Math 210, Final Exam, Spring 2010
Problem 1 Solution

1. The position vector

$$\overrightarrow{\mathbf{r}}(t) = t^3\,\hat{\mathbf{\imath}} + 18t\,\hat{\mathbf{\jmath}} + 3t^{-1}\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$$

describes the motion of a particle.

(a) Find the position at time $t = 2$.

(b) Find the velocity at time $t = 2$.

(c) Find the acceleration at time $t = 2$.

(d) Find the length of the path traveled by the particle during the time $1 \le t \le 2$.

Solution:

(a) The position at time $t = 2$ is:

$$\boxed{\overrightarrow{\mathbf{r}}(2) = 2^3\,\hat{\mathbf{\imath}} + 18(2)\,\hat{\mathbf{\jmath}} + 3(2)^{-1}\,\hat{\mathbf{k}} = 8\,\hat{\mathbf{\imath}} + 36\,\hat{\mathbf{\jmath}} + \frac{3}{2}\,\hat{\mathbf{k}}}$$

(b) The velocity is the derivative of position.

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{r}}'(t) = 3t^2\,\hat{\mathbf{\imath}} + 18\,\hat{\mathbf{\jmath}} - 3t^{-2}\,\hat{\mathbf{k}}$$

Therefore, the velocity at time $t = 2$ is:

$$\boxed{\overrightarrow{\mathbf{v}}(2) = 3(2)^2\,\hat{\mathbf{\imath}} + 18\,\hat{\mathbf{\jmath}} - 3(2)^{-2}\,\hat{\mathbf{k}} = 12\,\hat{\mathbf{\imath}} + 18\,\hat{\mathbf{\jmath}} - \frac{3}{4}\,\hat{\mathbf{k}}}$$

(c) The acceleration is the derivative of velocity.

$$\overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{v}}'(t) = 6t\,\hat{\mathbf{\imath}} + 6t^{-3}\,\hat{\mathbf{k}}$$

Therefore, the acceleration at time $t = 2$ is:

$$\boxed{\overrightarrow{\mathbf{a}}(2) = 6(2)\,\hat{\mathbf{\imath}} + 6(2)^{-3}\,\hat{\mathbf{k}} = 12\,\hat{\mathbf{\imath}} + \frac{3}{4}\,\hat{\mathbf{k}}}$$

(d) The length of the path traveled by the particle is:

$$\begin{aligned} L &= \int_1^2 \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|\,dt \\ &= \int_1^2 \sqrt{(3t^2)^2 + 18^2 + (-3t^{-2})^2}\,dt \\ &= \int_1^2 \sqrt{9t^4 + 324 + 9t^{-4}}\,dt \end{aligned}$$

It turns out that a simple antiderivative of the integrand does not exist. There was a typo in the original problem. The $\hat{\mathbf{\jmath}}$-component of $\overrightarrow{\mathbf{r}}(t)$ should have been $\sqrt{18}t$ not $18t$.

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Problem 2 Solution

2. (a) For $f(x,y) = e^{(x+1)y}$ find the derivatives:

$$\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial y}, \qquad \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial^2 f}{\partial y^2}$$

(b) Find the gradient of f at the point $(2,3)$.

Solution:

(a) The first partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = ye^{(x+1)y}$$
$$\frac{\partial f}{\partial y} = (x+1)e^{(x+1)y}$$

The second derivatives are:

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(ye^{(x+1)y}\right) = y^2 e^{(x+1)y}$$
$$\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left((x+1)e^{(x+1)y}\right) = (x+1)^2 e^{(x+1)y}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left((x+1)e^{(x+1)y}\right) = e^{(x+1)y} + y(x+1)e^{(x+1)y}$$

(b) The gradient of f at $(2,3)$ is:

$$\begin{aligned} \vec{\nabla} f(2,3) &= \langle f_x(2,3), f_y(2,3) \rangle \\ &= \left\langle 3e^{(2+1)3}, (2+1)e^{(2+1)3} \right\rangle \\ &= \boxed{\left\langle 3e^9, 3e^9 \right\rangle} \end{aligned}$$

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Problem 3 Solution

3. (a) Find a potential function for the vector field

$$\overrightarrow{\mathbf{F}}(x,y,z) = (1-z)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}}$$

(b) Integrate $\overrightarrow{\mathbf{F}}$ over the straight line from $(1,0,1)$ to $(0,1,2)$.
[You may calculate this directly or you may use a potential function.]

Solution:

(a) By inspection, a potential function for the vector field $\overrightarrow{\mathbf{F}}$ is:

$$\varphi(x,y,z) = x - xz + \frac{1}{2}y^2$$

To verify, we calculate the gradient of φ:

$$\begin{aligned}\overrightarrow{\nabla}\varphi &= \varphi_x\,\hat{\mathbf{i}} + \varphi_y\,\hat{\mathbf{j}} + \varphi_z\,\hat{\mathbf{k}} \\ &= (1-z)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - x\,\hat{\mathbf{k}} \\ &= \overrightarrow{\mathbf{F}}\end{aligned}$$

(b) Using the Fundamental Theorem of Line Integrals, the value of the line integral is:

$$\begin{aligned}\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \varphi(0,1,2) - \varphi(1,0,1) \\ &= \left[0 - (0)(2) + \frac{1}{2}(1)^2\right] - \left[1 - (1)(1) + \frac{1}{2}(0)^2\right] \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

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Problem 4 Solution

4. (a) Find the critical points of the function $f(x, y) = x^3 - 3x - y^2$.

(b) Use the second derivative test to classify each critical point as a local maximum, local minimum, or saddle.

Solution:

(a) By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = x^3 - 3x - y^2$ are $f_x = 3x^2 - 3$ and $f_y = -2y$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = 3x^2 - 3 = 0 \qquad (1)$$
$$f_y = -2y = 0 \qquad (2)$$

The two solutions to Equation (1) are $x = \pm 1$. The only solution to Equation (2) is $y = 0$. Thus, the critical points are $\boxed{(1, 0)}$ and $\boxed{(-1, 0)}$.

(b) We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 6x, \quad f_{yy} = -2, \quad f_{xy} = 0$$

The discriminant function $D(x, y)$ is then:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2$$
$$D(x, y) = (6x)(-2) - (0)^2$$
$$D(x, y) = -12x$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	**Conclusion**
$(1, 0)$	-12	6	Saddle Point
$(-1, 0)$	12	-6	Local Maximum

Recall that (a, b) is a saddle point if $D(a, b) < 0$ and that (a, b) corresponds to a local maximum of f if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$.

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Problem 5 Solution

5. Find the maximum and minimum of the function $f(x, y) = (x-1)^2 + y^2$ subject to the constraint:

$$g(x, y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $(\frac{x}{3})^2 + (\frac{y}{2})^2 = 1$ is compact which guarantees the existence of absolute extrema of f. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 1$$

which, when applied to our functions f and g, give us:

$$2(x-1) = \lambda\left(\frac{2x}{9}\right) \tag{1}$$

$$2y = \lambda\left(\frac{y}{2}\right) \tag{2}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \tag{3}$$

From Equation (2) we observe that:

$$\begin{aligned} 2y &= \lambda\left(\frac{y}{2}\right) \\ 4y &= \lambda y \\ 4y - \lambda y &= 0 \\ y(4-\lambda) &= 0 \\ y = 0, \quad \text{or} \quad \lambda &= 4 \end{aligned}$$

If $y = 0$ then Equation (3) gives us:

$$\begin{aligned} \left(\frac{x}{3}\right)^2 + \left(\frac{0}{2}\right)^2 &= 1 \\ \frac{x^2}{9} &= 1 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

If $\lambda = 4$ then Equation (1) gives us:

$$\begin{aligned}
2(x-1) &= \lambda \left(\frac{2x}{9}\right) \\
2(x-1) &= 4 \left(\frac{2x}{9}\right) \\
x - 1 &= \frac{4x}{9} \\
\frac{5x}{9} &= 1 \\
x &= \frac{9}{5}
\end{aligned}$$

which, when plugged into Equation (3), gives us:

$$\begin{aligned}
\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \\
\left(\frac{9/5}{3}\right)^2 + \frac{y^2}{4} &= 1 \\
\frac{9}{25} + \frac{y^2}{4} &= 1 \\
\frac{y^2}{4} &= \frac{16}{25} \\
y^2 &= \frac{64}{25} \\
y &= \pm\frac{8}{5}
\end{aligned}$$

Thus, the points of interest are $(3, 0)$, $(-3, 0)$, $(\frac{9}{5}, \frac{8}{5})$, and $(\frac{9}{5}, -\frac{8}{5})$.

We now evaluate $f(x, y) = (x - 1)^2 + y^2$ at each point of interest.

$$\begin{aligned}
f(3, 0) &= (3 - 1)^2 + 0^2 = 4 \\
f(-3, 0) &= (-3 - 1)^2 + 0^2 = 16 \\
f(\tfrac{9}{5}, \tfrac{8}{5}) &= (\tfrac{9}{5} - 1)^2 + (\tfrac{8}{5})^2 = \tfrac{16}{5} \\
f(\tfrac{9}{5}, -\tfrac{8}{5}) &= (\tfrac{9}{5} - 1)^2 + (-\tfrac{8}{5})^2 = \tfrac{16}{5}
\end{aligned}$$

From the values above we observe that f attains an absolute maximum of 16 and an absolute minimum of $\frac{16}{5}$.

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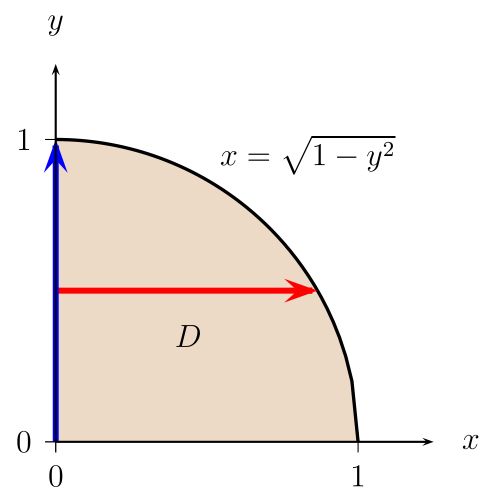
Problem 6 Solution

6. Compute the integral

$$\iint_R xy\,dx\,dy$$

over the quarter circle $R = \{(x, y) : 0 \le x,\ 0 \le y,\ x^2 + y^2 \le 1\}$. [You may use polar or Cartesian coordinates.]

Solution:



From the figure we see that the region D is bounded on the left by $x = 0$ and on the right by $x = \sqrt{1 - y^2}$. The projection of D onto the y-axis is the interval $0 \le y \le 1$. Using the order of integration $dx\,dy$ we have:

$$
\begin{aligned}
\iint_R xy\,dx\,dy &= \int_0^1 \int_0^{\sqrt{1-y^2}} xy\,dx\,dy \\
&= \int_0^1 \left[\frac{1}{2}x^2 y\right]_0^{\sqrt{1-y^2}} dy \\
&= \int_0^1 \frac{1}{2}\left(\sqrt{1-y^2}\right)^2 y\,dy \\
&= \frac{1}{2}\int_0^1 \left(1-y^2\right) y\,dy \\
&= \frac{1}{2}\int_0^1 \left(y-y^3\right)\,dy \\
&= \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 \\
&= \frac{1}{2}\left[\frac{1}{2}(1)^2 - \frac{1}{4}(1)^4\right] \\
&= \boxed{\frac{1}{8}}
\end{aligned}
$$

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Problem 7 Solution

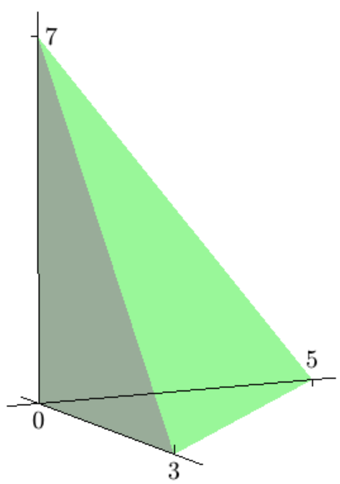
7. Compute the integral

$$\iiint_R 1\,dx\,dy\,dz$$

over the tetrahedron

$$R = \{(x, y, z) : 0 \le x,\ 0 \le y,\ 0 \le z,\ x/3 + y/5 + z/7 \le 1\}.$$

Solution: The region of integration is shown below.



The volume of the tetrahedron is

$$
\begin{aligned}
V &= \iiint_R 1\,dx\,dy\,dz \\
&= \int_0^3 \int_0^{5-5x/3} \int_0^{7-7x/3-7y/5} 1\,dz\,dy\,dx \\
&= \int_0^3 \int_0^{5-5x/3} \left(7 - \frac{7}{3}x - \frac{7}{5}y\right)\,dy\,dx \\
&= \int_0^3 \left[7y - \frac{7}{3}xy - \frac{7}{10}y^2\right]_0^{5-5x/3}\,dx \\
&= \int_0^3 \left[7\left(5 - \frac{5}{3}x\right) - \frac{7}{3}x\left(5 - \frac{5}{3}x\right) - \frac{7}{10}\left(5 - \frac{5}{3}x\right)^2\right]\,dx \\
&= \int_0^3 \left(35 - \frac{35}{3}x - \frac{35}{3}x + \frac{35}{9}x^2 - \frac{35}{2} + \frac{35}{3}x + \frac{35}{18}x^2\right)\,dx \\
&= \int_0^3 \left(\frac{35}{2} - \frac{35}{3}x + \frac{35}{18}x^2\right)\,dx \\
&= \left[\frac{35}{2}x - \frac{35}{6}x^2 + \frac{35}{54}x^3\right]_0^3 \\
&= \frac{35}{2}(3) - \frac{35}{6}(3)^2 + \frac{35}{54}(3)^3 \\
&= \boxed{\frac{35}{2}}
\end{aligned}
$$

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Problem 8 Solution

8. Find an equation for the tangent plane to the surface defined by $xy^2 + 2z^2 = 12$ at the point $(1, 2, 2)$.

Solution: We use the following formula for the equation for the tangent plane:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

because the equation for the surface is given in **implicit** form. Note that $\vec{\mathbf{n}} = \vec{\nabla} f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\rangle$ is a vector normal to the surface $f(x,y,z) = C$ and, thus, to the tangent plane at the point (a,b,c) on the surface.

The partial derivatives of $f(x,y,z) = xy^2 + 2z^2$ are:

$$\begin{aligned} f_x &= y^2 \\ f_y &= 2xy \\ f_z &= 4z \end{aligned}$$

Evaluating these derivatives at $(1,2,2)$ we get:

$$\begin{aligned} f_x(1,2,2) &= 2^2 = 4 \\ f_y(1,2,2) &= 2(1)(2) = 4 \\ f_z(1,2,2) &= 4(2) = 8 \end{aligned}$$

Thus, the tangent plane equation is:

$$\boxed{4(x-1) + 4(y-2) + 8(z-2) = 0}$$

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Problem 9 Solution

9. Compute the integral

$$\oint_C (3x^2 + y)\,dx + (x^2 + y^3)\,dy$$

over the counterclockwise boundary of the rectangle

$$R = \{(x, y) \,:\, 0 \le x \le 3,\ 0 \le y \le 2\}$$

using Green's theorem or otherwise.

Solution: Green's Theorem states that

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

where R is the region enclosed by C. The integrand of the double integral is:

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(x^2 + y^3\right) - \frac{\partial}{\partial y}\left(3x^2 + y\right) \\ &= 2x - 1 \end{aligned}$$

Thus, the value of the integral is:

$$\begin{aligned} \oint_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{s}} &= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \\ &= \iint_R (2x - 1)\,dA \\ &= \int_0^3 \int_0^2 (2x - 1)\,dy\,dx \\ &= \int_0^3 \Big[2xy - y\Big]_0^2\,dx \\ &= \int_0^3 \Big(4x - 2\Big)\,dx \\ &= \Big[2x^2 - 2x\Big]_0^3 \\ &= 2(3)^2 - 2(3) \\ &= \boxed{12} \end{aligned}$$