Math 210, Final Exam, Practice Fall 2009
Problem 1 Solution

1. A triangle has vertices at the points

$$A = (1, 1, 1),\ B = (1, -3, 4),\ \text{and } C = (2, -1, 3)$$

(a) Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Find an equation of the plane containing the triangle.

Solution:

(a) By definition, the angle between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is:

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\left|\overrightarrow{AB}\right|\right| \left|\left|\overrightarrow{AC}\right|\right|}$$

The vectors are $\overrightarrow{AB} = \langle 0, -4, 3\rangle$ and $\overrightarrow{AC} = \langle 1, -2, 2\rangle$. Thus, the cosine of the angle between them is:

$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{AB} \bullet \overrightarrow{BC}}{\left|\left|\overrightarrow{AB}\right|\right| \left|\left|\overrightarrow{BC}\right|\right|} \\
&= \frac{\langle 0, -4, 3\rangle \bullet \langle 1, -2, 2\rangle}{||\langle 0, -4, 3\rangle||\ ||\langle 1, 2, -1\rangle||} \\
&= \frac{(0)(1) + (-4)(-2) + (3)(2)}{\sqrt{0^2 + (-4)^2 + 3^2}\sqrt{1^2 + (-2)^2 + 2^2}} \\
&= \boxed{\frac{14}{15}}
\end{aligned}$$

(b) A vector perpendicular to the plane is the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ which both lie in the plane.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{AB} \times \overrightarrow{AC} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} -4 & 3 \\ -2 & 2 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 0 & -4 \\ 1 & -2 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(-4)(2) - (3)(-2)] - \hat{\mathbf{j}}\,[(0)(2) - (3)(1)] + \hat{\mathbf{k}}\,[(0)(-2) - (-4)(1)] \\
\overrightarrow{\mathbf{n}} &= -2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle -2, 3, 4\rangle
\end{aligned}$$

Using $A = (1, 1, 1)$ as a point on the plane, we have:

$$\boxed{-2(x-1) + 3(y-1) - 4(z-1) = 0}$$

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Problem 2 Solution

2. Find the critical points of the function $f(x,y) = x^2 + y^2 + x^2y + 1$ and classify each point as corresponding to either a saddle point, a local minimum, or a local maximum.

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = x^2 + y^2 + x^2y + 1$ are $f_x = 2x + 2xy$ and $f_y = 2y + x^2$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 2x + 2xy = 0 \tag{1}$$
$$f_y = 2y + x^2 = 0 \tag{2}$$

Factoring Equation (1) gives us:

$$\begin{aligned} 2x + 2xy &= 0 \\ 2x(1+y) &= 0 \\ x = 0, \text{ or } y &= -1 \end{aligned}$$

If $x = 0$ then Equation (2) gives us $y = 0$. If $y = -1$ then Equation (2) gives us:

$$\begin{aligned} 2(-1) + x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

Thus, the critical points are $\boxed{(0,0)}$, $\boxed{(\sqrt{2},-1)}$, and $\boxed{(-\sqrt{2},-1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = 2x$$

The discriminant function $D(x,y)$ is then:

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x,y) &= (2+2y)(2) - (2x)^2 \\ D(x,y) &= 4 + 4y - 4x^2 \end{aligned}$$

The values of $D(x,y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(0,0)$	4	2	Local Minimum
$(\sqrt{2},-1)$	-8	0	Saddle Point
$(-\sqrt{2},-1)$	-8	0	Saddle Point

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

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Problem 3 Solution

3. Find the directional derivative of the function $f(x,y) = e^x \sin(xy)$ at the point $(0,\pi)$ in the direction of $\overrightarrow{\mathbf{v}} = \langle 1,0\rangle$. In the direction of which unit vector is f increasing most rapidly at the point $(0,\pi)$?

Solution: By definition, the directional derivative of f at (a,b) in the direction of $\hat{\mathbf{u}}$ is:

$$D_{\mathbf{u}} f(a,b) = \overrightarrow{\nabla} f(a,b) \bullet \hat{\mathbf{u}}$$

The gradient of $f(x,y) = e^x \sin(xy)$ is:

$$\begin{aligned}\overrightarrow{\nabla} f &= \langle f_x, f_y\rangle \\ \overrightarrow{\nabla} f &= \langle e^x \sin(xy) + ye^x \cos(xy), xe^x \cos(xy)\rangle\end{aligned}$$

At the point $(0,\pi)$ we have:

$$\begin{aligned}\overrightarrow{\nabla} f(0,\pi) &= \left\langle e^0 \sin(0\cdot\pi) + \pi e^0 \cos(0\cdot\pi), 0\cdot e^0 \cos(0\cdot\pi)\right\rangle \\ \overrightarrow{\nabla} f(0,\pi) &= \langle \pi, 0\rangle\end{aligned}$$

The vector $\overrightarrow{\mathbf{v}} = \langle 1,0\rangle$ is already a unit vector. Therefore, the directional derivative is:

$$\begin{aligned}D_{\mathbf{v}} f(0,\pi) &= \overrightarrow{\nabla} f(0,\pi) \bullet \overrightarrow{\mathbf{v}} \\ &= \langle \pi, 0\rangle \bullet \langle 1, 0\rangle \\ &= \boxed{\pi}\end{aligned}$$

The direction of steepest ascent is:

$$\begin{aligned}\hat{\mathbf{u}} &= \frac{1}{\left\|\overrightarrow{\nabla} f(0,\pi)\right\|} \overrightarrow{\nabla} f(0,\pi) \\ &= \frac{1}{\pi} \langle \pi, 0\rangle \\ &= \boxed{\langle 1, 0\rangle}\end{aligned}$$

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Problem 4 Solution

4. Consider a space curve whose parameterization is given by:

$$\vec{\mathbf{r}}(t) = \left\langle \cos(\pi t), t^2, 1 \right\rangle$$

Find the unit tangent vector and curvature when $t = 2$.

Solution: The first two derivatives of $\vec{\mathbf{r}}(t)$ are:

$$\begin{aligned}
\vec{\mathbf{r}}'(t) &= \langle -\pi \sin(\pi t), 2t, 0 \rangle \\
\vec{\mathbf{r}}''(t) &= \left\langle -\pi^2 \cos(\pi t), 2, 0 \right\rangle
\end{aligned}$$

The unit tangent vector at $t = 2$ is:

$$\begin{aligned}
\vec{\mathbf{T}}(2) &= \frac{\vec{\mathbf{r}}'(2)}{\left|\left|\vec{\mathbf{r}}'(2)\right|\right|} \\
&= \frac{\langle -\pi \sin(2\pi), 2(2), 0 \rangle}{||\langle -\pi \sin(2\pi), 2(2), 0 \rangle||} \\
&= \frac{\langle 0, 4, 0 \rangle}{||\langle 0, 4, 0 \rangle||} \\
&= \frac{\langle 0, 4, 0 \rangle}{4} \\
&= \boxed{\langle 0, 1, 0 \rangle}
\end{aligned}$$

The curvature at $t = 2$ is:

$$\begin{aligned}
\kappa(2) &= \frac{\left|\left|\vec{\mathbf{r}}'(2) \times \vec{\mathbf{r}}''(2)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(2)\right|\right|^3} \\
&= \frac{||\langle -\pi \sin(2\pi), 4, 0 \rangle \times \langle -\pi^2 \cos(2\pi), 2, 0 \rangle||}{||\langle -\pi \sin(2\pi), 4, 0 \rangle||^3} \\
&= \frac{||\langle 0, 4, 0 \rangle \times \langle -\pi^2, 2, 0 \rangle||}{||\langle 0, 4, 0 \rangle||^3} \\
&= \frac{||\langle 0, 0, 4\pi^2 \rangle||}{4^3} \\
&= \frac{4\pi^2}{64} \\
&= \boxed{\frac{\pi^2}{16}}
\end{aligned}$$

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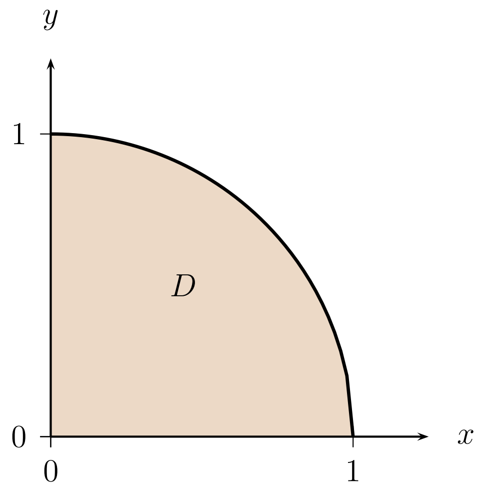
Problem 5 Solution

5. Evaluate $\iint_D e^{-(x^2+y^2)}\,dA$ where $D = \{(x,y)\,:\,x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

Solution:



From the figure we see that the region D is bounded above by $y = \sqrt{1-x^2}$ and below by $y = 0$. The projection of D onto the x-axis is the interval $0 \le x \le 1$. Since the region is a quarter-disk of radius 1, we will use polar coordinates to evaluate the integral. The region D is described in polar coordinates as $D = \{(r,\theta)\,:\,0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}$. The value of the integral is then:

$$
\begin{aligned}
\iint_D e^{-(x^2+y^2)}\,dA &= \int_0^{\pi/2}\int_0^1 e^{-r^2} r\,dr\,d\theta \\
&= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^1 d\theta \\
&= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-1^2} + \frac{1}{2}e^{-0^2}\right] d\theta \\
&= \int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}e^{-1}\right) d\theta \\
&= \left(\frac{1}{2} - \frac{1}{2}e^{-1}\right)\Big[\theta\Big]_0^{\pi/2} \\
&= \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{2}e^{-1}\right) \\
&= \boxed{\frac{\pi}{4}\left(1 - e^{-1}\right)}
\end{aligned}
$$

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Problem 6 Solution

6. Evaluate $\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}}$ where $\overrightarrow{\mathbf{F}} = \langle y+z, z+x, x+y \rangle$ and C is the line segment from $(1,1,0)$ to $(2,0,-1)$.

Solution: We note that the vector field $\overrightarrow{\mathbf{F}}$ is conservative. Letting $f = y+z$, $g = z+x$, and $h = x+y$ we have:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 1$$
$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} = 1$$
$$\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} = 1$$

By inspection, a potential function for the vector field is:

$$\varphi(x,y,z) = xy + xz + yz$$

Using the Fundamental Theorem of Line Integrals, the value of the line integral is:

$$\begin{aligned}\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \varphi(2,0,-1) - \varphi(1,1,0) \\ &= [(2)(0) + (2)(-1) + (0)(-1)] - [(1)(1) + (1)(0) + (1)(0)] \\ &= \boxed{-3}\end{aligned}$$

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Problem 7 Solution

7. Consider the paraboloid $z = 4 - x^2 - y^2$.

(a) Find an equation for the tangent plane to the paraboloid at the point $(1, 2, -1)$.

(b) Find the volume that is bounded by the paraboloid and the plane $z = 0$.

Solution:

(a) We use the following formula for the equation for the tangent plane:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

because the equation for the surface is given in **explicit** form. The partial derivatives of $f(x,y) = 4 - x^2 - y^2$ are:

$$f_x = -2x, \qquad f_y = -2y$$

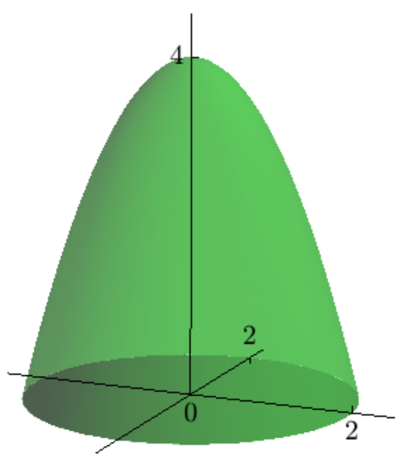
Evaluating these derivatives at $(1, 2)$ we get:

$$f_x(1,2) = -2, \qquad f_y(1,2) = -4$$

Thus, the tangent plane equation is:

$$\boxed{z = -1 - 2(x-1) - 4(y-2)}$$

(b) The region of integration is shown below.



The volume of the region can be obtained using either a double or a triple integral. In either case, we must be able to visualize the projection of the region onto the xy-plane. This region is the disk $D = \{(x, y) : x^2 + y^2 \leq 4\}$, the boundary of which is the intersection of the paraboloid $z = 4 - x^2 - y^2$ and the plane $z = 0$.

The double integral representing the volume is:

$$\text{Volume} = \iint_D (\text{top surface} - \text{bottom surface})\, dA$$

We will use polar coordinates to set up and evaluate the double integral. The top surface is then $z = 4 - r^2$ and the bottom surface is $z = 0$. The region D described in polar coordinates is $D = \{(r, \theta) : 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$. Thus, the volume is:

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^2 \left(4 - r^2 - 0\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(4r - r^3\right) dr\, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta \\
&= \int_0^{2\pi} 4\, d\theta \\
&= 4\theta\Big|_0^{2\pi} \\
&= \boxed{8\pi}
\end{aligned}$$

The triple integral representing the volume is:

$$\text{Volume} = \iiint_R 1\, dV$$

Using cylindrical coordinates we have:

$$\text{Volume} = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 1\, r\, dz\, dr\, d\theta$$

which evaluates to 8π.

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Problem 8 Solution

8. Let B be a constant and consider the vector field defined by:

$$\vec{\mathbf{F}} = \left\langle B\,xy + 1, x^2 + 2y \right\rangle$$

(a) For what value of B can we write $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ for some scalar function φ? Find such a function φ in this case.

(b) Using the value of B you found in part (a), evaluate the line integral of $\vec{\mathbf{F}}$ along any curve from $(1, 0)$ to $(-1, 0)$.

Solution:

(a) In order for the vector field $\vec{\mathbf{F}} = \langle f(x, y), g(x, y)\rangle$ to be conservative, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x, y) = B\,xy + 1$ and $g(x, y) = x^2 + 2y$ we get:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial x} \\ Bx &= 2x \\ B &= 2 \end{aligned}$$

If $\vec{\mathbf{F}} = \vec{\nabla}\varphi$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x, y) \tag{1}$$

$$\frac{\partial \varphi}{\partial y} = g(x, y) \tag{2}$$

Using $f(x, y) = 2xy + 1$ and integrating both sides of Equation (1) with respect to x we get:

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= f(x, y) \\ \frac{\partial \varphi}{\partial x} &= 2xy + 1 \\ \int \frac{\partial \varphi}{\partial x}\,dx &= \int (2xy + 1)\,dx \end{aligned}$$

$$\varphi(x, y) = x^2y + x + h(y) \tag{3}$$

We obtain the function $h(y)$ using Equation (2). Using $g(x,y) = x^2 + 2y$ we get the equation:

$$\begin{aligned} \frac{\partial \varphi}{\partial y} &= g(x,y) \\ \frac{\partial \varphi}{\partial y} &= x^2 + 2y \end{aligned}$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\begin{aligned} \frac{\partial}{\partial y}\left(x^2 y + x + h(y)\right) &= x^2 + 2y \\ x^2 + h'(y) &= x^2 + 2y \\ h'(y) &= 2y \end{aligned}$$

Now integrate both sides with respect to y to get:

$$\begin{aligned} \int h'(y)\,dy &= \int 2y\,dy \\ h(y) &= y^2 + C \end{aligned}$$

Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\varphi(x,y) = x^2 y + x + y^2}$$

(b) Using the Fundamental Theorem of Line Integrals, the value of the line integral is:

$$\begin{aligned} \int_C \overrightarrow{\mathbf{F}} \bullet d\,\overrightarrow{\mathbf{s}} &= \varphi(-1,0) - \varphi(1,0) \\ &= \left[(-1)^2(0) + (-1) + 0^2\right] - \left[(1)^2(0) + 1 + 0^2\right] \\ &= \boxed{-2} \end{aligned}$$

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Problem 9 Solution

9. Consider $f(x, y) = x\sin(x + 2y)$.

(a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, and f_{yy}.

(b) If $x = s^2 + t$ and $y = 2s + t^2$, compute the partials f_s and f_t.

Solution:

(a) The first and second partial derivatives are:

$$\begin{aligned} f_x &= \sin(x + 2y) + x\cos(x + 2y) \\ f_y &= 2x\cos(x + 2y) \\ f_{xx} &= \cos(x + 2y) + \cos(x + 2y) - x\sin(x + 2y) \\ f_{xy} &= 2\cos(x + 2y) - 2x\sin(x + 2y) \\ f_{yy} &= -4x\sin(x + 2y) \end{aligned}$$

(b) Using the Chain Rule, the partial derivatives f_s and f_t are:

$$\begin{aligned} f_s &= f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} \\ &= [\sin(x + 2y) + x\cos(x + 2y)]\,(2s) + [2x\cos(x + 2y)]\,(2) \\ &= \left[\sin(s^2 + t + 2(2s + t^2)) + (s^2 + t)\cos(s^2 + t + 2(2s + t^2))\right](2s) + \\ &\quad \left[2(s^2 + t)\cos(s^2 + t + 2(2s + t^2))\right](2) \\ f_t &= f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} \\ &= [\sin(x + 2y) + x\cos(x + 2y)]\,(1) + [2x\cos(x + 2y)]\,(2t) \\ &= \left[\sin(s^2 + t + 2(2s + t^2)) + (s^2 + t)\cos(s^2 + t + 2(2s + t^2))\right] + \\ &\quad \left[2(s^2 + t)\cos(s^2 + t + 2(2s + t^2))\right](2t) \end{aligned}$$

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Problem 10 Solution

10. Find the points on the ellipse $x^2 + xy + y^2 = 9$ where the distance from the origin is maximal and minimal. (Hint: Let $f(x, y) = x^2 + y^2$ be the function you want to extremize where (x, y) is a point on the ellipse.)

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + xy + y^2 = 9$ is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y) = x^2 + xy + y^2 = 9$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 9$$

which, when applied to our functions f and g, give us:

$$2x = \lambda (2x + y) \tag{1}$$
$$2y = \lambda (x + 2y) \tag{2}$$
$$x^2 + xy + y^2 = 9 \tag{3}$$

We begin by diving Equation (1) by Equation (2) to give us:

$$\begin{aligned} \frac{2x}{2y} &= \frac{\lambda(2x + y)}{\lambda(x + 2y)} \\ \frac{x}{y} &= \frac{2x + y}{x + 2y} \\ x(x + 2y) &= y(2x + y) \\ x^2 + 2xy &= 2xy + y^2 \\ x^2 &= y^2 \\ x &= \pm y \end{aligned}$$

If $x = y$ then Equation (3) gives us:

$$\begin{aligned} (y)^2 + (y)y + y^2 &= 9 \\ y^2 + y^2 + y^2 &= 9 \\ 3y^2 &= 9 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$$

Since $x = y$ we have $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$ as points of interest.

If $x = -y$ then Equation (3) gives us:

$$\begin{aligned} (-y)^2 + (-y)y + y^2 &= 9 \\ y^2 - y^2 + y^2 &= 9 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

Since $x = -y$ we have $(3, -3)$ and $(-3, 3)$ as points of interest.

We now evaluate $f(x, y) = x^2 + y^2$ at each point of interest.

$$\begin{aligned} f(\sqrt{3}, \sqrt{3}) &= (\sqrt{3})^2 + (\sqrt{3})^2 = 6 \\ f(-\sqrt{3}, -\sqrt{3}) &= (-\sqrt{3})^2 + (-\sqrt{3})^2 = 6 \\ f(3, -3) &= 3^2 + (-3)^2 = 18 \\ f(-3, 3) &= (-3)^2 + 3^2 = 18 \end{aligned}$$

From the values above we observe that f attains an absolute maximum of 18 and an absolute minimum of 6.

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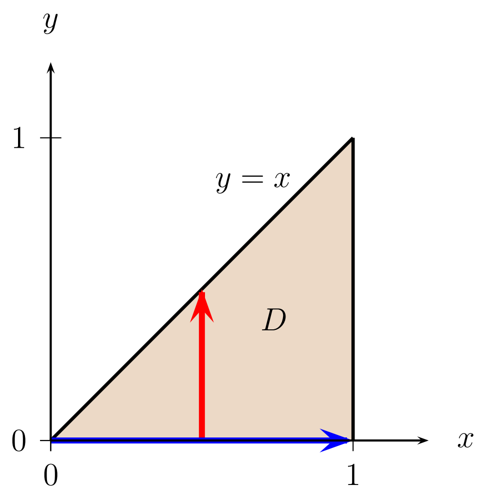
Problem 11 Solution

11. Sketch the region of integration for the integral below and evaluate the integral.

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

Solution:



From the figure we see that the region D is bounded above by $y = x$ and below by $y = 0$. The projection of D onto the x-axis is the interval $0 \le x \le 1$. Using the order of integration $dy\, dx$ we have:

$$\begin{aligned}
\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy &= \int_0^1 \int_0^x e^{-x^2}\, dy\, dx \\
&= \int_0^1 e^{-x^2} \Big[y\Big]_0^x\, dx \\
&= \int_0^1 xe^{-x^2}\, dx \\
&= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 \\
&= \left[-\frac{1}{2}e^{-1^2}\right] - \left[-\frac{1}{2}e^{-0^2}\right] \\
&= \boxed{\frac{1}{2} - \frac{1}{2}e^{-1}}
\end{aligned}$$

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Problem 12 Solution

12. Evaluate $\int_C f(x,y,z)\,ds$ where $f(x,y,z) = z\sqrt{x^2+y^2}$ and C is the helix $\vec{\mathbf{c}}(t) = (4\cos t, 4\sin t, 3t)$ for $0 \le t \le 2\pi$.

Solution: We use the following formula to evaluate the integral:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\left|\vec{\mathbf{c}}'(t)\right|\,dt$$

Using the fact that $x = 4\cos t$, $y = 4\sin t$, and $z = 3t$, the function $f(x,y,z)$ can be rewritten as:

$$\begin{aligned} f(x(t),y(t),z(t)) &= z(t)\sqrt{x(t)^2+y(t)^2} \\ &= (3t)\sqrt{(4\cos t)+(4\sin t)^2} \\ &= 3t\sqrt{16\cos^2 t + 16\sin^2 t} \\ &= 3t\cdot 4 \\ &= 12t \end{aligned}$$

The derivative $\vec{\mathbf{c}}'(t)$ and its magnitude are:

$$\begin{aligned} \vec{\mathbf{c}}'(t) &= \langle -4\sin t, 4\cos t, 3\rangle \\ \left|\vec{\mathbf{c}}'(t)\right| &= \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2} \\ &= 5 \end{aligned}$$

Therefore, the value of the line integral is:

$$\begin{aligned} \int_C f(x,y,z)\,ds &= \int_a^b f(x(t),y(t),z(t))\left|\vec{\mathbf{c}}'(t)\right|\,dt \\ &= \int_0^{2\pi} 12t\cdot 5\,dt \\ &= \int_0^{2\pi} 60t\,dt \\ &= \Big[30t^2\Big]_0^{2\pi} \\ &= \boxed{120\pi^2} \end{aligned}$$

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Problem 13 Solution

13. Consider the vectors $\vec{\mathbf{v}} = \langle 1, 2, a\rangle$ and $\vec{\mathbf{w}} = \langle 1, 1, 1\rangle$.

(a) Find the value of a such that $\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{w}}$.

(b) Find the two values of a such that the area of the parallelogram determined by $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ is equal to $\sqrt{6}$.

Solution:

(a) By definition, two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are perpendicular if and only if the dot product of the vectors is equal to zero.

$$\begin{aligned} \vec{\mathbf{v}} \bullet \vec{\mathbf{w}} &= 0 \\ \langle 1, 2, a\rangle \bullet \langle 1, 1, 1\rangle &= 0 \\ 1 + 2 + a &= 0 \\ &\boxed{a = -3} \end{aligned}$$

(b) By definition, the area of a parallelogram spanned by the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ is:

$$A = ||\vec{\mathbf{v}} \times \vec{\mathbf{w}}||$$

The cross product of $\vec{\mathbf{v}} = \langle 1, 2, a\rangle$ and $\vec{\mathbf{w}} = \langle 1, 1, 1\rangle$ is:

$$\begin{aligned} \vec{\mathbf{v}} \times \vec{\mathbf{w}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & a \\ 1 & 1 & 1 \end{vmatrix} \\ &= \langle 2 - a, a - 1, -1\rangle \end{aligned}$$

The area of the parallelogram is then:

$$\begin{aligned} A &= ||\vec{\mathbf{v}} \times \vec{\mathbf{w}}|| \\ &= ||\langle 2 - a, a - 1, -1\rangle|| \\ &= \sqrt{(2-a)^2 + (a-1)^2 + (-1)^2} \\ &= \sqrt{(a-2)^2 + (a-1)^2 + 1} \end{aligned}$$

In order for the area to be $\sqrt{6}$ it must be the case that:

$$\begin{aligned} \sqrt{(a-2)^2 + (a-1)^2 + 1} &= \sqrt{6} \\ (a-2)^2 + (a-1)^2 + 1 &= 6 \\ a^2 - 4a + 4 + a^2 - 2a + 1 + 1 &= 6 \\ 2a^2 - 6a + 6 &= 6 \\ 2a^2 - 6a &= 0 \\ 2a(a-3) &= 0 \\ a = 0 \quad \text{or} \quad a &= 3 \end{aligned}$$

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Problem 14 Solution

14. Consider a particle whose position vector is given by

$$\vec{\mathbf{r}}(t) = \left\langle \sin(\pi t), t^2, t+1 \right\rangle$$

(a) Find the velocity $\vec{\mathbf{r}}'(t)$ and the acceleration $\vec{\mathbf{r}}''(t)$.

(b) Set up the integral you would compute to find the distance traveled by the particle from $t = 0$ to $t = 4$. **Do not attempt to compute the integral.**

Solution:

(a) The velocity and acceleration vectors are:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle \pi \cos(\pi t), 2t, 1 \rangle$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \left\langle -\pi^2 \sin(\pi t), 2, 0 \right\rangle$$

(b) The distance traveled by the particle is:

$$\begin{aligned} L &= \int_0^4 \left|\left|\vec{\mathbf{r}}'(t)\right|\right| \, dt \\ &= \int_0^4 \sqrt{(\pi \cos(\pi t))^2 + (2t)^2 + 1^2}\, dt \\ &= \int_0^4 \sqrt{\pi^2 \cos^2(\pi t) + 4t^2 + 1}\, dt \end{aligned}$$

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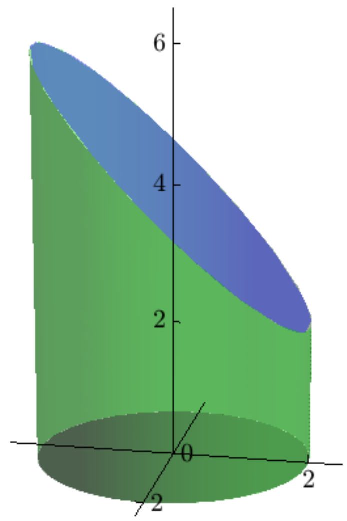
Problem 15 Solution

15. Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

Solution: The region R is plotted below.



The volume can be computed using either a double or a triple integral. The double integral formula for computing the volume of a region R bounded above by the surface $z = f(x, y)$ and below by the surface $z = g(x, y)$ with projection D onto the xy-plane is:

$$V = \iint_D (f(x, y) - g(x, y))\, dA$$

In this case, the top surface is $z = 4 - y = 4 - r \sin\theta$ in polar coordinates and the bottom surface is $z = 0$. The projection of R onto the xy-plane is a disk of radius 2, described in polar coordinates as $D = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. Thus, the volume formula is:

$$V = \int_0^{2\pi} \int_0^2 (4 - r \sin\theta - 0)\, r\, dr\, d\theta \tag{1}$$

The triple integral formula for computing the volume of R is:

$$V = \iint_D \left(\int_{g(x,y)}^{f(x,y)} 1\, dz \right) dA$$

Using cylindrical coordinates we have:

$$V = \int_0^{2\pi} \int_0^2 \int_0^{4 - r\sin\theta} 1\, r\, dz\, dr\, d\theta \tag{2}$$

Evaluating Equation (1) we get:

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (4 - r\sin\theta - 0)\; r\,dr\,d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{3} r^3 \sin\theta \right]_0^2 \, d\theta \\
&= \int_0^{2\pi} \left(8 - \frac{8}{3}\sin\theta \right) \, d\theta \\
&= \left[8\theta + \frac{8}{3}\cos\theta \right]_0^{2\pi} \\
&= \left[(8)(2\pi) + \frac{8}{3}\cos 2\pi \right] - \left[(8)(0) + \frac{8}{3}\cos 0 \right] \\
&= \boxed{16\pi}
\end{aligned}
$$

Note that Equation (2) will evaluate to the same answer.

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Problem 16 Solution

16. Use Green's Theorem to compute $\oint_C xy\,dx + y^5\,dy$ where C is the boundary of the triangle with vertices at $(0,0)$, $(2,0)$, $(2,1)$, oriented counterclockwise.

Solution: Green's Theorem states that

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

where D is the region enclosed by C. The integrand of the double integral is:

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x} y^5 - \frac{\partial}{\partial y} xy \\ &= 0 - x \\ &= -x \end{aligned}$$

Thus, the value of the integral is:

$$\begin{aligned} \oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \\ &= \iint_D (-x)\, dA \\ &= -\int_0^2 \int_0^{x/2} x\, dy\, dx \\ &= -\int_0^2 \Big[xy \Big]_0^{x/2} dx \\ &= -\int_0^2 \frac{1}{2} x^2\, dx \\ &= -\left[\frac{1}{6} x^3 \right]_0^2 \\ &= -\frac{1}{6}(2)^3 \\ &= \boxed{-\frac{4}{3}} \end{aligned}$$

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Problem 17 Solution

17. Consider the plane P containing the points $A = (1, 0, 0)$, $B = (2, 1, 1)$, and $C = (1, 0, 2)$.

(a) Find a unit vector perpendicular to P.

(b) Find the intersection of P with the line perpendicular to P that contains the point $D = (1, 1, 1)$.

Solution:

(a) A vector perpendicular to the plane is the cross product of $\overrightarrow{AB} = \langle 1, 1, 1\rangle$ and $\overrightarrow{BC} = \langle -1, -1, 1\rangle$ which both lie in the plane.

$$\begin{aligned}
\overrightarrow{\mathbf{n}} &= \overrightarrow{AB} \times \overrightarrow{BC} \\
\overrightarrow{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} \\
\overrightarrow{\mathbf{n}} &= \hat{\mathbf{i}}\,[(1)(1) - (1)(-1)] - \hat{\mathbf{j}}\,[(1)(1) - (1)(-1)] + \hat{\mathbf{k}}\,[(1)(-1) - (1)(-1)] \\
\overrightarrow{\mathbf{n}} &= 2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \\
\overrightarrow{\mathbf{n}} &= \langle 2, -2, 0\rangle
\end{aligned}$$

To make $\overrightarrow{\mathbf{n}}$ a unit vector we multiply by the reciprocal of its magnitude to get:

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{1}{||\overrightarrow{\mathbf{n}}||}\,\overrightarrow{\mathbf{n}} \\
&= \frac{1}{\sqrt{8}}\,\langle 2, -2, 0\rangle \\
&= \boxed{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle}
\end{aligned}$$

(b) To find the intersection of the plane P and the line perpendicular to P through $D = (1, 1, 1)$, we must form an equation for the plane and a set of parametric equations for the line. Using A as a point on the plane and the vector $\overrightarrow{\mathbf{n}} = \langle 2, -2, 0\rangle$ which is perpendicular to plane, we have:

$$2(x - 1) - 2(y - 0) - 0(z - 0) = 0$$

as an equation for the plane and:

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 1 - 0t$$

as a set of parametric equations for the line. Cleaning up the plane equation and substituting the parametric equations of the line for x, y, and z we get:

$$\begin{aligned} 2(x-1)-2(y-0)-0(z-0) &= 0 \\ 2x-2-2y &= 0 \\ 2x-2y &= 2 \\ x-y &= 1 \\ (1+2t)-(1-2t) &= 1 \\ 1+2t-1+2t &= 1 \\ 4t &= 1 \\ t &= \frac{1}{4} \end{aligned}$$

Substituting this value of t into the parametric equations for the line gives us:

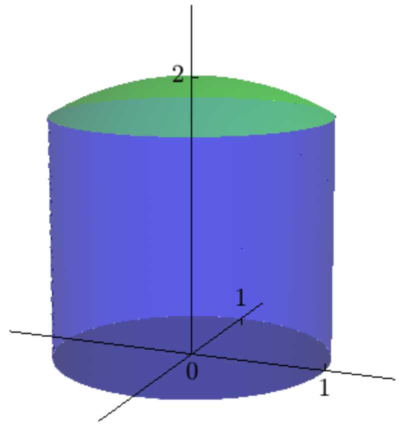
$$\begin{aligned} &x = 1+2t = 1+2\left(\frac{1}{4}\right) = \frac{3}{2} \\ &y = 1-2t = 1-2\left(\frac{1}{4}\right) = \frac{1}{2} \\ &z = 1 \end{aligned}$$

Thus, the point of intersection is $\boxed{\left(\frac{3}{2}, \frac{1}{2}, 1\right)}$.

Math 210, Final Exam, Practice Fall 2009
Problem 18 Solution

18. Use a triple integral to compute the volume of the region below the sphere $x^2+y^2+z^2=4$ and above the disk $x^2+y^2 \leq 1$ in the xy-plane.

Solution: The region of integration is shown below.



The equation for the sphere in cylindrical coordinates is $r^2+z^2=4 \implies z=\sqrt{4-r^2}$ since the region is above the xy-plane. Furthermore, the disk in the xy-plane is described by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ in cylindrical coordinates. Thus, the volume of the region is:

$$\begin{aligned} V &= \iiint_R 1\, dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{4-r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} \right]_0^1 d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}\left(4-1^2\right)^{3/2} + \frac{1}{3}\left(4-0^2\right)^{3/2} \right] d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\left(8-3\sqrt{3}\right) d\theta \\ &= \boxed{\frac{2\pi}{3}\left(8-3\sqrt{3}\right)} \end{aligned}$$

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Problem 19 Solution

19. Consider the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 4$.

(a) Write a parameterization $\Phi(u, v)$ for the cone, clearly indicating the domain of Φ.

(b) Find the surface area of the cone.

Solution:

(a) We begin by finding a parameterization of the paraboloid. Let $x = u\cos(v)$ and $y = u\sin(v)$, where we define u to be nonnegative. Then,

$$
\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
z &= \sqrt{(u\cos(v))^2 + (u\sin(v))^2} \\
z &= \sqrt{u^2\cos^2(v) + u^2\sin^2(v)} \\
z &= \sqrt{u^2} \\
z &= u
\end{aligned}
$$

Thus, we have $\vec{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), u\rangle$. To find the domain $\mathcal{R}$, we must determine the curve of intersection of the paraboloid and the plane $z = 4$. We do this by plugging $z = 4$ into the equation for the paraboloid to get:

$$
\begin{aligned}
\sqrt{x^2 + y^2} &= z \\
\sqrt{x^2 + y^2} &= 4 \\
x^2 + y^2 &= 16
\end{aligned}
$$

which describes a circle of radius 4. Thus, the domain $\mathcal{R}$ is the set of all points (x, y) satisfying $x^2 + y^2 \leq 4$. Using the fact that $x = u\cos(v)$ and $y = u\sin(v)$, this inequality becomes:

$$
\begin{aligned}
x^2 + y^2 &\leq 16 \\
(u\cos(v))^2 + (u\sin(v))^2 &\leq 16 \\
u^2 &\leq 16 \\
0 \leq u &\leq 4
\end{aligned}
$$

noting that, by definition, u must be nonnegative. The range of v-values is $0 \leq v \leq 2\pi$. Therefore, a parameterization of $\mathcal{S}$ is:

$$
\begin{aligned}
\vec{\mathbf{r}}(u, v) &= \langle u\cos(v), u\sin(v), u\rangle\,, \\
\mathcal{R} &= \left\{(u, v)\,\middle|\, 0 \leq u \leq 4,\ 0 \leq v \leq 2\pi\right\}
\end{aligned}
$$

(b) The formula for surface area we will use is:

$$S = \iint_{\mathcal{S}} dS = \iint_{\mathcal{R}} \left| \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v \right| dA$$

where the function $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ with domain $\mathcal{R}$ is a parameterization of the surface $\mathcal{S}$ and the vectors $\vec{\mathbf{t}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u}$ and $\vec{\mathbf{t}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v}$ are the tangent vectors.

The tangent vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ are then:

$$\begin{aligned}
\vec{\mathbf{t}}_u &= \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle \cos(v), \sin(v), 1 \rangle \\
\vec{\mathbf{t}}_v &= \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle -u\sin(v), u\cos(v), 0 \rangle
\end{aligned}$$

The cross product of these vectors is:

$$\begin{aligned}
\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= -u\cos(v)\,\hat{\mathbf{i}} - u\sin(v)\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}} \\
&= \langle -u\cos(v), -u\sin(v), u \rangle
\end{aligned}$$

The magnitude of the cross product is:

$$\begin{aligned}
\left| \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v \right| &= \sqrt{(-u\cos(v))^2 + (-u\sin(v))^2 + u^2} \\
&= \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} \\
&= \sqrt{u^2 + u^2} \\
&= u\sqrt{2}
\end{aligned}$$

We can now compute the surface area.

$$\begin{aligned}
S &= \iint_{\mathcal{R}} \left| \vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v \right| dA \\
&= \int_0^4 \int_0^{2\pi} u\sqrt{2}\, dv\, du \\
&= \int_0^4 \left[uv\sqrt{2} \right]_0^{2\pi} du \\
&= \int_0^4 2\pi\sqrt{2}u\, du \\
&= \left[\pi\sqrt{2}u^2 \right]_0^4 \\
&= \boxed{16\pi\sqrt{2}}
\end{aligned}$$

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Problem 20 Solution

20. Calculate $\int_C y\,dx + (x+z)\,dy + y\,dz$ along the curve given by $\vec{\mathbf{c}}(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.

Solution: We note that the vector field $\vec{\mathbf{F}}$ is conservative. Letting $f = y$, $g = x + z$, and $h = y$ we have:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 1$$
$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} = 0$$
$$\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} = 1$$

By inspection, a potential function for the vector field is:

$$\varphi(x, y, z) = xy + yz$$

Using the Fundamental Theorem of Line Integrals, the value of the line integral is:

$$\begin{aligned} \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}} &= \varphi(1,1,1) - \varphi(0,0,0) \\ &= [(1)(1) + (1)(1)] - [(0)(0) + (0)(0)] \\ &= \boxed{2} \end{aligned}$$

Note that the points $(1,1,1)$ and $(0,0,0)$ were obtained by plugging the endpoints of the interval $0 \le t \le 1$ into $\vec{\mathbf{c}}(t)$.