Math 210, Final Exam, Fall 2010
Problem 1 Solution

1. Let $\vec{\mathbf{u}} = \langle 1, -1, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 2, 1, 3 \rangle$.

   (a) Is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ acute, obtuse, or right?

   (b) Find an equation for the plane through $(1, -1, 2)$ containing $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Solution:

(a) The cosine of the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is:

$$\cos\theta = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||\ ||\vec{\mathbf{v}}||}$$

Since the magnitudes of the vectors are positive, the sign of the dot product will determine whether the angle is acute, obtuse, or right. The dot product is:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \langle 1, -1, 0 \rangle \bullet \langle 2, 1, 3 \rangle = 1$$

Since the dot product is positive we know that $\cos\theta > 0$ and, thus, the angle is **acute**.

(b) A vector perpendicular to the plane is the cross product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ which both lie in the plane.

$$\begin{aligned}
\vec{\mathbf{n}} &= \vec{\mathbf{u}} \times \vec{\mathbf{v}} \\
\vec{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}}\,[(-1)(3) - (0)(1)] - \hat{\mathbf{j}}\,[(1)(3) - (0)(2)] + \hat{\mathbf{k}}\,[(1)(1) - (-1)(2)] \\
\vec{\mathbf{n}} &= -3\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}} \\
\vec{\mathbf{n}} &= \langle -3, -3, 3 \rangle
\end{aligned}$$

Using $(1, -1, 2)$ as a point on the plane, we have:

$$\boxed{-3(x-1) - 3(y+1) + 3(z-2) = 0}$$

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Problem 2 Solution

2. The curve $\overrightarrow{\mathbf{r}}(t) = \langle 2\sin(t), 2\cos(t), -t\rangle$ describes the movement of a particle in $\mathbb{R}^3$.

 (a) Find the velocity and the acceleration of the particle as a function of t.

 (b) Find the tangent line to the curve at time $t = \pi/4$.

 (c) Find the distance traveled between time $t = 0$ and $t = \pi$.

Solution:

(a) The velocity and acceleration vectors are:

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{r}}'(t) = \langle 2\cos(t), -2\sin(t), -1\rangle$$
$$\overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{v}}'(t) = \langle -2\sin(t), -2\cos(t), 0\rangle$$

(b) A vector equation for the line tangent to $\overrightarrow{\mathbf{r}}(t)$ at t_0 is:

$$\overrightarrow{\mathbf{L}}(t) = \overrightarrow{\mathbf{r}}(t_0) + \overrightarrow{\mathbf{r}}'(t_0)(t - t_0)$$

Applying this formula to our vectors $\overrightarrow{\mathbf{r}}(t)$ and $\overrightarrow{\mathbf{r}}'(t)$ at $t_0 = \pi/4$ we have:

$$\overrightarrow{\mathbf{L}}(t) = \overrightarrow{\mathbf{r}}\left(\frac{\pi}{4}\right) + \overrightarrow{\mathbf{r}}'\left(\frac{\pi}{4}\right)\left(t - \frac{\pi}{4}\right)$$
$$\overrightarrow{\mathbf{L}}(t) = \left\langle 2\sin\left(\frac{\pi}{4}\right), 2\cos\left(\frac{\pi}{4}\right), -\frac{\pi}{4}\right\rangle + \left\langle 2\cos\left(\frac{\pi}{4}\right), -2\sin\left(\frac{\pi}{4}\right), -1\right\rangle\left(t - \frac{\pi}{4}\right)$$
$$\overrightarrow{\mathbf{L}}(t) = \left\langle \sqrt{2}, \sqrt{2}, -\frac{\pi}{4}\right\rangle + \left\langle \sqrt{2}, -\sqrt{2}, -1\right\rangle\left(t - \frac{\pi}{4}\right)$$

(c) The distance traveled by the particle is:

$$\begin{aligned} L &= \int_0^\pi \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right| \, dt \\ &= \int_0^\pi \sqrt{(2\cos(t))^2 + (-2\sin(t))^2 + (-1)^2}\, dt \\ &= \int_0^\pi \sqrt{4\cos^2(t) + 4\sin^2(t) + 1}\, dt \\ &= \int_0^\pi \sqrt{4+1}\, dt \\ &= \int_0^\pi \sqrt{5}\, dt \\ &= \boxed{\pi\sqrt{5}} \end{aligned}$$

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Problem 3 Solution

3. Use Green's Theorem to compute $\oint_C y\,dx + x^2 y\,dy$ where C traces the triangle with vertices $(0,0)$, $(2,0)$, $(1,1)$ traversed in this order.

Solution: Green's Theorem states that

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

where D is the region enclosed by C. The integrand of the double integral is:

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\, x^2 y - \frac{\partial}{\partial y}\, y \\ &= 2xy - 1 \end{aligned}$$

Thus, the value of the integral is:

$$\begin{aligned} \oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \\ &= \iint_D (2xy - 1)\, dA \\ &= \int_0^1 \int_y^{-y+2} (2xy - 1)\, dx\, dy \\ &= \int_0^1 \Big[x^2 y - x \Big]_y^{-y+2} dy \\ &= \int_0^1 \Big[\Big((-y+2)^2 y - (-y+2) \Big) - \Big((y)^2 y - y \Big) \Big] dy \\ &= \int_0^1 \Big(y^3 - 4y^2 + 4y + y - 2 - y^3 + y \Big) dy \\ &= \int_0^1 \Big(-4y^2 + 6y - 2 \Big) dy \\ &= \left[-\frac{4}{3} y^3 + 3y^2 - 2y \right]_0^1 \\ &= -\frac{4}{3} + 3 - 2 \\ &= \boxed{-\frac{1}{3}} \end{aligned}$$

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Problem 4 Solution

4. Find the critical points of $z = x^3 + x^2 + y^2 - 2xy - 12x$ and use the second derivative test to classify them as local maxima, local minima or saddles.

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = x^3 + x^2 + y^2 - 2xy - 12x$ are $f_x = 3x^2 + 2x - 2y - 12$ and $f_y = 2y - 2x$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 3x^2 + 2x - 2y - 12 = 0 \qquad (1)$$
$$f_y = 2y - 2x = 0 \qquad (2)$$

Solving Equation (2) for y we get:

$$y = x \qquad (3)$$

Substituting this into Equation (1) and solving for x we get:

$$\begin{aligned} 3x^2 + 2x - 2y - 12 &= 0 \\ 3x^2 + 2x - 2x - 12 &= 0 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ \iff \quad x = -2 \ \text{ or } \ x &= 2 \end{aligned}$$

We find the corresponding y-values using Equation (3): $y = x$.

- If $x = 2$, then $y = 2$.
- If $x = -2$, then $y = -2$.

Thus, the critical points are $\boxed{(2, 2)}$ and $\boxed{(-2, -2)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 6x + 2, \quad f_{yy} = 2, \quad f_{xy} = -2$$

The discriminant function $D(x, y)$ is then:

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - f_{xy}^2 \\ D(x, y) &= (6x + 2)(2) - (-2)^2 \\ D(x, y) &= 12x \end{aligned}$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(2,2)$	24	14	Local Minimum
$(-2,-2)$	-24	-10	Saddle Point

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

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Problem 5 Solution

5. Consider the vector field $\overrightarrow{\mathbf{F}} = \langle cx^2y^2 - e^y, 2x^3y - xe^y \rangle$ on $\mathbb{R}^2$ where c is a constant.

(a) Find the value for c that makes $\overrightarrow{\mathbf{F}}$ a conservative vector field.

(b) With c as in (a) find a function $\phi(x, y)$ so that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\phi$.

Solution:

(a) In order for the vector field $\overrightarrow{\mathbf{F}} = \langle f(x, y), g(x, y) \rangle$ to be conservative, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x, y) = cx^2y^2 - e^y$ and $g(x, y) = 2x^3y - xe^y$ we get:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial x} \\ 2cx^2y - e^y &= 6x^2y - e^y \\ 2cx^2y &= 6x^2y \\ c &= 3 \end{aligned}$$

(b) If $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x, y) \quad (1)$$

$$\frac{\partial \varphi}{\partial y} = g(x, y) \quad (2)$$

Using $f(x, y) = 3x^2y^2 - e^y$ and integrating both sides of Equation (1) with respect to x we get:

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= f(x, y) \\ \frac{\partial \varphi}{\partial x} &= 3x^2y^2 - e^y \\ \int \frac{\partial \varphi}{\partial x}\, dx &= \int \left(3x^2y^2 - e^y\right) dx \\ \varphi(x, y) &= x^3y^2 - xe^y + h(y) \quad (3) \end{aligned}$$

We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = 2x^3y - xe^y$ we get the equation:

$$\begin{aligned} \frac{\partial \varphi}{\partial y} &= g(x, y) \\ \frac{\partial \varphi}{\partial y} &= 2x^3y - xe^y \end{aligned}$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\begin{aligned}\frac{\partial}{\partial y}\left(x^3y^2 - xe^y + h(y)\right) &= 2x^3y - xe^y \\ 2x^3y - xe^y + h'(y) &= 2x^3y - xe^y \\ h'(y) &= 0\end{aligned}$$

which implies that $h(y) = 0$. Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\varphi(x, y) = x^3y^2 - xe^y}$$

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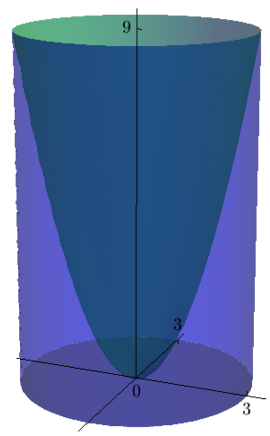
Problem 6 Solution

6. Compute the volume of the region in $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 9$, and the plane $z = 0$.

Solution: The region R is plotted below.



The volume can be computed using either a double or a triple integral. The double integral formula for computing the volume of a region R bounded above by the surface $z = f(x, y)$ and below by the surface $z = g(x, y)$ with projection D onto the xy-plane is:

$$V = \iint_D (f(x, y) - g(x, y))\, dA$$

In this case, the top surface is $z = x^2 + y^2 = r^2$ in polar coordinates and the bottom surface is $z = 0$. The projection of R onto the xy-plane is a disk of radius 3, described in polar coordinates as $D = \{(r, \theta) : 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$. Thus, the volume formula is:

$$V = \int_0^{2\pi} \int_0^3 \left(r^2 - 0\right) r\, dr\, d\theta \tag{1}$$

The triple integral formula for computing the volume of R is:

$$V = \iint_D \left(\int_{g(x,y)}^{f(x,y)} 1\, dz \right) dA$$

Using cylindrical coordinates we have:

$$V = \int_0^{2\pi} \int_0^3 \int_0^{r^2} 1\, r\, dz\, dr\, d\theta \tag{2}$$

Evaluating Equation (1) we get:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \left(r^2 - 0\right) \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^3 \, d\theta \\ &= \int_0^{2\pi} \frac{81}{4} \, d\theta \\ &= \left[\frac{81}{4}\right]_0^{2\pi} \\ &= \boxed{\frac{81\pi}{2}} \end{aligned}$$

Note that Equation (2) will evaluate to the same answer.

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Problem 7 Solution

7. Given the function $f(x,y) = xy^2 + y\cos(x)$ find:

(a) the gradient $\vec{\nabla} f$ at the point $P = (0,1)$.

(b) the directional derivative $D_{\mathbf{v}} f(0,1)$, where $\vec{\mathbf{f}}$ is the unit vector from $P = (0,1)$ *towards* $Q = (2,3)$.

Solution:

(a) The gradient of f is:

$$\begin{aligned}\vec{\nabla} f &= \langle f_x, f_y \rangle \\ &= \left\langle y^2 - y\sin(x), 2xy + \cos(x) \right\rangle\end{aligned}$$

At the point $P = (0,1)$ we have:

$$\begin{aligned}\vec{\nabla} f(0,1) &= \left\langle 1^2 - (1)\sin(0), 2(0)(1) + \cos(0) \right\rangle \\ &= \boxed{\langle 1, 1 \rangle}\end{aligned}$$

(b) The unit vector $\vec{\mathbf{v}}$ that points from $P = (0,1)$ *towards* $Q = (2,3)$ is:

$$\begin{aligned}\vec{\mathbf{v}} &= \frac{\overrightarrow{PQ}}{\left|\left|\overrightarrow{PQ}\right|\right|} \\ &= \frac{\langle 2, 2 \rangle}{||\langle 2, 2 \rangle||} \\ &= \frac{\langle 2, 2 \rangle}{2\sqrt{2}} \\ &= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle\end{aligned}$$

Thus, the directional derivative $D_{\mathbf{v}} f(0,1)$ is:

$$\begin{aligned}D_{\mathbf{v}} f(0,1) &= \vec{\nabla} f(0,1) \bullet \vec{\mathbf{v}} \\ &= \langle 1, 1 \rangle \bullet \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \boxed{\sqrt{2}}\end{aligned}$$

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Problem 8 Solution

8. Use the method of Lagrange multipliers to find points where $f(x, y) = xy$ attains its maximum and minimum subject to the constraint: $x^2 + 4y^2 = 2$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + 4y^2 = 2$ is compact which guarantees the existence of absolute extrema of f. Then let $g(x, y) = x^2 + 4y^2$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 2$$

which, when applied to our functions f and g, give us:

$$y = \lambda (2x) \tag{1}$$
$$x = \lambda (8y) \tag{2}$$
$$x^2 + 4y^2 = 2 \tag{3}$$

Dividing Equation (1) by Equation (2) gives us:

$$\begin{aligned} \frac{y}{x} &= \frac{\lambda(2x)}{\lambda(8y)} \\ \frac{y}{x} &= \frac{x}{4y} \\ 4y^2 &= x^2 \end{aligned}$$

Combining this result with Equation (3) and solving for x gives us:

$$\begin{aligned} x^2 + 4y^2 &= 2 \\ x^2 + x^2 &= 2 \\ 2x^2 &= 2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

When $x = 1$ we have:

$$\begin{aligned} 4y^2 &= x^2 \\ 4y^2 &= 1^2 \\ y^2 &= \frac{1}{4} \\ y &= \pm\frac{1}{2} \end{aligned}$$

We obtain the same values of y when $x = -1$. Therefore, the points of interest are $(1, \frac{1}{2})$, $(1, -\frac{1}{2})$, $(1, -\frac{1}{2})$, and $(-1, -\frac{1}{2})$.

We now evaluate $f(x, y) = xy$ at each point of interest.

$$\begin{aligned} f(1, \tfrac{1}{2}) &= (1)(\tfrac{1}{2}) = \tfrac{1}{2} \\ f(1, -\tfrac{1}{2}) &= (1)(-\tfrac{1}{2}) = -\tfrac{1}{2} \\ f(-1, \tfrac{1}{2}) &= (-1)(\tfrac{1}{2}) = -\tfrac{1}{2} \\ f(-1, -\tfrac{1}{2}) &= (-1)(-\tfrac{1}{2}) = \tfrac{1}{2} \end{aligned}$$

From the values above we observe that f attains an absolute maximum of $\frac{1}{2}$ and an absolute minimum of $-\frac{1}{2}$.

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Problem 9 Solution

9. Given the function $f(x,y) = xe^{xy}$ compute the partial derivatives:

$$\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial y}, \qquad \frac{\partial^2 f}{\partial x \partial x}, \qquad \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial^2 f}{\partial y \partial y}$$

Solution: The first partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy} = e^{xy}(1+xy)$$
$$\frac{\partial f}{\partial y} = x^2e^{xy}$$

The second derivatives are:

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(e^{xy}(1+xy)\right) = ye^{xy}(1+xy) + ye^{xy}$$
$$\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(x^2e^{xy}\right) = x^3e^{xy}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2e^{xy}\right) = 2xe^{xy} + x^2ye^{xy}$$