Math 210, Final Exam, Fall 2007
Problem 1 Solution

1. (a) Compute the integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}}$ where C is the circle $x^2+y^2=1$ of radius 1 centered at the origin, traversed counterclockwise, starting and ending at the point $(1,0)$ for

$$\vec{\mathbf{F}} = \langle P, Q\rangle = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$$

(b) For the vector field in part (a), we know that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ (*you are not to check this!*. Is $\vec{\mathbf{F}}$ conservative? Explain your answer.

Solution: (a) We evaluate the vector line integral using the formula:

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}} = \int_a^b \vec{\mathbf{F}} \bullet \vec{\mathbf{r}}'(t)\,dt$$

A parameterization of C is $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t)\rangle$, $0 \le t \le 2\pi$. The derivative is $\vec{\mathbf{r}}'(t) = \langle -\sin(t), \cos(t)\rangle$. Using the fact that $x = \cos(t)$ and $y = \sin(t)$ from the parameterization, the vector field $\vec{\mathbf{F}}$ written in terms of t is:

$$\vec{\mathbf{F}} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$$

$$\vec{\mathbf{F}} = \left\langle -\frac{\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}\right\rangle$$

$$\vec{\mathbf{F}} = \langle -\sin(t), \cos(t)\rangle$$

Thus, the value of the line integral is:

$$\begin{aligned}
\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}} &= \int_0^{2\pi} \vec{\mathbf{F}} \bullet \vec{\mathbf{r}}'(t)\,dt \\
&= \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \bullet \langle -\sin(t), \cos(t)\rangle\; dt \\
&= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t)\right)\; dt \\
&= \int_0^{2\pi} 1\,dt \\
&= \boxed{2\pi}
\end{aligned}$$

(b) The vector field is **NOT** conservative. If it were, then the integral $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}}$ would be 0. However, as we saw in part (a), the value of the integral is 2π.

In this problem, the vector field $\vec{\mathbf{F}}$ is undefined at the origin $(0,0)$. Thus, the domain of $\vec{\mathbf{F}}$ is not simply connected which means that $\vec{\mathbf{F}}$ is not conservative.

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Problem 2 Solution

2. A particle is traveling in $\mathbb{R}^3$, with position given at time t, for $0 \leq t \leq 3$ by

$$\vec{\mathbf{r}}(t) = \langle 1 + t, e^t, t^2 \rangle$$

(a) find the velocity of the particle at time t

(b) find the speed of the particle at time t

(c) find the acceleration of the particle at time t

(d) Write down an integral, *but do NOT attempt to compute it*, for the distance traveled by the particle between times $t = 0$ and $t = 3$.

Solution:

(a) The velocity is the derivative of position.

$$\boxed{\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle 1, e^t, 2t \rangle}$$

(b) The speed is the magnitude of velocity.

$$v(t) = ||\vec{\mathbf{v}}(t)||$$
$$v(t) = \sqrt{1^2 + (e^t)^2 + (2t)^2}$$
$$\boxed{v(t) = \sqrt{1 + e^{2t} + 4t^2}}$$

(c) The acceleration is the derivative of velocity.

$$\boxed{\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \langle 0, e^t, 2 \rangle}$$

(d) The distance traveled by the particle is:

$$L = \int_0^3 ||\vec{\mathbf{r}}'(t)||\ dt$$
$$\boxed{L = \int_0^3 \sqrt{1 + e^{2t} + 4t^2}\, dt}$$

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Problem 3 Solution

3. (a) Find the equation of the tangent plane to the surface $ze^{x^2-y^2} = 2$ at the point $(1, -1, 2)$.

(b) If $f(x, y, z) = ze^{x^2-y^2}$ is the same function as in part (a), compute the directional derivative of f at the point $(1, -1, 2)$ in the direction of $\langle 2, 2, 1\rangle$.

Solution: (a) We use the following formula for the equation for the tangent plane:

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

because the surface equation is given in **implicit** form. Note that $\vec{\mathbf{n}} = \vec{\nabla} f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)\rangle$ is a vector normal to the surface $f(x, y, z) = C$ and, thus, to the tangent plane at the point (a, b, c) on the surface.

The partial derivatives of $f(x, y, z) = ze^{x^2-y^2}$ are:

$$\begin{aligned} f_x &= 2xze^{x^2-y^2} \\ f_y &= -2yze^{x^2-y^2} \\ f_z &= e^{x^2-y^2} \end{aligned}$$

Evaluating the partial derivatives at $(1, -1, 2)$ we have:

$$\begin{aligned} f_x(1, -1, 2) &= 2(1)(2)e^{1^2-(-1)^2} = 4 \\ f_y(1, -1, 2) &= -2(-1)(2)e^{1^2-(-1)^2} = 4 \\ f_z(1, -1, 2) &= e^{1^2-(-1)^2} = 1 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{4(x - 1) + 4(y + 1) + (z - 2) = 0}$$

(b) By definition, the directional derivative of $f(x, y, z)$ at $(1, -1, 2)$ in the direction of $\hat{\mathbf{u}}$ is:

$$D_{\hat{\mathbf{u}}} f(1, -1, 2) = \vec{\nabla} f(1, -1, 2) \bullet \hat{\mathbf{u}}$$

From part (b), we have $\vec{\nabla} f(1, -1, 2) = \langle 4, 4, 1\rangle$. Recalling that $\hat{\mathbf{u}}$ must be a unit vector, we multiply $\langle 2, 2, 1\rangle$ by the reciprocal of its magnitude.

$$\hat{\mathbf{u}} = \frac{1}{|\langle 2, 2, 1\rangle|} \langle 2, 2, 1\rangle = \frac{1}{3} \langle 2, 2, 1\rangle$$

Therefore, the directional derivative is:

$$D_{\hat{\mathbf{u}}} f(1,-1,2) = \vec{\nabla} f(1,-1,2) \bullet \hat{\mathbf{u}}$$
$$D_{\hat{\mathbf{u}}} f(1,-1,2) = \langle 4,4,1 \rangle \bullet \frac{1}{3} \langle 2,2,1 \rangle$$
$$D_{\hat{\mathbf{u}}} f(1,-1,2) = \frac{1}{3} [(4)(2) + (4)(2) + (1)(1)]$$
$$\boxed{D_{\hat{\mathbf{u}}} f(1,-1,2) = \frac{17}{3}}$$

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Problem 4 Solution

4. Find the critical points of the function $f(x,y) = xy - \dfrac{x^2}{2} + \dfrac{y^3}{3} - 2y$ and determine which are local maxima, local minima, or saddles.

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = xy - \frac{x^2}{2} + \frac{y^3}{3} - 2y$ are $f_x = y - x$ and $f_y = x + y^2 - 2$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = y - x = 0 \tag{1}$$
$$f_y = x + y^2 - 2 = 0 \tag{2}$$

Solving Equation (1) for y we get:

$$y = x \tag{3}$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned} x + y^2 - 2 &= 0 \\ x + (x)^2 - 2 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ \iff \quad x = -2 \quad &\text{or} \quad x = 1 \end{aligned}$$

We find the corresponding y-values using Equation (3): $y = x$.

- If $x = -2$, then $y = -2$.
- If $x = 1$, then $y = 1$.

Thus, the critical points are $\boxed{(-2,-2)}$ and $\boxed{(1,1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = -1, \quad f_{yy} = 2y, \quad f_{xy} = 1$$

The discriminant function $D(x,y)$ is then:

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x,y) &= (-1)(2y) - (1)^2 \\ D(x,y) &= -2y - 1 \end{aligned}$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(-2, -2)$	3	-1	Local Maximum
$(1, 1)$	-3	-1	Saddle Point

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local maximum of f if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$.

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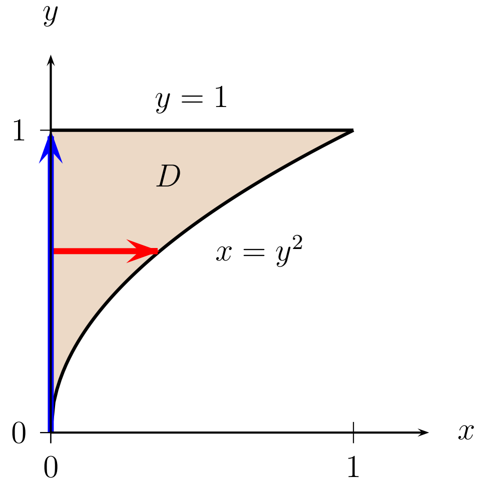
Problem 5 Solution

5. Compute the integral $\iint_D e^{y^3}\,dA$ where D is the region in the 1st quadrant of the xy plane bounded by the y-axis, the parabola $x = y^2$, and the line $y = 1$. (Hint: it makes a difference in which order you do this integral).

Solution:



From the figure we see that the region D is bounded on the left by $= 0$ and on the right by $x = y^2$. The projection of D onto the y-axis is the interval $0 \le y \le 1$. Using the order of integration $dx\,dy$ we have:

$$\begin{aligned}
\iint_D e^{y^3}\,dA &= \int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy \\
&= \int_0^1 e^{y^3}\Big[x\Big]_0^{y^2}\,dy \\
&= \int_0^1 y^2 e^{y^3}\,dy \\
&= \frac{1}{3}e^{y^3}\Big|_0^1 \\
&= \frac{1}{3}e^{1^3} - \frac{1}{3}e^{0^3} \\
&= \boxed{\frac{1}{3}(e-1)}
\end{aligned}$$

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Problem 6 Solution

6. (a) Let P be the parallelogram in the xy plane with vertices $A = (1, 1)$, $B = (2, 3)$, $C = (1, 4)$, and $D = (0, 2)$. Compute the area of P.

(b) Let C be the closed curve which is the boundary of the parallelogram P of part (a), traversed counterclockwise, i.e. it consists of the directed line segments AB, BC, CD, DA. Use Green's theorem to compute $\oint_C -y\,dx + x\,dy$.

Solution:

(a) The area of a parallelogram spanned by two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is, by definition:

$$A = ||\vec{\mathbf{u}} \times \vec{\mathbf{v}}||$$

Let $\vec{\mathbf{u}} = \overrightarrow{AB} = \langle 1, 2\rangle$ and $\vec{\mathbf{v}} = \overrightarrow{BC} = \langle -1, 1\rangle$. The cross product of these two vectors is $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle 0, 0, 3\rangle$. Thus, the area of the parallelogram is

$$A = ||\vec{\mathbf{u}} \times \vec{\mathbf{v}}|| = ||\langle 0, 0, 3\rangle|| = \boxed{3}$$

(b) Green's theorem states that:

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this problem we have $P = -y$ and $Q = x$ giving us:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

Thus, the line integral is:

$$\begin{aligned}\oint_C -y\,dx + x\,dy &= \iint_D 2\,dA \\ &= 2\iint_D 1\,dA \\ &= 2 \times (\text{Area of } D) \\ &= 2 \times 3 \\ &= \boxed{6}\end{aligned}$$

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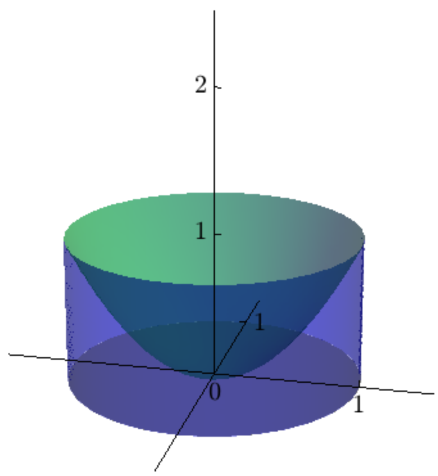
Problem 7 Solution

7. Let B be the region in $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$, the plane $z = 0$, and the cylinder $x^2 + y^2 = 1$. Draw a sketch of the region and compute the integral $\iiint_B x^2 \, dV$.

Solution: The region D is plotted below.



We use Cylindrical Coordinates to evaluate the triple integral. First, the integrand becomes:

$$f(x, y, z) = x^2$$
$$f(r, \theta, z) = (r \cos \theta)^2$$

Next, the equations for the paraboloid and cylinder are then:

$$\text{Paraboloid}: \ z = r^2$$
$$\text{Cylinder}: \ r = 1$$

The surface that bounds D from below is $z = 0$ (the xy-plane) and the surface that bounds D from above is $z = r^2$ (the paraboloid). The projection of the region D onto the xy-plane is the disk $0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi$.

Finally, using the fact that $dV = r \, dz \, dr \, d\theta$ in Cylindrical Coordinates, the value of the triple integral is:

$$
\begin{aligned}
\iiint_D x^2\,dV &= \int_0^{2\pi}\int_0^1\int_0^{r^2} (r\cos\theta)^2 r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1\int_0^{r^2} r^3\cos^2\theta\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\Big[z\Big]_0^{r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 r^5\cos^2\theta\,dr\,d\theta \\
&= \int_0^{2\pi}\cos^2\theta\left[\frac{1}{6}r^6\right]_0^1 d\theta \\
&= \frac{1}{6}\int_0^{2\pi}\cos^2\theta\,d\theta \\
&= \frac{1}{6}\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\
&= \boxed{\frac{\pi}{6}}
\end{aligned}
$$

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Problem 8 Solution

8. Consider the vector field $\overrightarrow{\mathbf{F}} = \langle x^2y, yz, z^3 \rangle$.

(a) Compute $\mathbf{Curl}(\overrightarrow{\mathbf{F}})$.

(b) Is $\overrightarrow{\mathbf{F}}$ a conservative vector field? Explain your answer.

(c) Compute $\mathbf{Div}(\overrightarrow{\mathbf{F}})$.

(d) Compute $\mathbf{Div}(\mathbf{Curl}(\overrightarrow{\mathbf{F}}))$.

Solution:

(a) The curl of the vector field is:

$$\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & yz & z^3 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}z^3 - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{i}} - \left(\frac{\partial}{\partial x}z^3 - \frac{\partial}{\partial z}x^2y\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}yz - \frac{\partial}{\partial y}x^2y\right)\hat{\mathbf{k}} \\
&= (0-y)\,\hat{\mathbf{i}} - (0-0)\,\hat{\mathbf{j}} + \left(0-x^2\right)\hat{\mathbf{k}} \\
&= \boxed{\left\langle -y, 0, -x^2 \right\rangle}
\end{aligned}$$

(b) Since $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \neq \overrightarrow{\mathbf{0}}$, the vector field is not conservative.

(c) The divergence of the vector field is:

$$\begin{aligned}
\overrightarrow{\nabla} \bullet \overrightarrow{\mathbf{F}} &= \frac{\partial}{\partial x}x^2y + \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}z^3 \\
&= \boxed{2xy + z + 3z^2}
\end{aligned}$$

(d) The divergence of the curl of the vector field is:

$$\begin{aligned}
\overrightarrow{\nabla} \bullet \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}}\right) &= \overrightarrow{\nabla} \bullet \left\langle -y, 0, -x^2 \right\rangle \\
&= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2) \\
&= 0 + 0 + 0 \\
&= \boxed{0}
\end{aligned}$$

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Problem 9 Solution

9. Let S be the surface which is the part of the plane $2x - 2y + z = 5$ above the square in the xy plane with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Compute the integral $\iint_S \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{S}}$ where $d\overrightarrow{\mathbf{S}}$ is the upward pointing normal and $\overrightarrow{\mathbf{F}} = \langle x, y, z \rangle$.

Solution: The formula we will use to compute the surface integral of the vector field $\overrightarrow{\mathbf{F}}$ is:

$$\iint_S \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{S}} = \iint_R \overrightarrow{\mathbf{F}} \bullet \left(\overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v\right) dA$$

where the function $\overrightarrow{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ with domain R is a parameterization of the surface S and the vectors $\overrightarrow{\mathbf{T}}_u = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial u}$ and $\overrightarrow{\mathbf{T}}_v = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial v}$ are the tangent vectors.

We begin by finding a parameterization of the plane. Let $x = u$ and $y = v$. Then, $z = 5 - 2u + 2v$ using the equation for the plane. Thus, we have $\overrightarrow{\mathbf{r}}(u,v) = \langle u, v, 5 - 2u + 2v \rangle$. Furthermore, the domain R is the set of all points (u,v) satisfying $0 \le u \le 1$ and $0 \le v \le 1$. Therefore, a parameterization of S is:

$$\begin{aligned} &\overrightarrow{\mathbf{r}}(u,v) = \langle u, v, 5 - 2u + 2v \rangle\,, \\ &R = \left\{(u,v) \,\middle|\, 0 \le u \le 1,\ 0 \le v \le 1\right\} \end{aligned}$$

The tangent vectors $\overrightarrow{\mathbf{T}}_u$ and $\overrightarrow{\mathbf{T}}_v$ are then:

$$\begin{aligned} \overrightarrow{\mathbf{T}}_u &= \frac{\partial \overrightarrow{\mathbf{r}}}{\partial u} = \langle 1, 0, -2 \rangle \\ \overrightarrow{\mathbf{T}}_v &= \frac{\partial \overrightarrow{\mathbf{r}}}{\partial v} = \langle 0, 1, 2 \rangle \end{aligned}$$

The cross product of these vectors is:

$$\begin{aligned} \overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{vmatrix} \\ &= \langle 2, -2, 1 \rangle \end{aligned}$$

The vector field $\overrightarrow{\mathbf{F}} = \langle x, y, z \rangle$ written in terms of u and v is:

$$\begin{aligned} \overrightarrow{\mathbf{F}} &= \langle x, y, z \rangle \\ \overrightarrow{\mathbf{F}} &= \langle u, v, 5 - 2u + 2v \rangle \end{aligned}$$

Before computing the surface integral, we note that $\overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v$ points upward, as desired, since the third component of the vector is positive.

The value of the surface integral is:

$$\begin{aligned}
\iint_S \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{S}} &= \iint_R \overrightarrow{\mathbf{F}} \bullet \left(\overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v\right)\, dA \\
&= \iint_R \langle u, v, 5 - 2u + 2v\rangle \bullet \langle 2, -2, 1\rangle\, dA \\
&= \iint_R (2u - 2v + 5 - 2u + 2v)\, dA \\
&= \iint_R 5\, dA \\
&= 5 \iint_R 1\, dA \\
&= 5 \times (\text{Area of } R) \\
&= 5 \times 1 \\
&= \boxed{5}
\end{aligned}$$