Math 210, Exam 2, Spring 2008
Problem 1 Solution

1. Complete each of the following:

(a) Compute the directional derivative $D_{\hat{\mathbf{u}}}$ of the function $f(x, y) = xy^2 + \ln(xy)$ at the point $(1, 1)$ in the direction of $\vec{\mathbf{v}} = \langle 1, 2 \rangle$.

(b) Use the Chain Rule to compute $\dfrac{\partial w}{\partial s}$ when $s = 1$, $t = 2$ if

$$w(x, y) = x + x^2 y^3, \;\; x(s, t) = st, \;\; y(s, t) = s^2$$

Solution:

(a) By definition, the directional derivative of $f(x, y)$ at $(1, 1)$ in the direction of $\hat{\mathbf{u}}$ is:

$$D_{\hat{\mathbf{u}}} f(1, 1) = \vec{\nabla} f(1, 1) \cdot \hat{\mathbf{u}}$$

The gradient of f is:

$$\vec{\nabla} f(x, y) = \langle f_x, f_y \rangle = \left\langle y^2 + \frac{1}{x}, 2xy + \frac{1}{y} \right\rangle$$

Evaluating at the point $(1, 1)$ we get:

$$\vec{\nabla} f(1, 1) = \left\langle 1^2 + \frac{1}{1}, 2(1)(1) + \frac{1}{1} \right\rangle = \langle 2, 3 \rangle$$

Recalling that $\hat{\mathbf{u}}$ must be a unit vector, we multiply $\langle 1, 2 \rangle$ by the reciprocal of its magnitude.

$$\hat{\mathbf{u}} = \frac{1}{|\vec{\mathbf{v}}|} \vec{\mathbf{v}} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$$

Therefore, the directional derivative is:

$$D_{\hat{\mathbf{u}}} f(1, 1) = \vec{\nabla} f(1, 1) \cdot \hat{\mathbf{u}}$$
$$D_{\hat{\mathbf{u}}} f(1, 1) = \langle 2, 3 \rangle \cdot \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$$
$$D_{\hat{\mathbf{u}}} f(1, 1) = \frac{1}{\sqrt{5}} [(2)(1) + (3)(2)]$$
$$\boxed{D_{\hat{\mathbf{u}}} f(1, 1) = \frac{8}{\sqrt{5}}}$$

(b) Using the Chain Rule, the derivative $\frac{\partial w}{\partial s}$ can be expressed as follows:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \tag{1}$$

The partial derivatives on the right hand side of the above equation are:

$$\frac{\partial w}{\partial x} = 1 + 2xy^3 \qquad \frac{\partial x}{\partial s} = t$$
$$\frac{\partial w}{\partial y} = 3x^2y^2 \qquad \frac{\partial y}{\partial s} = 2s$$

Plugging these into Equation (1) we get:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial w}{\partial s} = \left(1 + 2xy^3\right)(t) + \left(3x^2y^2\right)(2s)$$

We must evaluate the derivative at $s = 1$ and $t = 2$. The values of x and y at this point are:

$$x(1,2) = 2, \quad y(1,2) = 1$$

Thus, the value of the derivative is:

$$\left.\frac{\partial w}{\partial s}\right|_{s=1,t=2} = \left(1 + 2(2)(1)^3\right)(2) + \left(3(2)^2(1)^2\right)(2(1))$$
$$\boxed{\left.\frac{\partial w}{\partial s}\right|_{s=1,t=2} = 34}$$

Math 210, Exam 2, Spring 2008
Problem 2 Solution

2. Consider the function

$$f(x, y) = x^2 - xy - y^2$$

(a) Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(1, 1, -1)$.

(b) Use the linearization of $f(x, y)$ about $(1, 1)$ to estimate $f(1.1, 0.95)$.

Solution:

(a) We will use the following formula for the plane tangent to $f(x, y) = x^2 - xy - y^2$ at the point $(1, 1, -1)$:

$$z = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

The first partial derivatives of f are:

$$f_x = 2x - y$$
$$f_y = -x - 2y$$

At the point $(1, 1)$ we have:

$$f(1, 1) = -1$$
$$f_x(1, 1) = 2(1) - 1 = 1$$
$$f_y(1, 1) = -1 - 2(1) = -3$$

Thus, an equation for the tangent plane is:

$$\boxed{z = -1 + (x - 1) - 3(y - 1)}$$

(b) The linearization of $f(x, y)$ about $(1, 1)$ has the exact same form as the equation for the tangent plane. That is,

$$L(x, y) = -1 + (x - 1) - 3(y - 1)$$

The value of $f(1.1, 0.95)$ is estimated to be the value of $L(1.1, 0.95)$:

$$f(1.1, 0.95) \approx L(1.1, 0.95)$$
$$f(1.1, 0.95) \approx -1 + (1.1 - 1) - 3(0.95 - 1)$$
$$\boxed{f(1.1, 0.95) \approx -0.75}$$

Math 210, Exam 2, Spring 2008
Problem 3 Solution

3. Find the critical points of $f(x, y)$ and specify for each whether it corresponds to a local maximum, local minimum or saddle point, given that the partial derivatives of f are:

$$f_x = 2x - 4y, \quad f_y = -4x + 5y + 3y^2$$

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives $f_x = 2x - 4y$ and $f_y = -4x + 5y + 3y^2$ exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 2x - 4y = 0 \qquad (1)$$
$$f_y = -4x + 5y + 3y^2 = 0 \qquad (2)$$

Solving Equation (1) for x we get:

$$x = 2y \qquad (3)$$

Substituting this into Equation (2) and solving for y we get:

$$\begin{aligned} -4x + 5y + 3y^2 &= 0 \\ -4(2y) + 5y + 3y^2 &= 0 \\ 3y^2 - 3y &= 0 \\ 3y(y-1) &= 0 \\ \iff \quad y = 0 \quad \text{or} \quad y &= 1 \end{aligned}$$

We find the corresponding x-values using Equation (3): $x = 2y$.

- If $y = 0$, then $x = 2(0) = 0$.
- If $y = 1$, then $x = 2(1) = 2$.

Thus, the critical points are $\boxed{(0, 0)}$ and $\boxed{(2, 1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 2, \quad f_{yy} = 5 + 6y, \quad f_{xy} = -4$$

The discriminant function $D(x, y)$ is then:

$$
\begin{aligned}
D(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\
D(x, y) &= (2)(5 + 6y) - (-4)^2 \\
D(x, y) &= 12y - 6
\end{aligned}
$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(0, 0)$	-6	2	Saddle Point
$(2, 1)$	6	2	Local Minimum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local minimum of f if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

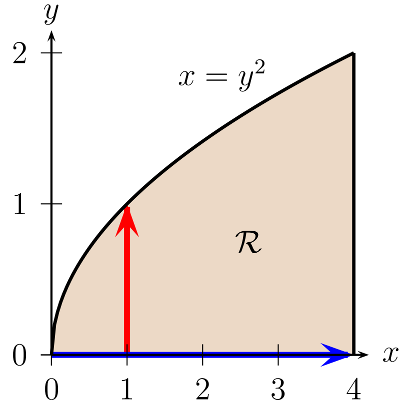
Math 210, Exam 2, Spring 2008
Problem 4 Solution

4. For the following integral:

$$\int_0^2 \int_{y^2}^4 \frac{y^3}{x} e^{x^2} \, dx \, dy$$

sketch the region of integration, reverse the order of integration, and evaluate the resulting integral.

Solution: The region of integration $\mathcal{R}$ is sketched below:



The region $\mathcal{R}$ can be described as follows:

$$\mathcal{R} = \left\{(x,y) \,:\, 0 \le y \le \sqrt{x},\ 0 \le x \le 4\right\}$$

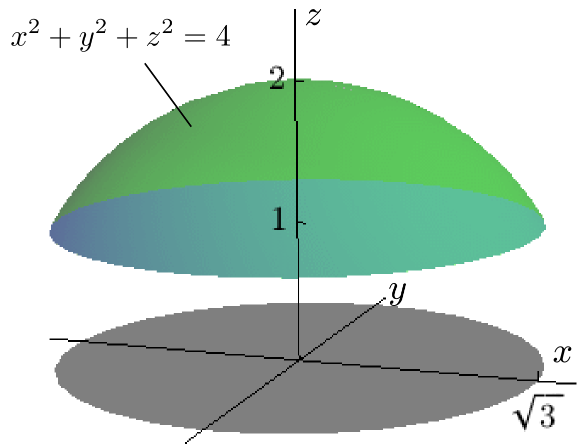
where $y = 0$ is the bottom curve and $y = \sqrt{x}$ is the top curve, obtained by solving the equation $x = y^2$ for y in terms of x. The projection of $\mathcal{R}$ onto the x-axis is the interval $0 \le x \le 4$. Therefore, the value of the integral is:

$$\begin{aligned}
\int_0^2 \int_{y^2}^4 \frac{y^3}{x} e^{x^2} \, dx \, dy &= \int_0^4 \int_0^{\sqrt{x}} \frac{y^3}{x} e^{x^2} \, dx \, dy \\
&= \int_0^4 \frac{1}{x} e^{x^2} \left[\frac{1}{4} y^4\right]_0^{\sqrt{x}} dx \\
&= \int_0^4 \frac{1}{x} e^{x^2} \left[\frac{1}{4} x^2\right] dx \\
&= \frac{1}{4} \int_0^4 x e^{x^2} \, dx \\
&= \frac{1}{4} \left[\frac{1}{2} e^{x^2}\right]_0^4 \\
&= \frac{1}{4} \left[\frac{1}{2} e^{4^2} - \frac{1}{2} e^{0^2}\right] \\
&= \boxed{\frac{1}{8}\left(e^{16} - 1\right)}
\end{aligned}$$

Math 210, Exam 2, Spring 2008
Problem 5 Solution

5. Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = 1$. (Hint: use cylindrical coordinates)

Solution: The region is plotted below.



The hint tells us that we should consider finding the volume using a triple integral. Thus, we use the formula:

$$V = \iiint_D 1\,dV$$

In cylindrical coordinates, the equation for the sphere is $z = \sqrt{4 - r^2}$, taking the positive root because the region is above the xy-plane. The projection of the region onto the xy-plane is the disk $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$, where the radius of the disk was obtained by finding the intersection of the plane $z = 1$ with the sphere:

$$\begin{aligned} z &= z \\ 1 &= \sqrt{4 - r^2} \\ 1 &= 4 - r^2 \\ r^2 &= 3 \\ r &= \sqrt{3} \end{aligned}$$

Using the fact that $dV = r\,dz\,dr\,d\theta$ in cylindrical coordinates, the volume is:

$$
\begin{aligned}
V &= \iiint_D 1\, dV \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} r \Big[z \Big]_1^{\sqrt{4-r^2}} dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} r \left(\sqrt{4-r^2} - 1 \right) dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3} \left(4-r^2\right)^{3/2} - \frac{1}{2} r^2 \right]_0^{\sqrt{3}} d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3} \left(4-3\right)^{3/2} - \frac{1}{2}(3) + \frac{1}{3}(4-0)^{3/2} + \frac{1}{2}(0) \right] d\theta \\
&= \int_0^{2\pi} \frac{5}{6}\, d\theta \\
&= \left[\frac{5}{6}\theta \right]_0^{2\pi} \\
&= \boxed{\frac{5\pi}{3}}
\end{aligned}
$$