Math 210, Final Exam, Spring 2008
Problem 1 Solution

1. Consider the vector field $\overrightarrow{\mathbf{F}} = \langle y^2, 2xy + 2y \rangle$.

   (a) Show that $\overrightarrow{\mathbf{F}}$ is conservative.

   (b) Find a potential function φ such that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$.

   (c) Compute $\displaystyle\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}}$ along any path C from $(-1, 2)$ to $(3, 0)$.

Solution:

(a) In order for the vector field $\overrightarrow{\mathbf{F}} = \langle f(x, y), g(x, y) \rangle$ to be conservative, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x, y) = y^2$ and $g(x, y) = 2xy + 2y$ we get:

$$\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = 2y$$

Thus, the vector field is conservative.

(b) If $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x, y) \qquad (1)$$

$$\frac{\partial \varphi}{\partial y} = g(x, y) \qquad (2)$$

Using $f(x, y) = y^2$ and integrating both sides of Equation (1) with respect to x we get:

$$\frac{\partial \varphi}{\partial x} = f(x, y)$$

$$\frac{\partial \varphi}{\partial x} = y^2$$

$$\int \frac{\partial \varphi}{\partial x}\, dx = \int \left(y^2\right)\, dx$$

$$\varphi(x, y) = xy^2 + h(y) \qquad (3)$$

We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = 2xy + 2y$ we get the equation:

$$\frac{\partial \varphi}{\partial y} = g(x, y)$$

$$\frac{\partial \varphi}{\partial y} = 2xy + 2y$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\begin{aligned}\frac{\partial}{\partial y}\left(xy^2 + h(y)\right) &= 2xy + 2y\\ 2xy + h'(y) &= 2xy + 2y\\ h'(y) &= 2y\end{aligned}$$

Now integrate both sides with respect to y to get:

$$\begin{aligned}\int h'(y)\,dy &= \int 2y\,dy\\ h(y) &= y^2 + C\end{aligned}$$

Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\varphi(x, y) = xy^2 + y^2}$$

(c) Using the Fundamental Theorem of Line Integrals, we have:

$$\begin{aligned}\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \varphi(3, 0) - \varphi(-1, 2)\\ &= \left[3(0)^2 + 0^2\right] - \left[(-1)(2)^2 + 2^2\right]\\ &= \boxed{0}\end{aligned}$$

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Problem 2 Solution

2. Complete each of the following:

(a) Consider a particle whose position vector is given by:

$$\overrightarrow{\mathbf{r}}(t) = \left\langle \sin(\pi t), t^2, t+1 \right\rangle$$

Find the velocity, speed, and acceleration of the particle at $t = 2$.

(b) Find the directional derivative $D_{\mathbf{u}}f$ of the function $f(x, y) = e^{x+y}\sin(xy)$ at the point $(\pi, 1)$ in the direction of $\overrightarrow{\mathbf{v}} = \langle 4, 0 \rangle$.

Solution:

(a) The velocity, speed, and acceleration functions are:

$$\begin{aligned} \overrightarrow{\mathbf{v}}(t) &= \overrightarrow{\mathbf{r}}'(t) = \langle \pi\cos(\pi t), 2t, 1 \rangle \\ v(t) &= \left|\left|\overrightarrow{\mathbf{v}}(t)\right|\right| = \sqrt{\pi^2\cos^2(\pi t) + 4t^2 + 1} \\ \overrightarrow{\mathbf{a}}(t) &= \overrightarrow{\mathbf{v}}'(t) = \left\langle -\pi^2\sin(\pi t), 2, 0 \right\rangle \end{aligned}$$

At $t = 2$ we have:

$$\begin{aligned} \overrightarrow{\mathbf{v}}(2) &= \langle \pi, 4, 1 \rangle \\ v(t) &= \sqrt{\pi^2 + 17} \\ \overrightarrow{\mathbf{a}}(t) &= \langle 0, 2, 0 \rangle \end{aligned}$$

(b) The directional derivative of $f(x, y)$ at (a, b) in the direction of the unit vector $\hat{\mathbf{u}}$ is, by definition:

$$D_{\mathbf{u}}f(a, b) = \overrightarrow{\nabla} f(a, b) \bullet \hat{\mathbf{u}}$$

The gradient of $f(x, y)$ is:

$$\overrightarrow{\nabla} f = \langle f_x, f_y \rangle = \left\langle e^{x+y}(\sin(xy) + y\cos(xy)), e^{x+y}(\sin(xy) + x\cos(xy)) \right\rangle$$

At the point $(\pi, 1)$ we have:

$$\overrightarrow{\nabla} f(\pi, 1) = \left\langle -e^{\pi+1}, -\pi e^{\pi+1} \right\rangle$$

The unit vector $\hat{\mathbf{u}}$ in the direction of $\overrightarrow{\mathbf{v}} = \langle 4, 0 \rangle$ is:

$$\hat{\mathbf{u}} = \frac{1}{\left|\left|\overrightarrow{\mathbf{v}}\right|\right|}\overrightarrow{\mathbf{v}} = \frac{1}{4}\langle 4, 0 \rangle = \langle 1, 0 \rangle$$

Thus, the directional derivative is:

$$\begin{aligned} D_{\mathbf{u}}f(\pi, 1) &= \overrightarrow{\nabla} f(\pi, 1) \bullet \hat{\mathbf{u}} \\ &= \left\langle -e^{\pi+1}, -\pi e^{\pi+1} \right\rangle \bullet \langle 1, 0 \rangle \\ &= \boxed{-e^{\pi+1}} \end{aligned}$$

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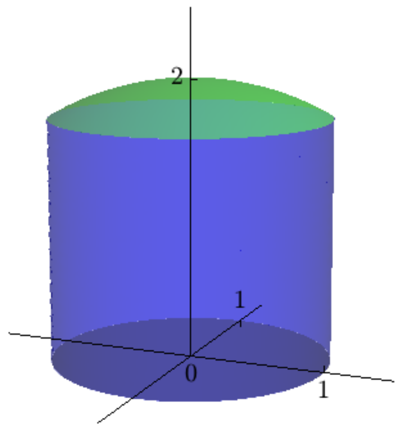
Problem 3 Solution

3. Use a triple integral to compute the volume of the region below the sphere $x^2+y^2+z^2=4$ and above the disk $x^2+y^2 \le 1$ in the xy-plane. (Hint: Use cylindrical coordinates.)

Solution: The region of integration is shown below.



The equation for the sphere in cylindrical coordinates is $r^2+z^2=4 \quad \Longrightarrow \quad z=\sqrt{4-r^2}$ since the region is above the xy-plane. Furthermore, the disk in the xy-plane is described by $0 \le r \le 1$, $0 \le \theta \le 2\pi$ in cylindrical coordinates. Thus, the volume of the region is:

$$\begin{aligned}
V &= \iiint_R 1\, dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 r\sqrt{4-r^2}\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} \right]_0^1 d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}\left(4-1^2\right)^{3/2} + \frac{1}{3}\left(4-0^2\right)^{3/2} \right] d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left(8-3\sqrt{3}\right) d\theta \\
&= \boxed{\frac{2\pi}{3}\left(8-3\sqrt{3}\right)}
\end{aligned}$$

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Problem 4 Solution

4. Find the critical points of the function $f(x, y) = x^2 + y^2 + x^2y + 1$ and classify each point as corresponding to either a saddle point, a local minimum, or a local maximum.

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = x^2 + y^2 + x^2y + 1$ are $f_x = 2x + 2xy$ and $f_y = 2y + x^2$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 2x + 2xy = 0 \qquad (1)$$
$$f_y = 2y + x^2 = 0 \qquad (2)$$

Factoring Equation (1) gives us:

$$\begin{aligned} 2x + 2xy &= 0 \\ 2x(1 + y) &= 0 \\ x = 0, \ \text{or } y &= -1 \end{aligned}$$

If $x = 0$ then Equation (2) gives us $y = 0$. If $y = -1$ then Equation (2) gives us:

$$\begin{aligned} 2(-1) + x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

Thus, the critical points are $\boxed{(0, 0)}$, $\boxed{(\sqrt{2}, -1)}$, and $\boxed{(-\sqrt{2}, -1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = 2x$$

The discriminant function $D(x, y)$ is then:

$$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x, y) &= (2 + 2y)(2) - (2x)^2 \\ D(x, y) &= 4 + 4y - 4x^2 \end{aligned}$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(0,0)$	4	2	Local Minimum
$(\sqrt{2},-1)$	-8	0	Saddle Point
$(-\sqrt{2},-1)$	-8	0	Saddle Point

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

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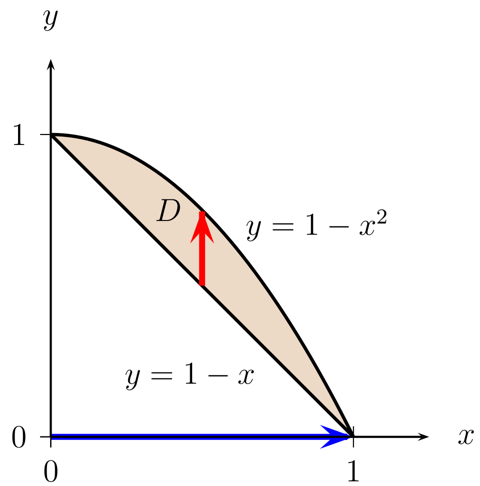
Problem 5 Solution

5. Compute the integral $\iint_D (x+3)\, dA$ where D is the region bounded by the curves $y = 1-x$ and $y = 1 - x^2$.

Solution:



From the figure we see that the region D is bounded above by $y = 1 - x^2$ and below by $y = 1 - x$. The projection of D onto the x-axis is the interval $0 \le x \le 1$. Using the order of integration $dy\, dx$ we have:

$$\begin{aligned}
\iint_D (x+3)\, dA &= \int_0^1 \int_{1-x}^{1-x^2} (x+3)\, dy\, dx \\
&= \int_0^1 (x+3) \Big[y \Big]_{1-x}^{1-x^2} dx \\
&= \int_0^1 (x+3)[(1-x^2) - (1-x)]\, dx \\
&= \int_0^1 (x+3)(x - x^2)\, dx \\
&= \int_0^1 \left(-x^3 - 2x^2 + 3x\right)\, dx \\
&= \left[-\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 \\
&= \boxed{\frac{7}{12}}
\end{aligned}$$

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Problem 6 Solution

6. Let S be the portion of the plane $x+y+z=6$ that lies above the square $0 \le x \le 2$, $1 \le y \le 3$ in the xy plane. Compute the integral $\iint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}} = \langle x, y, z\rangle$ and the normal vector to S points upward.

Solution: The formula we will use to compute the surface integral of the vector field $\vec{\mathbf{F}}$ is:

$$\iint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iint_R \vec{\mathbf{F}} \bullet \left(\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v\right)\, dA$$

where the function $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with domain R is a parameterization of the surface S and the vectors $\vec{\mathbf{T}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u}$ and $\vec{\mathbf{T}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v}$ are the tangent vectors.

We begin by finding a parameterization of the plane. Let $x = u$ and $y = v$. Then, $z = 6-x-y$ using the equation for the plane. Thus, we have $\vec{\mathbf{r}}(u,v) = \langle u, v, 6-u-v\rangle$. Furthermore, the domain R is the set of all points (u,v) satisfying $0 \le u \le 2$ and $1 \le v \le 3$. Therefore, a parameterization of S is:

$$\vec{\mathbf{r}}(u,v) = \langle u, v, 6-u-v\rangle\,,$$
$$R = \left\{(u,v)\,\middle|\, 0 \le u \le 2,\ 1 \le v \le 3\right\}$$

The tangent vectors $\vec{\mathbf{T}}_u$ and $\vec{\mathbf{T}}_v$ are then:

$$\vec{\mathbf{T}}_u = \frac{\partial \vec{\mathbf{r}}}{\partial u} = \langle 1, 0, -1\rangle$$
$$\vec{\mathbf{T}}_v = \frac{\partial \vec{\mathbf{r}}}{\partial v} = \langle 0, 1, -1\rangle$$

The cross product of these vectors is:

$$\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$
$$= \langle 1, 1, 1\rangle$$

The vector field $\vec{\mathbf{F}} = \langle x, y, z\rangle$ written in terms of u and v is:

$$\vec{\mathbf{F}} = \langle x, y, z\rangle$$
$$\vec{\mathbf{F}} = \langle u, v, 6-u-v\rangle$$

Before computing the surface integral, we note that $\vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v$ points upward, as desired, since the third component of the vector is positive.

The value of the surface integral is:

$$\begin{aligned}
\iint_S \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{S}} &= \iint_R \overrightarrow{\mathbf{F}} \bullet \left(\overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v\right) \, dA \\
&= \iint_R \langle u, v, 6-u-v \rangle \bullet \langle 1, 1, 1 \rangle \, dA \\
&= \iint_R (u + v + 6 - u - v) \, dA \\
&= \iint_R 6 \, dA \\
&= 6 \iint_R 1 \, dA \\
&= 6 \times (\text{Area of } R) \\
&= 6 \times 4 \\
&= \boxed{24}
\end{aligned}$$

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Problem 7 Solution

7. Find an equation for the plane that contains the points $(1,2,1)$, $(-3,0,1)$, and $(2,2,0)$.

Solution: A vector $\overrightarrow{\mathbf{n}}$ perpendicular to the plane is the cross product of any two non-parallel vectors that lie in the plane. Let $\overrightarrow{\mathbf{u}} = \langle -3-1, 0-2, 1-1\rangle = \langle -4, -2, 0\rangle$ be the vector from $(1,2,1)$ to $(-3,0,1)$ and $\overrightarrow{\mathbf{v}} = \langle 2-1, 2-2, 0-1\rangle = \langle 1, 0, -1\rangle$ be the vector from $(1,2,1)$ to $(2,2,0)$. Then the normal vector is:

$$\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}$$

$$\overrightarrow{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4 & -2 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\overrightarrow{\mathbf{n}} = \hat{\mathbf{i}} \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} -4 & 0 \\ 1 & -1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} -4 & -2 \\ 1 & 0 \end{vmatrix}$$

$$\overrightarrow{\mathbf{n}} = \hat{\mathbf{i}}\,[(-2)(-1) - (0)(0)] - \hat{\mathbf{j}}\,[(-4)(-1) - (0)(1)] + \hat{\mathbf{k}}\,[(-4)(0) - (-2)(1)]$$

$$\overrightarrow{\mathbf{n}} = 2\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$$

$$\overrightarrow{\mathbf{n}} = \langle 2, -4, 2\rangle$$

Using $(1,2,1)$ as a point on the plane, we have:

$$\boxed{2(x-1) - 4(y-2) + 2(z-1) = 0}$$

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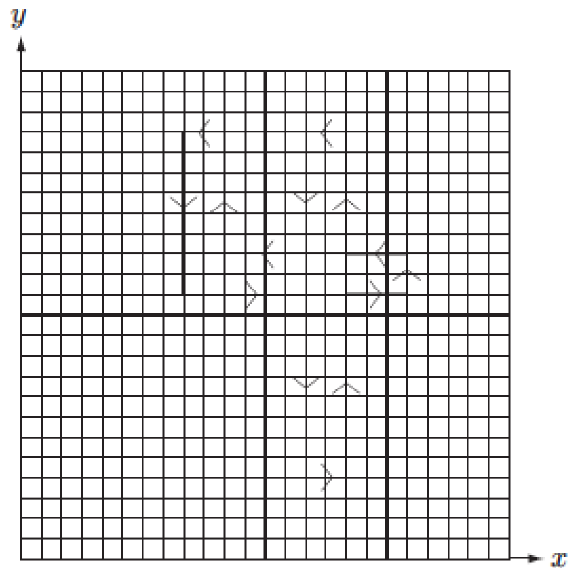
Problem 8 Solution

8. Consider the vector field $\overrightarrow{\mathbf{F}} = \langle e^{2x} + y, 4x + \sin(y^2) \rangle$ and the curve C shown below. Use Green's Theorem to compute $\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}}$. (Note: Each square in the grid has a side of length $\frac{1}{4}$.)



Solution: Green's Theorem states that

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

where D is the region enclosed by C. The integrand of the double integral is:

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(4x + \sin(y^2)\right) - \frac{\partial}{\partial y}\left(e^{2x} + y\right) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Thus, the value of the integral is:

$$\begin{aligned} \oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \\ &= \iint_D 3\, dA \\ &= 3 \iint_D 1\, dA \\ &= 3 \times (\text{area of } D) \\ &= 3 \times 4 \\ &= \boxed{12} \end{aligned}$$

Note that D consists of 64 squares and each has an area of $(\frac{1}{4})^2 = \frac{1}{16}$ so the area of D is $64 \times \frac{1}{16} = 4$.