**Math 210, Exam 2, Spring 2012**
**Problem 1 Solution**

1. Consider the integral $\iint_R \left(x^2+y^2\right)^{3/2}\, dA$ where $R = \{(x,y)\ :\ 1 \le x^2+y^2 \le 4,\ y \ge 0\}$.

   (a) Rewrite this as an integral in polar coordinates.

   (b) Compute the integral.

**Solution**:

(a) The region $R$ is described as $\{(r,\theta)\ :\ 1 \le r \le 2,\ 0 \le \theta \le \pi\}$ in polar coordinates. Therefore, the integral becomes

$$\iint_R \left(x^2+y^2\right)^{3/2}\, dA = \int_0^{\pi}\int_1^2 \left(r^2\right)^{3/2} r\, dr\, d\theta.$$

(b) The value of the integral is

$$\begin{aligned}\int_0^{\pi}\int_1^2 \left(r^2\right)^{3/2} r\, dr\, d\theta &= \int_0^{\pi}\int_1^2 r^4\, dr\, d\theta,\\ &= \pi\left[\frac{r^5}{5}\right]_1^2,\\ &= \frac{31\pi}{5}.\end{aligned}$$

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**Problem 2 Solution**

2. Let $f(x, y) = \dfrac{x+1}{y+1}$.

(a) Compute the gradient $\nabla f(x, y)$.

(b) Compute the directional derivative of $f$ at the point $(2, 0)$ in the direction of the unit vector $u = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

(c) Find the equation of the tangent plane of the surface $z = f(x, y)$ at the point $(1, 1, 1)$.

**Solution**:

(a) The first partial derivatives of $f$ are

$$f_x = \frac{\partial}{\partial x} \frac{x+1}{y+1} \qquad f_y = \frac{\partial}{\partial y} \frac{x+1}{y+1},$$

$$f_x = \frac{1}{y+1} \frac{\partial}{\partial x}(x+1) \qquad f_y = (x+1) \frac{\partial}{\partial y} \frac{1}{y+1},$$

$$f_x = \frac{1}{y+1} \qquad f_y = -\frac{x+1}{(y+1)^2}.$$

Therefore, the gradient of $f$ is

$$\vec{\nabla} f = \left\langle \frac{1}{y+1}, -\frac{x+1}{(y+1)^2} \right\rangle$$

(b) By definition, the directional derivative of a function $f(x, y)$ at the point $(a, b)$ in the direction of the unit vector $\hat{\mathbf{u}}$ is

$$D_{\hat{\mathbf{u}}} f(a, b) = \vec{\nabla} f(a, b) \bullet \hat{\mathbf{u}}$$

The gradient of $f$ evaluated at $(2, 0)$ is

$$\vec{\nabla} f(2, 0) = \left\langle \frac{1}{0+1}, -\frac{2+1}{(0+1)^2} \right\rangle = \langle 1, -3 \rangle$$

Thus, the directional derivative is

$$D_{\hat{\mathbf{u}}} f(2, 0) = \langle 1, -3 \rangle \bullet \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = -\frac{1}{2} + \frac{3\sqrt{3}}{2}$$

(c) For the surface $z = f(x, y)$, an equation for the tangent plane at $(x, y) = (a, b)$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The partial derivatives evaluated at $(1, 1)$ are

$$f_x(1, 1) = \frac{1}{2}, \qquad f_y(1, 1) = -\frac{1}{2}$$

Therefore, an equation for the tangent plane is

$$z = 1 + \frac{1}{2}(x - 1) - \frac{1}{2}(y - 1)$$

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**Problem 3 Solution**

3. (a) Write a double integral that represents the volume below the surface $z = e^{-x^2}$ and above the triangle in the $xy$-plane with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$.

(b) Compute the integral from part (a).

**Solution**:

(a) The volume of the region below the surface $z = f(x, y)$ and above the region $D$ in the $xy$-plane is given by the double integral

$$V = \iint_D f(x, y)\, dA$$

In this problem, the region $D$ is described as $\{(x, y) : 0 \le y \le x,\ 0 \le x \le 1\}$. Therefore, the volume integral is

$$V = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

Note that $D$ can also be described as $\{(x, y) : y \le x \le 1,\ 0 \le y \le 1\}$. However, the corresponding volume integral

$$V = \int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

cannot be evaluated easily.

(b) We evaluate the first integral found in part (a).

$$V = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx,$$

$$V = \int_0^1 e^{-x^2} \Big[ y \Big]_0^x\, dx,$$

$$V = \int_0^1 x e^{-x^2}\, dx,$$

$$V = \left[ -\frac{1}{2} e^{-x^2} \right]_0^1,$$

$$V = \frac{1}{2}\left(1 - e^{-1}\right).$$

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# Problem 4 Solution

4. Compute the triple integral $\int_1^2 \int_0^1 \int_1^{e^y} \frac{e^y}{xz^2}\, dxdydz$

**Solution**: The value of the integral is computed as follows:

$$\begin{aligned}
\int_1^2 \int_0^1 \int_1^{e^y} \frac{e^y}{xz^2}\, dxdydz &= \int_1^2 \int_0^1 \frac{e^y}{z^2} \Big[ \ln(x) \Big]_1^{e^y} dy\, dz, \\
&= \int_1^2 \int_0^1 \frac{e^y}{z^2} \Big[ \ln(e^y) - \ln(1) \Big] dy\, dz, \\
&= \int_1^2 \int_0^1 \frac{e^y}{z^2} \cdot y\, dy\, dz, \\
&= \int_1^2 \frac{1}{z^2} \Big[ ye^y - e^y \Big]_0^1 dz, \\
&= \int_1^2 \frac{1}{z^2}\, dz, \\
&= \left[ -\frac{1}{z} \right]_1^2, \\
&= \frac{1}{2}
\end{aligned}$$

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**Problem 5 Solution**

5. Consider the function $f(x, y) = 2 + x^2 - y^2 - y$.

(a) Find the critical points of $f$ and classify each one as a local maximum, local minimum, or saddle point.

(b) Find the absolute maximum and minimum values of $f$ on the disk

$$D = \{(x, y) \,:\, x^2 + y^2 \leq 1\}$$

and the points where these extreme values occur.

**Solution**:

(a) The critical points of $f$ are the set of points $(a, b)$ that satisfy $f_x(a, b) = 0$ and $f_y(a, b) = 0$ simultaneously. The system of equations

$$f_x = 2x = 0,$$
$$f_y = -2y - 1 = 0$$

has the solution $x = 0$, $y = -\frac{1}{2}$. Therefore, $(0, -\frac{1}{2})$ is the only critical point. We classify this point using the Second Derivative Test. The second partial derivatives of $f$ are

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{xy} = 0$$

Therefore, the discriminant function $D(x, y)$ is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -4$$

Since $D(0, -\frac{1}{2}) = -4 < 0$, we know that $(0, -\frac{1}{2})$ corresponds to a saddle point.

(b) The absolute extrema will occur either at a critical point in the interior of $D$ or on the boundary. We've already found the critical point in part (a). We must now consider the boundary which is the curve $x^2 + y^2 = 1$.

We can proceed in one of two ways: (1) solve the boundary equation for $x^2$ and plug into the function $f$ to reduce it to a function of one variable or (2) use Lagrange Multpliers. We'll use the former method.

The boundary equation gives us $x^2 = 1 - y^2$. Plugging into $f$ we find that

$$f(y) = 2 + (1 - y^2) - y^2 - y = 3 - y - 2y^2$$

where $y \in [-1, 1]$. The critical points of $f$ in its domain are found by solving the equation $f'(y) = 0$.

$$f'(y) = -1 - 4y = 0 \qquad \Longleftrightarrow \qquad y = -\frac{1}{4}$$

The value of $f$ at the critical point $y = -\frac{1}{4}$ and at the endpoints of the domain are

$$f(-\tfrac{1}{4}) = \tfrac{25}{8}, \quad f(-1) = 2, \quad f(1) = 0$$

The value of $f$ at the critical point in the interior of $D$ is

$$f(0, -\tfrac{1}{2}) = \tfrac{9}{4}$$

The largest of the above values of the function is $\frac{25}{8}$ (the absolute maximum) while the smallest is 0 (the absolute minimum). The corresponding points are $(\pm\frac{\sqrt{15}}{4}, -\frac{1}{4})$ and $(0, 1)$.