Math 210, Final Exam, Spring 2009
Problem 1 Solution

1. Let $f(x, y, z) = (x^2 + y)z + x\cos(y^2 - z)$.

   (a) Find the gradient $\vec{\nabla} f$ at the point $P = (0, 1, 1)$.

   (b) Find the directional derivative $D_{\mathbf{v}} f(0, 1, 1)$ where $\vec{\mathbf{v}}$ is the unit vector from P *towards* $Q = (2, 3, 0)$.

Solution:

(a) The gradient of f is:

$$\begin{aligned}\vec{\nabla} f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle 2xz + \cos(y^2 - z), z - 2xy\sin(y^2 - z), x^2 + y + x\sin(y^2 - z) \right\rangle\end{aligned}$$

At the point $P = (0, 1, 1)$ we have:

$$\begin{aligned}\vec{\nabla} f(0, 1, 1) &= \left\langle 2(0)(1) + \cos(1^2 - 1), 1 - 2(0)(1)\sin(1^1 - 1), 0^2 + 1 + (0)\sin(1^1 - 1) \right\rangle \\ &= \boxed{\langle 1, 1, 1 \rangle}\end{aligned}$$

(b) The unit vector $\vec{\mathbf{v}}$ that points from $P = (0, 1, 1)$ *towards* $Q = (2, 3, 0)$ is:

$$\begin{aligned}\vec{\mathbf{v}} &= \frac{\overrightarrow{PQ}}{\left|\left|\overrightarrow{PQ}\right|\right|} \\ &= \frac{\langle 2, 2, -1 \rangle}{||\langle 2, 2, -1 \rangle||} \\ &= \frac{\langle 2, 2, -1 \rangle}{3} \\ &= \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle\end{aligned}$$

Thus, the directional derivative $D_{\mathbf{v}} f(0, 1, 1)$ is:

$$\begin{aligned}D_{\mathbf{v}} f(0, 1, 1) &= \vec{\nabla} f(0, 1, 1) \bullet \vec{\mathbf{v}} \\ &= \langle 1, 1, 1 \rangle \bullet \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle \\ &= \frac{2}{3} + \frac{2}{3} - \frac{1}{3} \\ &= \boxed{1}\end{aligned}$$

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Problem 2 Solution

2. Consider the vector fields $\overrightarrow{\mathbf{F}} = \langle ye^{xy} + y^2, xe^{xy} + 2xy \rangle$ and $\overrightarrow{\mathbf{G}} = \langle xe^{xy}, ye^{xy} \rangle$.

(a) Which of the two vector fields is conservative and which is not? (justify)

(b) Find a potential function ϕ for the conservative among the vector fields.

Solution:

(a) In order for the vector field $\overrightarrow{\mathbf{H}} = \langle f(x,y), g(x,y) \rangle$ to be conservative, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

- $\overrightarrow{\mathbf{F}}$: Using $f(x,y) = ye^{xy} + y^2$ and $g(x,y) = xe^{xy} + 2xy$ we have:

$$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy} + 2y, \quad \frac{\partial g}{\partial x} = e^{xy} + xye^{xy} + 2y$$

verifying that $\overrightarrow{\mathbf{F}}$ is conservative.

- $\overrightarrow{\mathbf{G}}$: Using $f(x,y) = xe^{xy}$ and $g(x,y) = ye^{xy}$ we have:

$$\frac{\partial f}{\partial y} = x^2 e^{xy}, \quad \frac{\partial g}{\partial x} = y^2 e^{xy}$$

verifying that $\overrightarrow{\mathbf{G}}$ is **not** conservative.

(b) If $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\phi$, then it must be the case that:

$$\frac{\partial \phi}{\partial x} = f(x,y) \tag{1}$$
$$\frac{\partial \phi}{\partial y} = g(x,y) \tag{2}$$

Using $f(x,y) = ye^{xy} + y^2$ and integrating both sides of Equation (1) with respect to x we get:

$$\frac{\partial \varphi}{\partial x} = f(x,y)$$
$$\frac{\partial \varphi}{\partial x} = ye^{xy} + y^2$$
$$\int \frac{\partial \varphi}{\partial x}\, dx = \int \left(ye^{xy} + y^2\right)\, dx$$
$$\varphi(x,y) = e^{xy} + xy^2 + h(y) \tag{3}$$

We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = xe^{xy} + 2xy$ we get the equation:

$$\frac{\partial \varphi}{\partial y} = g(x, y)$$
$$\frac{\partial \varphi}{\partial y} = xe^{xy} + 2xy$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\frac{\partial}{\partial y}\left(e^{xy} + xy^2 + h(y)\right) = xe^{xy} + 2xy$$
$$xe^{xy} + 2xy + h'(y) = xe^{xy} + 2xy$$
$$h'(y) = 0$$

which gives us $h(y) = C$. Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\phi(x, y) = e^{xy} + xy^2}$$

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Problem 3 Solution

3. Use Green's theorem to compute

$$\oint_C xy^2\,dx + (x-y)\,dy$$

where C traces the triangle with vertices $(0,0)$, $(1,0)$, $(0,2)$ traversed in this order.

Solution: Green's Theorem states that

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

where D is the region enclosed by C. The integrand of the double integral is:

$$\begin{aligned}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial y}xy^2 \\ &= 1 - 2xy\end{aligned}$$

Thus, the value of the integral is:

$$\begin{aligned}\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \\ &= \iint_D (1-2xy)\,dA \\ &= \int_0^1 \int_0^{-2x+2} (1-2xy)\,dy\,dx \\ &= \int_0^1 \left[y - xy^2\right]_0^{-2x+2} dx \\ &= \int_0^1 \left[(-2x+2) - x(-2x+2)^2\right] dx \\ &= \int_0^1 \left(-2x + 2 - 4x^3 + 8x^2 - 4x\right) dx \\ &= \int_0^1 \left(-4x^3 + 8x^2 - 6x + 2\right) dx \\ &= \left[-x^4 + \frac{8}{3}x^3 - 3x^2 + 2x\right]_0^1 \\ &= -1 + \frac{8}{3} - 3 + 2 \\ &= \boxed{\frac{2}{3}}\end{aligned}$$

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Problem 4 Solution

4. Let $\vec{\mathbf{u}} = \langle 1, 2, 3\rangle$ and $\vec{\mathbf{v}} = \langle 2, -1, 0\rangle$.

(a) What can be said about the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$: acute/obtuse/right?

(b) Find an equation for the plane through $(1, 1, 1)$ containing $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Solution:

(a) The angle is determined by the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$:

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \langle 1, 2, 3\rangle \bullet \langle 2, -1, 0\rangle = (1)(2) + (2)(-1) + (3)(0) = 0$$

Since the dot product is zero, the vectors are perpendicular. Thus, the angle between the two vectors is a **right** angle.

(b) A vector perpendicular to the plane is the cross product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ which both lie in the plane.

$$\begin{aligned}
\vec{\mathbf{n}} &= \vec{\mathbf{u}} \times \vec{\mathbf{v}} \\
\vec{\mathbf{n}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 2 & -1 & 0 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}} \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\
\vec{\mathbf{n}} &= \hat{\mathbf{i}}\,[(2)(0) - (3)(-1)] - \hat{\mathbf{j}}\,[(1)(0) - (3)(2)] + \hat{\mathbf{k}}\,[(1)(-1) - (2)(2)] \\
\vec{\mathbf{n}} &= 3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}} \\
\vec{\mathbf{n}} &= \langle 3, 6, -5\rangle
\end{aligned}$$

Using $(1, 1, 1)$ as a point on the plane, we have:

$$\boxed{3(x - 1) + 6(y - 1) - 5(z - 1) = 0}$$

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Problem 5 Solution

5. Find the equation of the tangent plane to the level surface $e^{xz} + (x+y)^3 - yz = 3$ at the point $(0, 2, 3)$.

Solution: We use the following formula for the equation for the tangent plane:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

because the equation for the surface is given in **implicit** form. Note that $\vec{\mathbf{n}} = \vec{\nabla} f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\rangle$ is a vector normal to the surface $f(x,y,z) = C$ and, thus, to the tangent plane at the point (a,b,c) on the surface.

The partial derivatives of $f(x,y,z) = e^{xz} + (x+y)^3 - yz$ are:

$$\begin{aligned} f_x &= ze^{xz} + 3(x+y)^2 \\ f_y &= 3(x+y)^2 - z \\ f_z &= xe^{xz} - y \end{aligned}$$

Evaluating these derivatives at $(0, 2, 3)$ we get:

$$\begin{aligned} f_x(0,2,3) &= 3e^{(0)(3)} + 3(0+2)^2 = 15 \\ f_y(0,2,3) &= 3(0+2)^2 - 3 = 9 \\ f_z(0,2,3) &= (0)e^{(0)(3)} - 2 = -2 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{15(x-0) + 9(y-2) - 2(z-3) = 0}$$

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Problem 6 Solution

6. Use the method of Lagrange multipliers to find points where $f(x, y) = x + 6y - 7$ attains its maximum and minimum on the ellipse $x^2 + 3y^2 = 13$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + 3y^2 = 13$ is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y) = x^2 + 3y^2 = 13$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 13$$

which, when applied to our functions f and g, give us:

$$1 = \lambda (2x) \qquad (1)$$
$$6 = \lambda (6y) \qquad (2)$$
$$x^2 + 3y^2 = 13 \qquad (3)$$

We begin by noting that Equation (1) gives us:

$$\begin{aligned} 1 &= \lambda(2x) \\ x &= \frac{1}{2\lambda} \end{aligned}$$

and Equation (2) gives us:

$$\begin{aligned} 6 &= \lambda(6y) \\ y &= \frac{1}{\lambda} \end{aligned}$$

Plugging the above expressions for x and y into Equation (3) and solving for λ we get:

$$\begin{aligned} x^2 + 3y^2 &= 13 \\ \left(\frac{1}{2\lambda}\right)^2 + 3\left(\frac{1}{\lambda}\right)^2 &= 13 \\ \frac{1}{4\lambda^2} + \frac{3}{\lambda^2} &= 13 \\ \frac{1}{4\lambda^2} + \frac{12}{4\lambda^2} &= 13 \\ 1 + 12 &= 13(4\lambda^2) \\ 52\lambda^2 &= 13 \\ \lambda^2 &= \frac{1}{4} \\ \lambda &= \pm\frac{1}{2} \end{aligned}$$

When $\lambda = \frac{1}{2}$ we get $x = 1$ and $y = 2$. When $\lambda = -\frac{1}{2}$ we get $x = -1$ and $y = -2$. Thus, the points of interest are $(1, 2)$ and $(-1, -2)$.

We now evaluate $f(x, y) = x + 6y - 7$ at each point of interest.

$$f(1, 2) = 6$$
$$f(-1, -2) = -20$$

From the values above we observe that f attains an absolute maximum of 6 and an absolute minimum of -20.

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Problem 7 Solution

7. Find all critical values of $f(x,y) = x^3 + 2xy - 2y^2 - 10x$ and classify them into local maxima, local minima, and saddle points.

Solution: By definition, an interior point (a,b) in the domain of f is a **critical point** of f if either

(1) $f_x(a,b) = f_y(a,b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a,b).

The partial derivatives of $f(x,y) = x^3 + 2xy - 2y^2 - 10x$ are $f_x = 3x^2 + 2y - 10$ and $f_y = 2x - 4y$. These derivatives exist for all (x,y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 3x^2 + 2y - 10 = 0 \tag{1}$$
$$f_y = 2x - 4y = 0 \tag{2}$$

Solving Equation (2) for x we get:

$$x = 2y \tag{3}$$

Substituting this into Equation (1) and solving for y we get:

$$\begin{aligned} 3x^2 + 2y - 10 &= 0 \\ 3(2y)^2 + 2y - 10 &= 0 \\ 12y^2 + 2y - 10 &= 0 \\ 6y^2 + y - 5 &= 0 \\ (6y-5)(y+1) &= 0 \\ \iff \quad y = \frac{5}{6} \quad &\text{or} \quad y = -1 \end{aligned}$$

We find the corresponding x-values using Equation (3): $x = 2y$.

- If $y = \frac{5}{6}$, then $x = \frac{5}{3}$.
- If $y = -1$, then $x = -2$.

Thus, the critical points are $\boxed{(\frac{5}{3}, \frac{5}{6})}$ and $\boxed{(-2,-1)}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 6x, \quad f_{yy} = -4, \quad f_{xy} = 2$$

The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = (6x)(-4) - (2)^2$$
$$D(x, y) = -24x - 4$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(\frac{5}{3}, \frac{5}{6})$	-44	10	Saddle Point
$(-2, -1)$	44	-12	Local Maximum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local maximum of f if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$.

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Problem 8 Solution

8. Let C be the curve parameterized by $\vec{\mathbf{c}}(t) = \langle 3t, 2\cos(t), 2\sin(t)\rangle$ for $0 \leq t \leq 2\pi$.

(a) Find $\vec{\mathbf{c}}'(t)$ and $\vec{\mathbf{c}}''(t)$.

(b) Find the length of the curve.

Solution:

(a) The first two derivatives of $\vec{\mathbf{c}}(t)$ are:

$$\begin{aligned}\vec{\mathbf{c}}'(t) &= \langle 3, -2\sin(t), 2\cos(t)\rangle \\ \vec{\mathbf{c}}''(t) &= \langle 0, -2\cos(t), -2\sin(t)\rangle\end{aligned}$$

(b) The length of the curve is:

$$\begin{aligned}L &= \int_0^{2\pi} \left|\left|\vec{\mathbf{r}}'(t)\right|\right| \, dt \\ &= \int_0^{2\pi} \sqrt{3^2 + (-2\sin(t))^2 + (2\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{9 + 4\sin^2(t) + 4\cos^2(t)}\, dt \\ &= \int_0^{2\pi} \sqrt{9+4}\, dt \\ &= \int_0^{2\pi} \sqrt{13}\, dt \\ &= \boxed{2\pi\sqrt{13}}\end{aligned}$$

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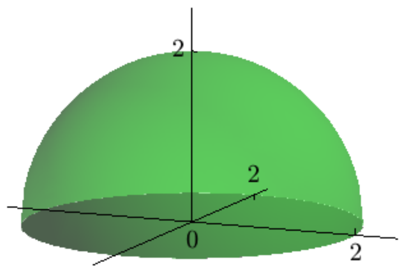
Problem 9 Solution

9. Let H be the upper semi-ball $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$. Compute

$$\iiint_H z\, dV$$

Solution: The region of integration is shown below.



The inequality describing the ball in cylindrical coordinates is $r^2 + z^2 \leq 4 \implies z \geq \sqrt{4 - r^2}$ since the region is above the xy-plane. The projection of H onto the xy-plane is the disk $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Thus, the value of the integral is:

$$\begin{aligned}
V &= \iiint_H z\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} z\, r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r \left[\frac{1}{2} z^2\right]_0^{\sqrt{4-r^2}} dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \frac{1}{2} r \left(4 - r^2\right) dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^2 \left(4r - r^3\right) dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4\right]_0^2 d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left(2(2)^2 - \frac{1}{4}(2)^4\right) d\theta \\
&= 2 \int_0^{2\pi} d\theta \\
&= \boxed{4\pi}
\end{aligned}$$

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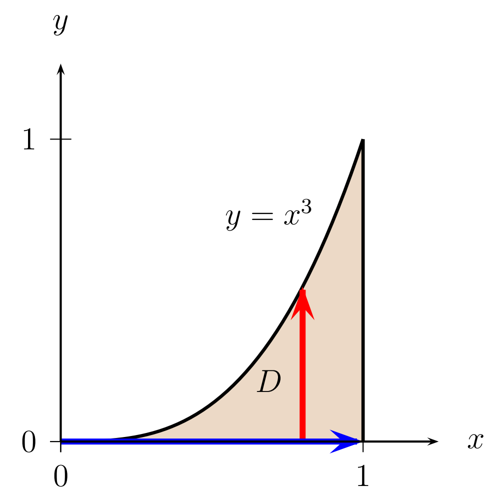
Problem 10 Solution

10. Change the order of integration and evaluate the iterated integral:

$$\int_0^1 \int_{y^{1/3}}^1 \left(xy + \sin\left(x^4\right)\right)\, dx\, dy.$$

Solution:



From the figure we see that the region D is bounded above by $y = x^3$ and below by $y = 0$. The projection of D onto the x-axis is the interval $0 \le x \le 1$. Using the order of integration $dy\, dx$ we have:

$$\begin{aligned}
\int_0^1 \int_{y^{1/3}}^1 \left(xy + \sin\left(x^4\right)\right)\, dx\, dy &= \int_0^1 \int_0^{x^3} \left(xy + \sin\left(x^4\right)\right)\, dy\, dx \\
&= \int_0^1 \left[\frac{1}{2}xy^2 + y\sin\left(x^4\right)\right]_0^{x^3} dx \\
&= \int_0^1 \left[\frac{1}{2}x\left(x^3\right)^2 + x^3\sin\left(x^4\right)\right] dx \\
&= \int_0^1 \left[\frac{1}{2}x^7 + x^3\sin\left(x^4\right)\right] dx \\
&= \left[\frac{1}{16}x^8 - \frac{1}{4}\cos\left(x^4\right)\right]_0^1 \\
&= \left[\frac{1}{16}(1)^8 - \frac{1}{4}\cos\left(1^4\right)\right] - \left[\frac{1}{16}(0)^8 - \frac{1}{4}\cos\left(0^4\right)\right] \\
&= \frac{1}{16} - \frac{1}{4}\cos(1) + \frac{1}{4} \\
&= \boxed{\frac{5}{16} - \frac{1}{4}\cos(1)}
\end{aligned}$$