Math 210, Exam 2, Spring 2010
Problem 1 Solution

1. Find and classify the critical points of the function

$$f(x, y) = x^3 + 3xy - y^3.$$

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = x^3 + 3xy - y^3$ are $f_x = 3x^2 + 3y$ and $f_y = 3x - 3y^2$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 3x^2 + 3y = 0 \quad (1)$$
$$f_y = 3x - 3y^2 = 0 \quad (2)$$

Solving Equation (1) for y we get:

$$y = -x^2 \quad (3)$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned} 3x - 3y^2 &= 0 \\ 3x - 3\left(-x^2\right)^2 &= 0 \\ 3x - 3x^4 &= 0 \\ 3x(1 - x^3) &= 0 \end{aligned}$$

We observe that the above equation is satisfied if either $x = 0$ or $x^3 - 1 = 0 \iff x = 1$. We find the corresponding y-values using Equation (3): $y = -x^2$.

- If $x = 0$, then $y = -0^2 = 0$.
- If $x = 1$, then $y = -(1)^2 = -1$.

Thus, the critical points are $\boxed{(0, 0)}$ and $\boxed{(1, -1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 6x, \quad f_{yy} = -6y, \quad f_{xy} = 3$$

The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = (6x)(-6y) - (3)^2$$
$$D(x, y) = -36xy - 9$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(0, 0)$	-9	0	Saddle Point
$(1, -1)$	27	6	Local Minimum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local minimum of f if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$.

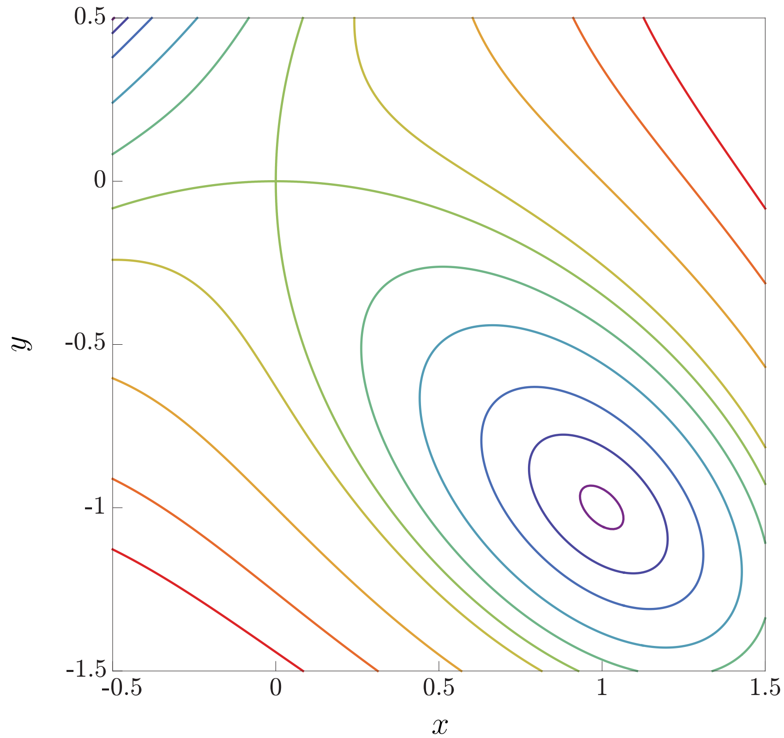

Figure 1: Pictured above are level curves of $f(x, y)$. Darker colors correspond to smaller values of $f(x, y)$. It is apparent that $(0, 0)$ is a saddle point and $(1, -1)$ corresponds to a local minimum.

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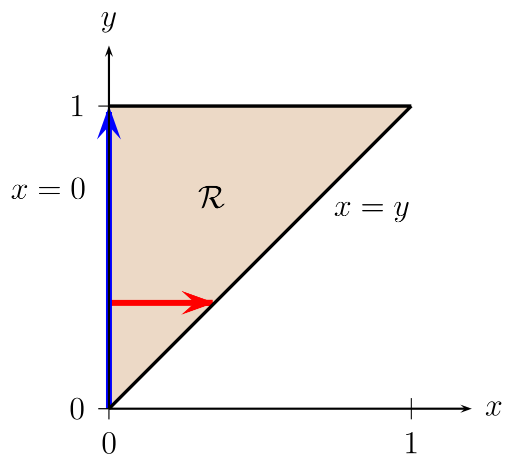
Problem 2 Solution

2. Sketch the region of integration and compute $\int_0^1 \int_0^y e^{-y^2}\,dx\,dy$.

Solution:



The integral is evaluated as follows:

$$\begin{aligned}\int_0^1 \int_0^y e^{-y^2}\,dx\,dy &= \int_0^2 \left[xe^{-y^2}\right]_0^y dy \\ &= \int_0^2 ye^{-y^2}\,dy \\ &= \left[-\frac{1}{2}e^{-y^2}\right]_0^1 \\ &= \left[-\frac{1}{2}e^{-1}\right] - \left[-\frac{1}{2}e^{0}\right] \\ &= \boxed{\frac{1}{2} - \frac{1}{2}e^{-1}}\end{aligned}$$

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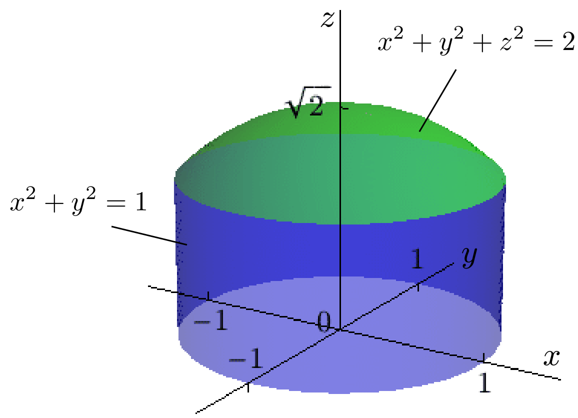
Problem 3 Solution

3. Compute

$$\iiint_A z\, dV$$

where A is the region inside the sphere $x^2+y^2+z^2=2$, inside the cylinder $x^2+y^2=1$, and above the xy-plane.

Solution: The region A is plotted below.



We use Cylindrical Coordinates to evaluate the triple integral. The equations for the sphere and cylinder are then:

$$\begin{aligned} \text{Sphere}: &\quad r^2+z^2=2 \quad \Rightarrow \quad z=\sqrt{2-r^2} \\ \text{Cylinder}: &\quad r^2=1 \quad \Rightarrow \quad r=1 \end{aligned}$$

The surface that bounds A from below is $z=0$ (the xy-plane) and the surface that bounds A from above is $z=\sqrt{2-r^2}$ (the sphere). The projection of the region A onto the xy-plane is the disk $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Using the fact that $dV = r\, dz\, dr\, d\theta$ in Cylindrical Coordinates, the value of the triple integral is:

$$
\begin{aligned}
\iiint_A z\,dV &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{2-r^2}} zr\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r\left[\frac{1}{2}z^2\right]_0^{\sqrt{2-r^2}} dr\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^1 r\left(2-r^2\right) dr\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\left[r^2-\frac{1}{4}r^4\right]_0^1 d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\frac{3}{4}\,d\theta \\
&= \frac{3}{8}\Big[\theta\Big]_0^{2\pi} \\
&= \boxed{\frac{3\pi}{4}}
\end{aligned}
$$

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Problem 4 Solution

4. Compute the integral of the field $\overrightarrow{\mathbf{F}}(x,y) = (x+y)\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}$ along the curve $\overrightarrow{\mathbf{c}}(\theta) = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$.

Solution: By definition, the line integral of a vector field $\overrightarrow{\mathbf{F}}$ along a curve $\mathcal{C}$ with parameterization $\overrightarrow{\mathbf{c}}(\theta) = \langle x(\theta), y(\theta)\rangle$, $a \le \theta \le b$ is given by the formula:

$$\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}}\, ds = \int_a^b \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{c}}'(\theta)\, d\theta$$

From the given parameterization $\overrightarrow{\mathbf{c}}(\theta) = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ we have:

$$\overrightarrow{\mathbf{c}}'(\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

and, using the fact that $x(\theta) = \cos\theta$ and $y(\theta) = \sin\theta$, the function $\overrightarrow{\mathbf{F}}$ can be rewritten as:

$$\begin{aligned}\overrightarrow{\mathbf{F}} &= (x+y)\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}\\ \overrightarrow{\mathbf{F}} &= (\cos\theta + \sin\theta)\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}\end{aligned}$$

Assuming an interval $0 \le \theta \le 2\pi$, the value of the line integral is then:

$$\begin{aligned}\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}}\, ds &= \int_a^b \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{c}}'(\theta)\, d\theta\\ &= \int_0^{2\pi} \left((\cos\theta + \sin\theta)\ \hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}\right) \cdot \left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right)\, d\theta\\ &= \int_0^{2\pi} (\cos\theta + \sin\theta)(-\sin\theta)\, d\theta\\ &= \int_0^{2\pi} (-\sin\theta\cos\theta - \sin^2\theta)\, d\theta\\ &= \left[\frac{1}{2}\cos^2\theta - \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}\\ &= \left[\frac{1}{2}\cos^2(2\pi) - \frac{1}{2}(2\pi) + \frac{1}{4}\sin(4\pi)\right] - \left[\frac{1}{2}\cos^2 0 - \frac{1}{2}(0) + \frac{1}{4}\sin(0)\right]\\ &= \boxed{-\pi}\end{aligned}$$

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Problem 5 Solution

5. Find the minimum and maximum of the function $f(x, y) = x + y^2$ subject to the condition $2x^2 + y^2 = 1$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $2x^2 + y^2 = 1$ is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y) = 2x^2 + y^2 = 1$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 1$$

which, when applied to our functions f and g, give us:

$$1 = \lambda\,(4x) \tag{1}$$
$$2y = \lambda\,(2y) \tag{2}$$
$$2x^2 + y^2 = 1 \tag{3}$$

We begin by noting that Equation (2) gives us:

$$\begin{aligned} 2y &= \lambda(2y) \\ 2y - \lambda(2y) &= 0 \\ 2y(1 - \lambda) &= 0 \end{aligned}$$

From this equation we either have $y = 0$ or $\lambda = 1$. Let's consider each case separately.

Case 1: Let $y = 0$. We find the corresponding x-values using Equation (3).

$$\begin{aligned} 2x^2 + y^2 &= 1 \\ 2x^2 + 0^2 &= 1 \\ x^2 &= \tfrac{1}{2} \\ x &= \pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

Thus, the points of interest are $(\frac{1}{\sqrt{2}}, 0)$ and $(-\frac{1}{\sqrt{2}}, 0)$.

Case 2: Let $\lambda = 1$. Plugging this into Equation (1) we get:

$$\begin{aligned} 1 &= \lambda(4x) \\ 1 &= 1(4x) \\ x &= \tfrac{1}{4} \end{aligned}$$

We find the corresponding y-values using Equation (3).

$$\begin{aligned} 2x^2 + y^2 &= 1 \\ 2\left(\tfrac{1}{4}\right)^2 + y^2 &= 1 \\ \tfrac{1}{8} + y^2 &= 1 \\ y^2 &= \tfrac{7}{8} \\ y &= \pm\sqrt{\tfrac{7}{8}} \end{aligned}$$

Thus, the points of interest are $(\frac{1}{4}, \sqrt{\frac{7}{8}})$ and $(\frac{1}{4}, -\sqrt{\frac{7}{8}})$.

We now evaluate $f(x,y) = x + y^2$ at each point of interest obtained by Cases 1 and 2.

$$f(\tfrac{1}{\sqrt{2}}, 0) = \tfrac{1}{\sqrt{2}}$$
$$f(-\tfrac{1}{\sqrt{2}}, 0) = -\tfrac{1}{\sqrt{2}}$$
$$f(\tfrac{1}{4}, \sqrt{\tfrac{7}{8}}) = \tfrac{9}{8}$$
$$f(\tfrac{1}{4}, -\sqrt{\tfrac{7}{8}}) = \tfrac{9}{8}$$

From the values above we observe that f attains an absolute maximum of $\frac{9}{8}$ and an absolute minimum of $-\frac{1}{\sqrt{2}}$.

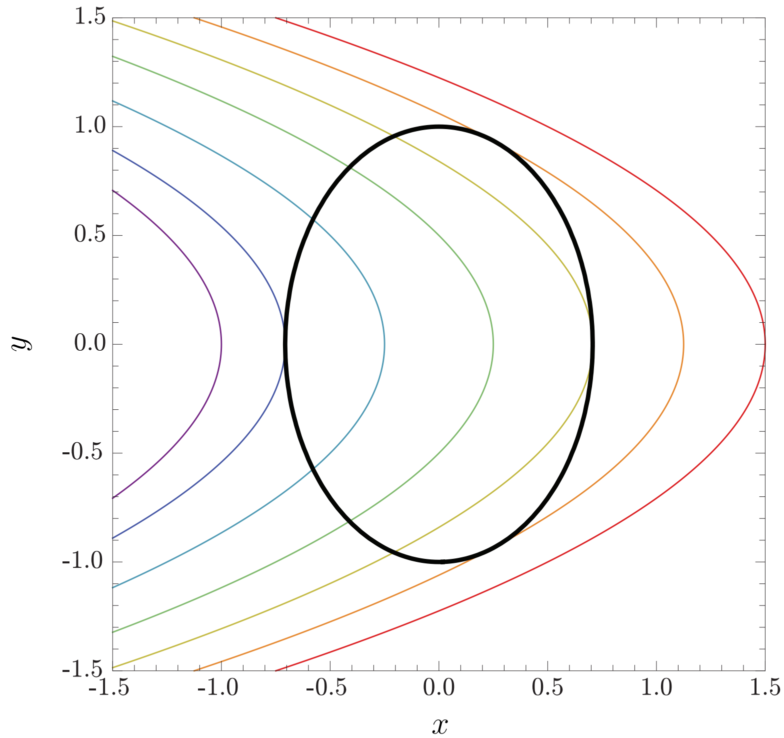

Figure 1: Shown in the figure are the level curves of $f(x,y) = x + y^2$ and the ellipse $2x^2 + y^2 = 1$ (thick, black curve). Darker colors correspond to smaller values of $f(x,y)$. Notice that (1) the parabola $f(x,y) = x + y^2 = \frac{9}{8}$ is tangent to the ellipse at the points $(\frac{1}{4}, \sqrt{\frac{7}{8}})$ and $(\frac{1}{4}, -\sqrt{\frac{7}{8}})$ which correspond to the absolute maximum and (2) the parabola $f(x,y) = x + y^2 = -\frac{1}{\sqrt{2}}$ is tangent to the ellipse at the point $(-\frac{1}{\sqrt{2}}, 0)$ which corresponds to the absolute minimum.

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Problem 6 Solution

6. For the vector field $\overrightarrow{\mathbf{F}}(x,y) = (x+y)\,\hat{\mathbf{i}} + (x-y)\,\hat{\mathbf{j}}$, find a function $\varphi(x,y)$ with $\operatorname{grad}\varphi = \overrightarrow{\mathbf{F}}$ or use the partial derivative test to show that such a function does not exist.

Solution: In order for the vector field $\overrightarrow{\mathbf{F}} = \langle f(x,y), g(x,y)\rangle$ to have a potential function $\varphi(x,y)$ such that $\operatorname{grad}\varphi = \overrightarrow{\mathbf{F}}$, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x,y) = x+y$ and $g(x,y) = x-y$ we get:

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = 1$$

which verifies the existence of a potential function for the given vector field.

If $\operatorname{grad}\varphi = \overrightarrow{\mathbf{F}}$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x,y) \tag{1}$$

$$\frac{\partial \varphi}{\partial y} = g(x,y) \tag{2}$$

Using $f(x,y) = x+y$ and integrating both sides of Equation (1) with respect to x we get:

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &= f(x,y) \\
\frac{\partial \varphi}{\partial x} &= x+y \\
\int \frac{\partial \varphi}{\partial x}\,dx &= \int (x+y)\,dx \\
\varphi(x,y) &= \frac{1}{2}x^2 + xy + h(y) \qquad (3)
\end{aligned}$$

We obtain the function $h(y)$ using Equation (2). Using $g(x,y) = x-y$ we get the equation:

$$\begin{aligned}
\frac{\partial \varphi}{\partial y} &= g(x,y) \\
\frac{\partial \varphi}{\partial y} &= x-y
\end{aligned}$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + xy + h(y)\right) &= x-y \\
x + h'(y) &= x-y \\
h'(y) &= -y
\end{aligned}$$

Now integrate both sides with respect to y to get:

$$\int h'(y)\,dy = \int -y\,dy$$
$$h(y) = -\frac{1}{2}y^2 + C$$

Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\varphi(x,y) = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2}$$