial derivatives of $F(x, y, z) = 3x^2 + y^2 - 4z^2$ are:

$$F_x = 6x, \quad F_y = 2y, \quad F_z = -8z$$

Evaluating these derivatives at $(1, -4, 3)$ we get:

$$\begin{aligned} F_x(1, -4, 3) &= 6(1) = 6 \\ F_y(1, -4, 3) &= 2(-4) = -8 \\ F_z(1, -4, 3) &= -8(3) = -24 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{6(x - 1) - 8(y + 4) - 24(z - 3) = 0}$$

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Problem 7 Solution

7. Let $f(x,y) = \frac{1}{3}x^3 + y^2 - xy$. Find all critical points of $f(x,y)$ and classify each as a local maximum, local minimum, or saddle point.

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = \frac{1}{3}x^3 + y^2 - xy$ are $f_x = x^2 - y$ and $f_y = 2y - x$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = x^2 - y = 0 \tag{1}$$
$$f_y = 2y - x = 0 \tag{2}$$

Solving Equation (1) for y we get:

$$y = x^2 \tag{3}$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned} 2y - x &= 0 \\ 2x^2 - x &= 0 \\ x(2x-1) &= 0 \\ \iff \quad x = 0 \quad \text{or} \quad x &= \frac{1}{2} \end{aligned}$$

We find the corresponding y-values using Equation (3): $y = x^2$.

- If $x = 0$, then $y = 0^2 = 0$.
- If $x = \frac{1}{2}$, then $y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Thus, the critical points are $\boxed{(0,0)}$ and $\boxed{(\frac{1}{2}, \frac{1}{4})}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 2x, \quad f_{yy} = 2, \quad f_{xy} = -1$$

The discriminant function $D(x,y)$ is then:

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x,y) &= (2x)(2) - (-1)^2 \\ D(x,y) &= 4x - 1 \end{aligned}$$

The values of $D(x,y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(0,0)$	-1	0	Saddle Point
$(\frac{1}{2},\frac{1}{4})$	1	1	Local Minimum

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

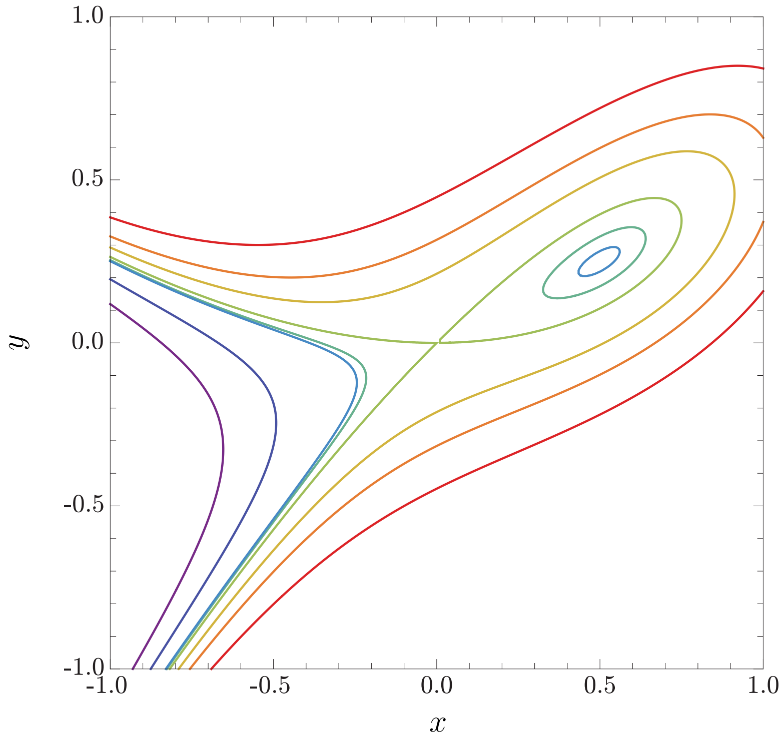

Figure 1: Picture above are level curves of $f(x,y)$. Darker colors correspond to smaller values of $f(x,y)$. It is apparent that $(0,0)$ is a saddle point and $(\frac{1}{2},\frac{1}{4})$ corresponds to a local minimum.

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Problem 8 Solution

8. Find the minimum and maximum of the function $f(x, y) = x^2 - y$ subject to the condition $x^2 + y^2 = 4$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + y^2 = 4$ is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y) = x^2 + y^2 = 4$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 4$$

which, when applied to our functions f and g, give us:

$$2x = \lambda\,(2x) \tag{1}$$
$$-1 = \lambda\,(2y) \tag{2}$$
$$x^2 + y^2 = 4 \tag{3}$$

We begin by noting that Equation (1) gives us:

$$\begin{aligned} 2x &= \lambda(2x) \\ 2x - \lambda(2x) &= 0 \\ 2x(1 - \lambda) &= 0 \end{aligned}$$

From this equation we either have $x = 0$ or $\lambda = 1$. Let's consider each case separately.

Case 1: Let $x = 0$. We find the corresponding y-values using Equation (3).

$$\begin{aligned} x^2 + y^2 &= 4 \\ 0^2 + y^2 &= 4 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

Thus, the points of interest are $(0, 2)$ and $(0, -2)$.

Case 2: Let $\lambda = 1$. Plugging this into Equation (2) we get:

$$\begin{aligned} -1 &= \lambda(2y) \\ -1 &= 1(2y) \\ y &= -\tfrac{1}{2} \end{aligned}$$

We find the corresponding x-values using Equation (3).

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 + \left(-\tfrac{1}{2}\right)^2 &= 4 \\ x^2 + \tfrac{1}{4} &= 4 \\ x^2 &= \tfrac{15}{4} \\ x &= \pm\tfrac{\sqrt{15}}{2} \end{aligned}$$

Thus, the points of interest are $(\frac{\sqrt{15}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{15}}{2}, -\frac{1}{2})$.

We now evaluate $f(x,y) = x^2 - y$ at each point of interest obtained by Cases 1 and 2.

$$f(0,2) = -2$$
$$f(0,-2) = 2$$
$$f(\tfrac{\sqrt{15}}{2}, -\tfrac{1}{2}) = \tfrac{17}{4}$$
$$f(-\tfrac{\sqrt{15}}{2}, -\tfrac{1}{2}) = \tfrac{17}{4}$$

From the values above we observe that f attains an absolute maximum of $\frac{17}{4}$ and an absolute minimum of -2.

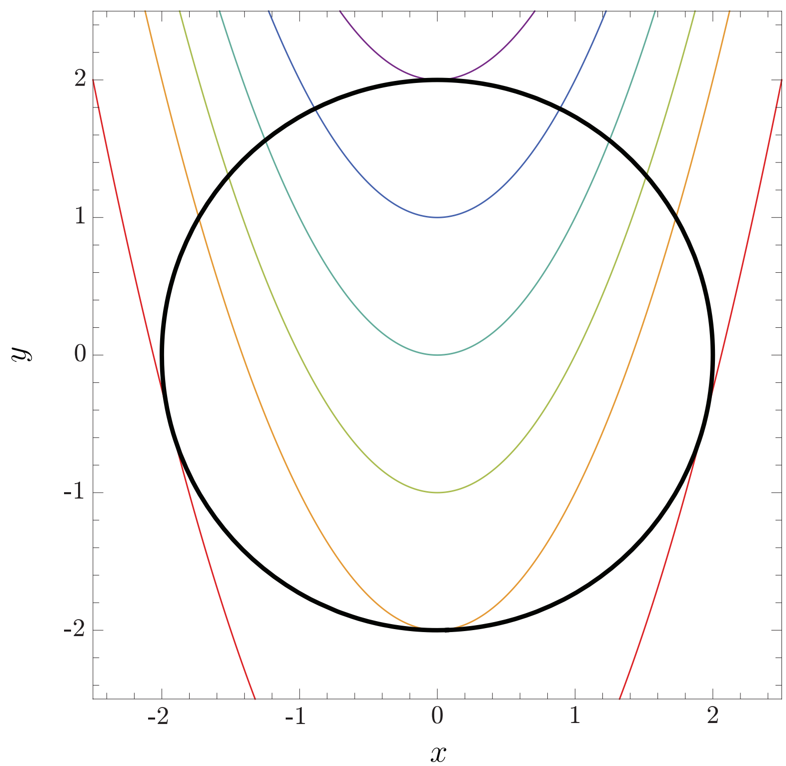

Figure 1: Shown in the figure are the level curves of $f(x,y) = x^2 - y$ and the circle $x^2 + y^2 = 4$ (thick, black curve). Darker colors correspond to smaller values of $f(x,y)$. Notice that (1) the parabola $f(x,y) = x^2 - y = \frac{17}{4}$ is tangent to the circle at the points $(\frac{\sqrt{15}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{15}}{2}, -\frac{1}{2})$ which correspond to the absolute maximum and (2) the parabola $f(x,y) = x^2 - y = -2$ is tangent to the circle at the point $(0,2)$ which corresponds to the absolute minimum.

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Problem 9 Solution

9. Use polar coordinates to find the volume of the region bounded by the paraboloid $z = 1 - x^2 - y^2$ in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$.

Solution: The volume formula we use is:

$$V = \iint_{\mathcal{D}} \left(1 - x^2 - y^2\right)\, dA$$

where $\mathcal{D}$ is the projection of the paraboloid onto the first quadrant in the xy-plane. We are asked to use polar coordinates:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ dA = r\,dr\,d\theta$$

1. First, we describe the region $\mathcal{D}$. Since $z \geq 0$ and $z = 1 - x^2 - y^2$ we know that:

   $$1 - x^2 - y^2 \geq 0$$
   $$x^2 + y^2 \leq 1$$

   Since the projection is in the first quadrant, the region $\mathcal{D}$ can be described in rectangular coordinates as:

   $$\mathcal{D} = \{(x, y)\ :\ x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$$

   or, equivalently, in polar coordinates as:

   $$\mathcal{D} = \left\{(r, \theta)\ :\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$$

2. Then, using the polar coordinate equations, the paraboloid $z = 1 - x^2 - y^2$ can be written in polar coordinates as:

   $$z = 1 - r^2$$

3. Finally, we compute the volume as follows:

   $$\begin{aligned} V &= \iint_{\mathcal{D}} \left(1 - x^2 - y^2\right)\, dA \\ &= \int_0^{\pi/2} \int_0^1 \left(1 - r^2\right) r\,dr\,d\theta \\ &= \int_0^{\pi/2} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta \\ &= \int_0^{\pi/2} \frac{1}{4}\,d\theta \\ &= \left[\frac{1}{4}\theta\right]_0^{\pi/2} \\ &= \boxed{\frac{\pi}{8}} \end{aligned}$$

Math 210, Exam 2, Practice Fall 2009
Problem 10 Solution

10. Find the minimum and maximum of the function

$$f(x, y, z) = x^2 - y^2 + 2z^2$$

on the surface of the sphere defined by the equation $x^2 + y^2 + z^2 = 1$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that the sphere is compact and that $f(x, y, z)$ is continuous on the sphere, which guarantees the existence of absolute extrema of f. Then, let $g(x, y, z) = x^2 + y^2 + z^2 = 1$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = 1$$

which, when applied to our functions f and g, give us:

$$2x = \lambda(2x) \qquad (1)$$
$$-2y = \lambda(2y) \qquad (2)$$
$$4z = \lambda(2z) \qquad (3)$$
$$x^2 + y^2 + z^2 = 1 \qquad (4)$$

From Equation (1) we can have either $x = 0$ or $\lambda = 1$.

- If $x = 0$ then we turn to Equation (2). In this case we either have $y = 0$ or $\lambda = -1$.
  - Suppose $y = 0$. Plugging $x = 0$ and $y = 0$ into Equation (4) we get:

    $$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ 0^2 + 0^2 + z^2 &= 1 \\ z^2 &= 1 \\ z &= \pm 1 \end{aligned}$$

    Thus, the points of interest are $(0, 0, 1)$ and $(0, 0, -1)$.
  - Now suppose $\lambda = -1$. Then Equation (3) gives us:

    $$\begin{aligned} 4z &= \lambda(2z) \\ 4z &= (-1)(2z) \\ 6z &= 0 \\ z &= 0 \end{aligned}$$

    Plugging $x = 0$ and $z = 0$ into Equation (4) we get:

    $$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ 0^2 + y^2 + 0^2 &= 1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

    Thus, the points of interest are $(0, 1, 0)$ and $(0, -1, 0)$.

- If $\lambda = 1$ then Equations (2) and (3) give us:

$$\begin{aligned} -2y &= \lambda(2y) \\ -2y &= (1)(2y) \\ -4y &= 0 \\ y &= 0 \end{aligned} \qquad\qquad \begin{aligned} 4z &= \lambda(2z) \\ 4z &= (1)(2z) \\ 2z &= 0 \\ z &= 0 \end{aligned}$$

  Plugging $y = 0$ and $z = 0$ into Equation (4) we get:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ x^2 + 0^2 + 0^2 &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

  Thus, the points of interest are $(1, 0, 0)$ and $(-1, 0, 0)$.

Evaluating $f(x, y, z)$ at all points of interest we find that:

$$\begin{aligned} f(1, 0, 0) &= 1 \\ f(-1, 0, 0) &= 1 \\ f(0, 1, 0) &= -1 \\ f(0, -1, 0) &= -1 \\ f(0, 0, 1) &= 2 \\ f(0, 0, -1) &= 2 \end{aligned}$$

Therefore, the absolute maximum value of f is 2 and the absolute minimum of f is -1.

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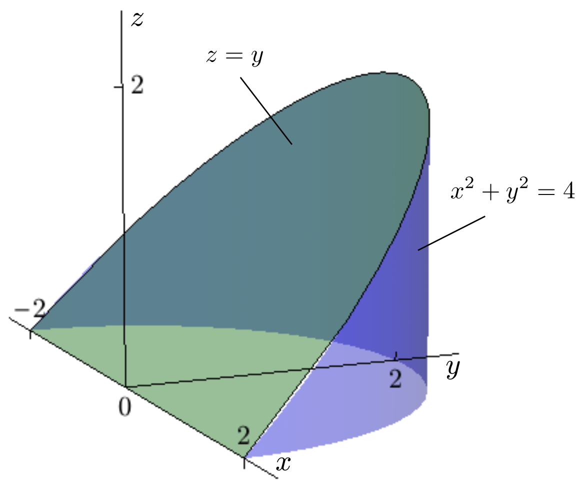
Problem 11 Solution

11. Using cylindrical coordinates, compute

$$\iiint_W \left(x^2+y^2\right)^{1/2}\, dV$$

where W is the region within the cylinder $x^2+y^2 \le 4$ and $0 \le z \le y$.

Solution: The region W is plotted below.



In cylindrical coordinates, the equations for the cylinder $x^2+y^2=4$ and the plane $z=y$ are:

$$\begin{aligned} \text{Cylinder}: &\quad r = 2 \\ \text{Plane}: &\quad z = r\sin\theta \end{aligned}$$

Furthermore, we can write the integrand in cylindrical coordinates as:

$$\begin{aligned} f(x,y,z) &= \left(x^2+y^2\right)^{1/2} \\ f(r,\theta,z) &= r \end{aligned}$$

The projection of W onto the xy-plane is the half-disk $0 \le r \le 2,\ 0 \le \theta \le \pi$. Using the fact that $dV = r\,dz\,dr\,d\theta$ in cylindrical coordinates, the value of the integral is:

$$
\begin{aligned}
\iiint_W \left(x^2+y^2\right)^{1/2}\, dV &= \int_0^\pi \int_0^2 \int_0^{r\sin\theta} r^2\, dz\, dr\, d\theta \\
&= \int_0^\pi \int_0^2 r^2 \Big[z\Big]_0^{r\sin\theta} dr\, d\theta \\
&= \int_0^\pi \int_0^2 r^3 \sin\theta\, dr\, d\theta \\
&= \int_0^\pi \sin\theta \left[\frac{1}{4}r^4\right]_0^2 d\theta \\
&= 4\int_0^\pi \sin\theta\, d\theta \\
&= 4\Big[-\cos\theta\Big]_0^\pi \\
&= 4\Big[-\cos\pi+\cos 0\Big] \\
&= \boxed{8}
\end{aligned}
$$

Math 210, Exam 2, Practice Fall 2009
Problem 12 Solution

12. Compute the integral $\iiint_B x^2\, dV$, where B is the unit ball

$$B = \{(x, y, z) \,:\, x^2 + y^2 + z^2 \leq 1\}$$

Solution: Due to the fact that B is a ball of radius 1, we use Spherical Coordinates to evaluate the integral. In Spherical Coordinates, the equation for the sphere is $\rho = 1$ and the integrand is:

$$f(x, y, z) = x^2$$
$$f(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta)^2$$

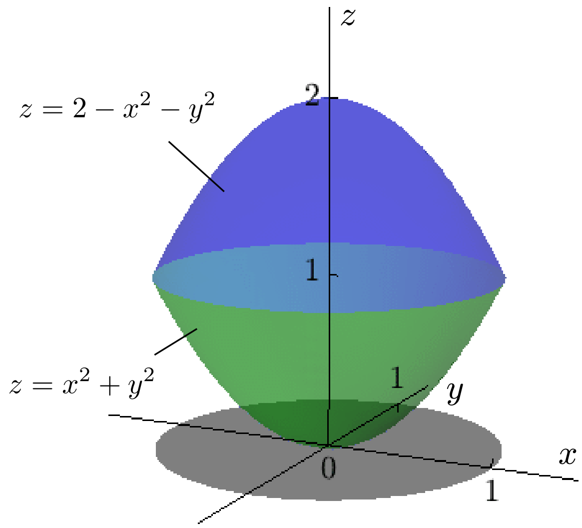
Using the fact that $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ in Spherical Coordinates, the value of the integral is:

$$\begin{aligned}
\iiint_B x^2\, dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho\sin\phi\cos\theta)^2\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4 \sin^3\phi \cos^2\theta\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta \left[\frac{1}{5}\rho^5\right]_0^1 d\phi\, d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\, d\phi\, d\theta \\
&= \frac{1}{5}\int_0^{2\pi} \cos^2\theta \left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} d\theta \\
&= \frac{1}{5}\int_0^{2\pi} \cos^2\theta \left[\left(\frac{1}{3}\cos^3\pi - \cos\pi\right) - \left(\frac{1}{3}\cos^3 0 - \cos 0\right)\right] d\theta \\
&= \frac{1}{5}\int_0^{2\pi} \frac{4}{3}\cos^2\theta\, d\theta \\
&= \frac{4}{15}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{4}{15}\left[\left(\frac{1}{2}(2\pi) + \frac{1}{4}\sin(4\pi)\right) - \left(\frac{1}{2}(0) + \frac{1}{4}\sin(0)\right)\right] \\
&= \boxed{\frac{4\pi}{15}}
\end{aligned}$$

Math 210, Exam 2, Practice Fall 2009
Problem 13 Solution

13. Find the volume of the region bounded below and above by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

Solution: The region is plotted below.



The volume may be computed using either a double integral or a triple integral. Using a triple integral, the formula is:

$$V = \iiint_R 1\, dV$$

Due to the shape of the boundary, we will use Cylindrical Coordinates. The paraboloids can be written in Cylindrical Coordinates as:

$$\begin{aligned} \text{Paraboloid 1}: &\ \ z = r^2 \\ \text{Paraboloid 2}: &\ \ z = 2 - r^2 \end{aligned}$$

The region R is bounded above by $z = 2 - r^2$ and below by $z = r^2$. The projection of R onto the xy-plane is the disk $0 \le r \le 1,\ 0 \le \theta \le 2\pi$. The radius of the disk is obtained by determining the intersection of the two surfaces:

$$\begin{aligned} z &= z \\ r^2 &= 2 - r^2 \\ r^2 &= 1 \\ r &= 1 \end{aligned}$$

Using the fact that $dV = r\,dz\,dr\,d\theta$ in Cylindrical Coordinates, the volume is:

$$\begin{aligned}
V &= \iiint_R 1\,dV \\
&= \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r\Big[z\Big]_{r^2}^{2-r^2} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r\left(2-r^2-r^2\right)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(2r-2r^3\right)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[r^2-\frac{1}{2}r^4\right]_0^1 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} d\theta \\
&= \frac{1}{2}\Big[\theta\Big]_0^{2\pi} \\
&= \boxed{\pi}
\end{aligned}$$

Math 210, Exam 2, Practice Fall 2009
Problem 14 Solution

14. Let $f(x,y) = e^{xy}$ and (r,θ) be polar coordinates. Find $\frac{\partial f}{\partial r}$. Express your answer in terms of the variables x and y.

Solution: First, the equations for x and y in polar coordinates are defined as:

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

Using the Chain Rule, the derivative $\frac{\partial f}{\partial r}$ can be expressed as follows:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{2}$$

The partial derivatives on the right hand side of the above equation are:

$$\frac{\partial f}{\partial x} = ye^{xy} \qquad \frac{\partial x}{\partial r} = \cos\theta$$
$$\frac{\partial f}{\partial y} = xe^{xy} \qquad \frac{\partial y}{\partial r} = \sin\theta$$

Plugging these into Equation (2) and using Equations (1) we get:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial f}{\partial r} = ye^{xy}\cos\theta + xe^{xy}\sin\theta$$
$$\frac{\partial f}{\partial r} = e^{xy}(y\cos\theta + x\sin\theta)$$

Using the fact that:

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad r = \sqrt{x^2+y^2}$$

we can write our answer in terms of x and y:

$$\frac{\partial f}{\partial r} = e^{xy}(y\cos\theta + x\sin\theta)$$
$$\frac{\partial f}{\partial r} = ye^{xy}\left(y\cdot\frac{x}{r} + x\cdot\frac{y}{r}\right)$$
$$\frac{\partial f}{\partial r} = ye^{xy}\left(y\cdot\frac{x}{\sqrt{x^2+y^2}} + x\cdot\frac{y}{\sqrt{x^2+y^2}}\right)$$
$$\boxed{\frac{\partial f}{\partial r} = \frac{2xy}{\sqrt{x^2+y^2}}e^{xy}}$$

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Problem 15 Solution

15. Compute the average value of the function $f(x, y) = 2 + x - y$ on the quarter disk $A = \{(x, y) : x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\}$.

Solution: We use the following formula to compute the average value of f:

$$\bar{f} = \frac{\iint_A f(x, y)\, dA}{\iint_A 1\, dA}$$

Since the region A is a quarter circle, we use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$. The region A can then be described as:

$$A = \left\{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$$

and the function f written in polar coordinates is:

$$f(r, \theta) = 2 + r\cos\theta - r\sin\theta$$

The double integral of f over A is then:

$$\begin{aligned}
\iint_A f(x, y)\, dA &= \int_0^{\pi/2} \int_0^1 (2 + r\cos\theta - r\sin\theta)\, r\, dr\, d\theta \\
&= \int_0^{\pi/2} \left[r^2 + \frac{1}{3}r^3\cos\theta - \frac{1}{3}r^3\sin\theta\right]_0^1 d\theta \\
&= \int_0^{\pi/2} \left(1 + \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta \\
&= \left[\theta + \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{\pi/2} \\
&= \left[\frac{\pi}{2} + \frac{1}{3}\sin\frac{\pi}{2} + \frac{1}{3}\cos\frac{\pi}{2}\right] - \left[0 + \frac{1}{3}\sin 0 + \frac{1}{3}\cos 0\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

We recognize that the double integral $\iint_A 1\, dA$ represents the area of A. Since A is a quarter circle of radius 1, the area is $\frac{\pi}{4}$. Thus, the average value of f is:

$$\boxed{\bar{f} = \frac{\iint_A f(x, y)\, dA}{\iint_A 1\, dA} = \frac{\frac{\pi}{2}}{\frac{\pi}{4}} = 2}$$

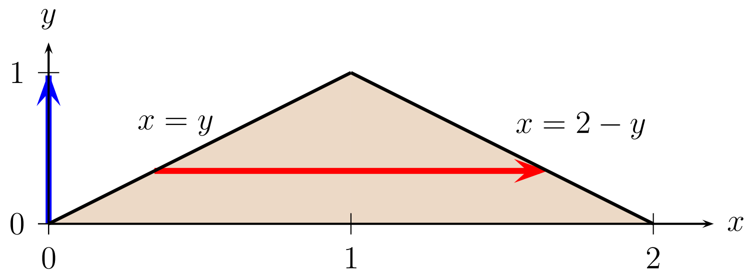
Math 210, Exam 2, Practice Fall 2009
Problem 16 Solution

16. Compute the integral

$$\iint_D \frac{x}{y+1}\,dA$$

where D is the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

Solution:



The integral is evaluated as follows:

$$\begin{aligned}
\iint_D \frac{x}{y+1}\,dA &= \int_0^1 \int_y^{2-y} \frac{x}{y+1}\,dx\,dy \\
&= \int_0^1 \frac{1}{y+1}\left[\frac{x^2}{2}\right]_y^{2-y} \\
&= \int_0^1 \frac{1}{y+1}\left[\frac{(2-y)^2}{2} - \frac{y^2}{2}\right]\,dy \\
&= \frac{1}{2}\int_0^1 \frac{1}{y+1}\left(4-4y+y^2-y^2\right)\,dy \\
&= \frac{1}{2}\int_0^1 \frac{1}{y+1}(4-4y)\,dy \\
&= 2\int_0^1 \frac{1-y}{1+y}\,dy \\
&= 2\int_0^1 \left(\frac{2}{1+y} - 1\right)\,dy \\
&= 2\Big[2\ln(1+y) - y\Big]_0^1 \\
&= 2\Big[2\ln(1+1) - 1\Big] - 2\Big[2\ln(1+0) - 0\Big] \\
&= \boxed{4\ln(2) - 2}
\end{aligned}$$

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Problem 17 Solution

17. Let $f(x, y) = x^2 - x + y^2$, and let $\mathcal{D}$ be the bounded region defined by the inequalities $x \geq 0$ and $x \leq 1 - y^2$.

(a) Find and classify the critical points of $f(x, y)$.

(b) Sketch the region $\mathcal{D}$.

(c) Find the absolute maximum and minimum values of f on the region $\mathcal{D}$, and list the points where these values occur.

Solution: First we note that the domain of $f(x, y)$ is bounded and closed, i.e. compact, and that $f(x, y)$ is continuous on the domain. Thus, we are guaranteed to have absolute extrema.

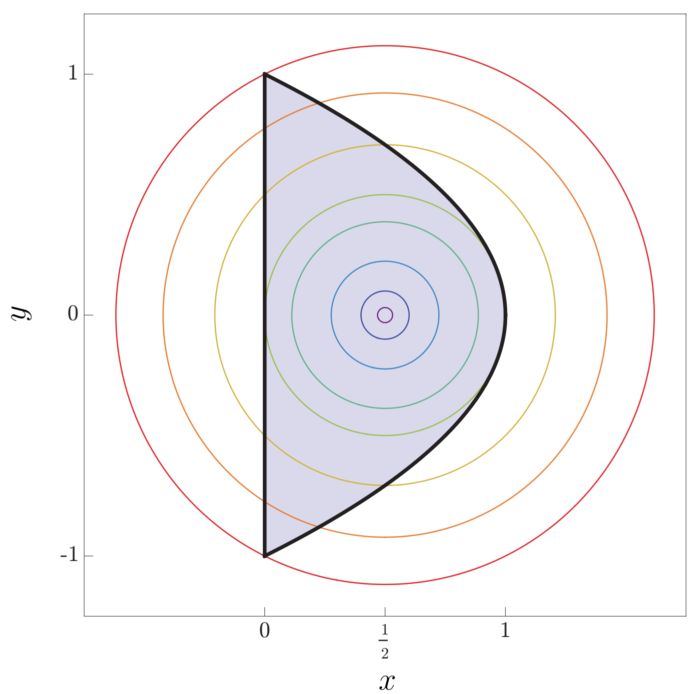
(a) The partial derivatives of f are $f_x = 2x - 1$ and $f_y = 2y$. The critical points of f are all solutions to the system of equations:

$$\begin{aligned} f_x &= 2x - 1 = 0 \\ f_y &= 2y = 0 \end{aligned}$$

The only solution is $x = \frac{1}{2}$ and $y = 0$, which is an interior point of $\mathcal{D}$. The function value at the critical point is:

$$\boxed{f(\tfrac{1}{2}, 0) = -\tfrac{1}{4}}$$

(b) The region $\mathcal{D}$ (shaded) is plotted below along with level curves of $f(x, y)$.



(c) We must now determine the minimum and maximum values of f on the boundary of $\mathcal{D}$. To do this, we must consider each part of the boundary separately:

Part I : Let this part be the line segment between $(0,-1)$ and $(0,1)$. On this part we have $x=0$ and $-1 \le y \le 1$. We now use the fact that $x=0$ to rewrite $f(x,y)$ as a function of one variable that we call $g_I(y)$.

$$\begin{aligned} f(x,y) &= x^2 - x + y^2 \\ g_I(y) &= 0^2 - 0 + y^2 \\ g_I(y) &= y^2 \end{aligned}$$

The critical points of $g_I(y)$ are:

$$\begin{aligned} g_I'(y) &= 0 \\ 2y &= 0 \\ y &= 0 \end{aligned}$$

Evaluating $g_I(y)$ at the critical point $y=0$ and at the endpoints of the interval $-1 \le y \le 1$, we find that:

$$g_I(0) = 0, \quad g_I(-1) = 1, \quad g_I(1) = 1$$

Note that these correspond to the function values:

$$\boxed{f(0,0) = 0, \quad f(0,-1) = 1, \quad f(0,1) = 1}$$

Part II : Let this part be the parabola $x = 1 - y^2$ on the interval $-1 \le y \le 1$. We now use the fact that $x = 1 - y^2$ to rewrite $f(x,y)$ as a function of one variable that we call $g_{II}(y)$.

$$\begin{aligned} f(x,y) &= x^2 - x + y^2 \\ g_{II}(y) &= \left(1 - y^2\right)^2 - \left(1 - y^2\right) + y^2 \\ g_{II}(y) &= 1 - 2y^2 + y^4 - 1 + y^2 + y^2 \\ g_{II}(y) &= y^4 \end{aligned}$$

The critical points of $g_{II}(y)$ are:

$$\begin{aligned} g_{II}'(y) &= 0 \\ 4y^3 &= 0 \\ y &= 0 \end{aligned}$$

Evaluating $g_{II}(y)$ at the critical point $y=0$ and at the endpoints of the interval $-1 \le y \le 1$, we find that:

$$g_{II}(0) = 0, \quad g_{II}(-1) = 1, \quad g_{II}(1) = 1$$

Note that these correspond to the function values:

$$\boxed{f(1,0) = 0, \quad f(0,-1) = 1, \quad f(0,1) = 1}$$

Finally, after comparing these values of f we find that the **absolute maximum** of f is 1 at the points $(0, -1)$ and $(0, 1)$ and that the **absolute minimum** of f is $-\frac{1}{4}$ at the point $(\frac{1}{2}, 0)$.

Note: In the figure from part (b) we see that the level curves of f are circles centered at $(\frac{1}{2}, 0)$. It is clear that the absolute minimum of f occurs at $(\frac{1}{2}, 0)$ and that the absolute maximum of f occurs at $(0, -1)$ and $(0, 1)$, which are points on the largest circle centered at $(\frac{1}{2}, 0)$ that contains points in $\mathcal{D}$.

Math 210, Exam 2, Practice Fall 2009
Problem 18 Solution

18. Consider the function $F(x,y) = x^2 e^{4x-y^2}$. Find the direction (unit vector) in which F has the fastest growth at the point $(1,2)$.

Solution: The direction in which F has the fastest growth at the point $(1,2)$ is the direction of **steepest ascent**:

$$\hat{\mathbf{u}} = \frac{1}{\left|\vec{\nabla} F(1,2)\right|} \vec{\nabla} F(1,2)$$

The gradient of F is:

$$\vec{\nabla} F = \langle F_x, F_y \rangle$$
$$\vec{\nabla} F = \left\langle 2xe^{4x-y^2} + 4x^2 e^{4x-y^2}, -2x^2 y e^{4x-y^2} \right\rangle$$

and its value at the point $(1,2)$ is:

$$\vec{\nabla} F(1,2) = \langle 6, -4 \rangle$$

Thus, the direction of steepest ascent is:

$$\begin{aligned} \hat{\mathbf{u}} &= \frac{1}{\left|\vec{\nabla} F(1,2)\right|} \vec{\nabla} F(1,2) \\ &= \frac{1}{|6,-4|} \langle 6, -4 \rangle \\ &= \boxed{\frac{1}{\sqrt{13}} \langle 3, -2 \rangle} \end{aligned}$$

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Problem 19 Solution

19. Let $\vec{\mathbf{r}}(t) = \langle e^{-t}, \cos(t) \rangle$ describe movement of a point in the plane, and let $f(x, y) = x^2y - e^{x+y}$. Use the chain rule to compute the derivative of $f(\vec{\mathbf{r}}(t))$ at time $t = 0$.

Solution: We use the Chain Rule for Paths formula:

$$\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right) = \vec{\nabla} f \cdot \vec{\mathbf{r}}'(t)$$

where the gradient of f is:

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \left\langle 2xy - e^{x+y}, x^2 - e^{x+y} \right\rangle$$

and the derivative $\vec{\mathbf{r}}'(t)$ is:

$$\vec{\mathbf{r}}'(t) = \left\langle -e^{-t}, -\sin(t) \right\rangle$$

Taking the dot product of these vectors gives us the derivative of $f(\vec{\mathbf{r}}(t))$.

$$\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right) = \vec{\nabla} f \cdot \vec{\mathbf{r}}'(t)$$

$$\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right) = \left\langle 2xy - e^{x+y}, x^2 - e^{x+y} \right\rangle \cdot \left\langle -e^{-t}, -\sin(t) \right\rangle$$

$$\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right) = -e^{-t}\left(2xy - e^{x+y}\right) - \sin(t)\left(x^2 - e^{x+y}\right)$$

At $t = 0$ we know that $\vec{\mathbf{r}}(0) = \langle 1, 1 \rangle$ which tells us that $x = 1$ and $y = 1$. Therefore, plugging $t = 0$, $x = 1$, and $y = 1$ into the derivative we find that:

$$\left.\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right)\right|_{t=0} = -e^{-0}\left(2(1)(1) - e^{1+1}\right) - \sin(0)\left(1^2 - e^{1+1}\right)$$

$$\boxed{\left.\frac{d}{dt} f\left(\vec{\mathbf{r}}(t)\right)\right|_{t=0} = e^2 - 2}$$

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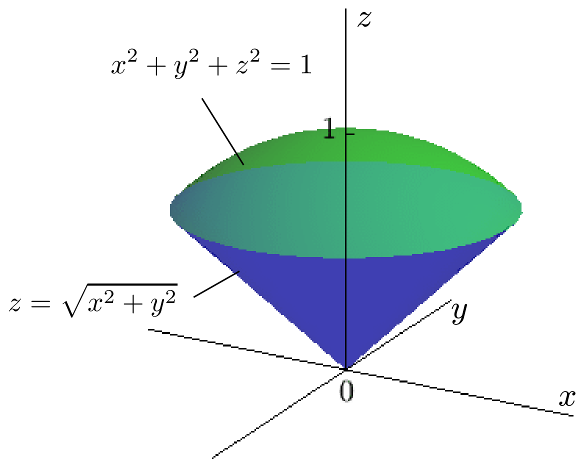
Problem 20 Solution

20. Let the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ describe the density in the region $A = \left\{x^2 + y^2 + z^2 \leq 1,\ \sqrt{x^2 + y^2} \leq z\right\}$. Use spherical coordinates to compute its mass.

Solution: The region A is plotted below.



The mass of the region is given by the triple integral:

$$\text{mass} = \iiint_A f(x, y, z)\, dV$$

In Spherical Coordinates, the equations for the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$ are:

$$\text{Sphere}: \quad \rho = 1$$
$$\text{Cone}: \quad \phi = \frac{\pi}{4}$$

and the density function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ is:

$$\text{density}: \quad f(\rho, \phi, \theta) = \rho$$

Using the fact that $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ in Spherical Coordinates, the mass of the region is:

$$
\begin{aligned}
\text{mass} &= \iiint_A f(x,y,z)\,dV \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\left(\rho^2\sin\phi\right)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\sin\phi\left[\frac{1}{4}\rho^4\right]_0^1\,d\phi\,d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi/4}\,d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\left[-\cos\frac{\pi}{4}-(-\cos 0)\right]\,d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\left(-\frac{\sqrt{2}}{2}+1\right)\,d\theta \\
&= \frac{1}{4}\left(1-\frac{\sqrt{2}}{2}\right)\Big[\theta\Big]_0^{2\pi} \\
&= \boxed{\frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)}
\end{aligned}
$$