Math 210, Final Exam, Spring 2012
Problem 1 Solution

1. Consider three position vectors (tails are the origin):

$$\vec{\mathbf{u}} = \langle 1, 0, 0\rangle$$
$$\vec{\mathbf{v}} = \langle 4, 0, 2\rangle$$
$$\vec{\mathbf{w}} = \langle 0, 1, 1\rangle$$

(a) Find an equation of the plane passing through the tips of $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

(b) Find an equation of the line perpendicular to the plane from part (a) and passing through the origin.

Solution:

(a) Since the tails of the given vectors are at the origin, the tips of the vectors are the points $U = (1, 0, 0)$, $V = (4, 0, 2)$, and $W = (0, 1, 1)$, respectively. The plane containing the tips has $\vec{\mathbf{n}} = \overrightarrow{UV} \times \overrightarrow{UW}$ as a normal vector. Since $\overrightarrow{UV} = \langle 3, 0, 2\rangle$ and $\overrightarrow{UW} = \langle -1, 1, 1\rangle$, the normal vector is

$$\vec{\mathbf{n}} = \overrightarrow{UV} \times \overrightarrow{UW} = \langle -2, -5, 3\rangle$$

Using $U = (1, 0, 0)$ as a point on the plane, an equation for the plane is

$$-2(x - 1) - 5(y - 0) + 3(z - 0) = 0$$

(b) The line perpendicular to the plane in part (a) is parallel to the plane's normal vector. Thus, since $\langle -2, -5, 3\rangle$ is parallel to the line and the origin $(0, 0, 0)$ is on the line, the vector equation for the line is

$$\vec{\mathbf{r}}(t) = \langle 0, 0, 0\rangle + t\,\langle -2, -5, 3\rangle$$

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Problem 2 Solution

2. Consider the curve $\vec{\mathbf{r}}(t) = \langle t, t^3 \rangle$, $-\infty < t < \infty$.

(a) Find the curvature $\kappa(t)$.

(b) Find all values of t where $\kappa(t) = 0$.

(c) Compute the limits

$$\lim_{t\to\infty} \kappa(t), \qquad \lim_{t\to-\infty} \kappa(t)$$

(d) What do the limits in part (c) say about the curve $\vec{\mathbf{r}}(t)$?

Solution:

(a) By definition, the curvature of a curve parametrized by $\vec{\mathbf{r}}(t)$ is given by the formula

$$\kappa(t) = \frac{\left|\left|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(t)\right|\right|^3}$$

The first two derivatives of $\vec{\mathbf{r}}(t)$ are $\vec{\mathbf{r}}'(t) = \langle 1, 3t^2 \rangle$ and $\vec{\mathbf{r}}''(t) = \langle 0, 6t \rangle$ and their cross product is $\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t) = 6t\,\hat{\mathbf{k}}$. Thus, the curvature of $\vec{\mathbf{r}}(t)$ is

$$\kappa(t) = \frac{\left|\left|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)\right|\right|}{\left|\left|\vec{\mathbf{r}}'(t)\right|\right|^3},$$

$$\kappa(t) = \frac{\left|\left|6t\,\hat{\mathbf{k}}\right|\right|}{\left|\left|\langle 1, 3t^2 \rangle\right|\right|^3},$$

$$\kappa(t) = \frac{6|t|}{(1 + 9t^4)^{3/2}}$$

(b) The curvature is 0 when $t = 0$.

(c) The limits of $\kappa(t)$ as $t \to \pm\infty$ are

$$\lim_{t\to\pm\infty} \kappa(t) = \lim_{t\to\pm\infty} \frac{6|t|}{(1 + 9t^4)^{3/2}} = 0$$

(d) Lines are curves of zero curvature. Thus, the limits in part (c) suggest that $\vec{\mathbf{r}}(t)$ behaves linearly as $t \to \pm\infty$.

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Problem 3 Solution

3. Given the function of two variables $G(x, y) = \sqrt{y - x^2}$

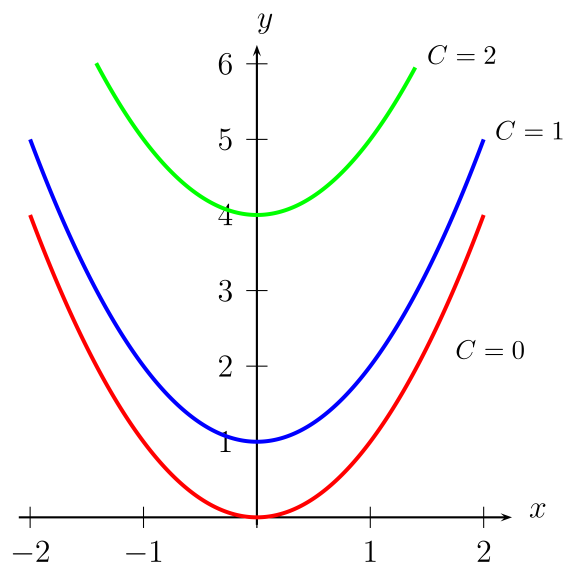
(a) Determine the domain of G.

(b) Sketch the level curves $G = 0$, $G = 1$, and $G = 2$ all on one coordinate grid. What kind of curves are they?

(c) At the point $(1, 2)$, find the direction in which G has its maximum rate of increase. Also determine this maximum rate.

Solution:

(a) The domain of G is the set of all pairs (x, y) such that $y - x^2 \geq 0$.

(b) 

(c) The direction of maximum rate of increase of $G(x, y)$ at the point $(1, 2)$ is, by definition,

$$\hat{\mathbf{u}} = \frac{\vec{\nabla} G(1, 2)}{\left|\left|\vec{\nabla} G(1, 2)\right|\right|}$$

The gradient of G is

$$\vec{\nabla} G = \langle G_x, G_y \rangle = \left\langle -\frac{x}{\sqrt{y - x^2}}, \frac{1}{2\sqrt{y - x^2}} \right\rangle$$

The value of $\vec{\nabla} G$ at the point $(1, 2)$ is $\vec{\nabla} G(1, 2) = \left\langle -1, \frac{1}{2} \right\rangle$ and its magnitude is $\left|\left|\vec{\nabla} G(1, 2)\right|\right| = \frac{\sqrt{5}}{2}$. Thus, the direction of maximum rate of increase of G at $(1, 2)$ is

$$\hat{\mathbf{u}} = \frac{\left\langle -1, \frac{1}{2} \right\rangle}{\frac{\sqrt{5}}{2}}$$

The maximum rate of increase, by definition, is $\left|\left|\vec{\nabla} G(1,2)\right|\right| = \frac{\sqrt{5}}{2}$.

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Problem 4 Solution

4. Find absolute maximum and minimum of the function $f(x, y) = xy - x$ over the region $R = \{x^2 + y^2 \leq 4\}$. Also, find the points where these extremes occur.

Solution: First, the region R is closed and bounded (i.e. compact) and f is defined at every point in R. Therefore, we are guaranteed to find absolute extrema. Next, we look for all critical points of f in R. These will be points for which the first derivatives of f vanish. Thus, we must solve the system of equations:

$$\begin{aligned} f_x &= y - 1 = 0, \\ f_y &= x = 0 \end{aligned}$$

which has $x = 0$ and $y = 1$ as the only solution. We must now determine the extreme values of f on the boundary of R which is the circle $x^2 + y^2 = 4$. We will resort to using the method of Lagrange multipliers to find these values. The following system of equations must then be solved:

$$\begin{aligned} f_x &= \lambda g_x, \\ f_y &= \lambda g_y, \\ g(x, y) &= 0 \end{aligned}$$

where $g(x, y) = x^2 + y^2 - 4$. Evaluate the partial derivatives we then have

$$y - 1 = \lambda(2x), \tag{1}$$

$$x = \lambda(2y), \tag{2}$$

$$x^2 + y^2 = 4. \tag{3}$$

Dividing Equation (1) by Equation (2) and simplifying gives us

$$\begin{aligned} \frac{y-1}{x} &= \frac{\lambda(2x)}{\lambda(2y)}, \\ \frac{y-1}{x} &= \frac{x}{y}, \\ y(y-1) &= x^2, \\ x^2 &= y^2 - y \end{aligned}$$

Substituting $y^2 - y$ for x^2 in Equation (3) and solving for x we get

$$
\begin{aligned}
x^2 + y^2 &= 4, \\
y^2 - y + y^2 &= 4, \\
2y^2 - y - 4 &= 0
\end{aligned}
$$

which has the two solutions

$$y_{1,2} = \frac{1 \pm \sqrt{33}}{4}$$

Let y_1 be the positive solution and y_2 the negative one. If $y = y_1$ then the corresponding x-values are $x_{11,12} = \pm\sqrt{y_1^2 - y_1}$. Similarly, if $y = y_2$ then the corresponding x-values are $x_{21,22} = \pm\sqrt{y_2^2 - y_2}$.

We must now evaluate $f(x, y)$ at the critical point $(0, 1)$ and at all critical points on the boundary of R.

$$
\begin{aligned}
f(0,1) &= 0, \\
f(x_{11}, y_1) &= x_{11}(y_1 - 1) = (y_1 - 1)\sqrt{y_1^2 - y_1} = \sqrt{y_1}(y_1 - 1)^{3/2} \\
f(x_{12}, y_1) &= x_{12}(y_1 - 1) = -(y_1 - 1)\sqrt{y_1^2 - y_1} = -\sqrt{y_1}(y_1 - 1)^{3/2} \\
f(x_{21}, y_2) &= x_{21}(y_2 - 1) = (y_2 - 1)\sqrt{y_2^2 - y_2} = \sqrt{-y_2}(1 - y_2)^{3/2} \\
f(x_{22}, y_2) &= x_{22}(y_2 - 1) = -(y_2 - 1)\sqrt{y_2^2 - y_2} = -\sqrt{-y_2}(1 - y_2)^{3/2}
\end{aligned}
$$

A calculator would be useful here but isn't necessary. We can estimate $\sqrt{33}$ to be 5.75 using a linear approximation of $F(x) = \sqrt{x}$ about $x = 36$ giving us $y_1 \approx 1.6875$ and $y_2 \approx -1.1875$. One can then show that $f(x_{21}, y_2)$ is the absolute maximum and $f(x_{22}, y_2)$ is the absolute minimum of f on R.

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Problem 5 Solution

5. Consider the integral

$$\int_0^1 \int_{x^2}^1 x \cos\left(y^2\right)\, dy\, dx.$$

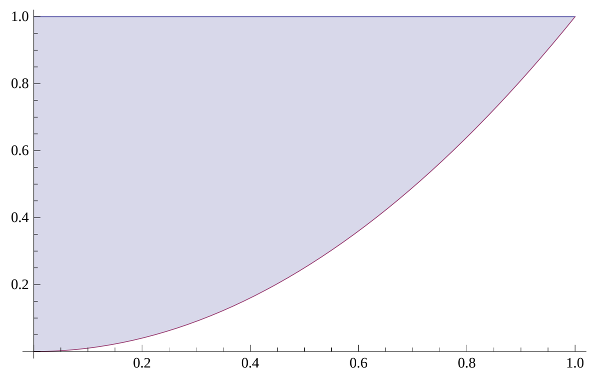
(a) Sketch the region of integration.

(b) Reverse the order of integration properly.

(c) Evaluate the integral from part (b).

Solution:

(a) The region of integration is sketched below.



(b) Upon switching the order of integration we obtain

$$\int_0^1 \int_0^{\sqrt{y}} x \cos\left(y^2\right)\, dx\, dy$$

(c) Evaluating the above double integral we get

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{y}} x \cos\left(y^2\right)\, dx\, dy &= \int_0^1 \left[\frac{1}{2}x^2 \cos\left(y^2\right)\right]_0^{\sqrt{y}} dy,\\ &= \frac{1}{2}\int_0^1 y \cos\left(y^2\right)\, dy,\\ &= \frac{1}{2}\left[\frac{1}{2}\sin\left(y^2\right)\right]_0^1,\\ &= \frac{1}{4}\sin(1)\end{aligned}$$

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Problem 6 Solution

6. Consider the following vector field in space

$$\overrightarrow{\mathbf{F}} = \langle x+y, x+z, y \rangle .$$

(a) Check that this field is conservative.

(b) Find a potential of $\overrightarrow{\mathbf{F}}$.

(c) Evaluate the following line integral

$$\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{r}},$$

where C is a contour originating at $(0,0,0)$ and terminating at $(0,1,1)$.

Solution:

(a) Let $P = x+y$, $Q = x+z$, and $R = y$. Given that $P_y = Q_x = 1$, $P_z = R_x = 0$, and $Q_z = R_y = 1$ we know that $\overrightarrow{\mathbf{F}}$ is conservative by the cross-partials test.

(b) By inspection, a potential function for $\overrightarrow{\mathbf{F}}$ is $\varphi(x,y,z) = \frac{1}{2}x^2 + xy + yz$.

(c) Using the Fundamental Theorem of Line Integrals, we obtain

$$\begin{aligned}
\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{r}} &= \varphi(0,1,1) - \varphi(0,0,0), \\
&= \left(\frac{1}{2}(0)^2 + 0\cdot 1 + 1\cdot 1\right) - \left(\frac{1}{2}(0)^2 + 0\cdot 0 + 0\cdot 0\right), \\
&= 1
\end{aligned}$$

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Problem 7 Solution

7. Compute the circulation of the vector field

$$\vec{\mathbf{H}} = \langle -y^3, x^3 \rangle$$

over the boundary of the region $D = \{x^2 + y^2 \le 1, y \ge 0\}$.

Solution: The boundary of D is a simple, closed curve oriented counter clockwise. Therefore, we may use Green's Theorem to compute the circulation:

$$\oint_{\partial D} \vec{\mathbf{H}} \bullet d\vec{\mathbf{r}} = \iint_D (Q_x - P_y)\, dA$$

Letting $P = -y^3$ and $Q = x^3$ we get $Q_x = 3x^2$ and $P = -3y^2$. Therefore, $Q_x - P_y = 3(x^2 + y^2)$. Since D is a half-disk, we will use polar coordinates to evaluate the double integral above. The integrand then becomes $3r^2$, $dA = r\,dr\,d\theta$, and the region D can be described as $\{0 \le r \le 1,\ 0 \le \theta \le \pi\}$. Thus, the circulation is

$$\begin{aligned}\oint_{\partial D} \vec{\mathbf{H}} \bullet d\vec{\mathbf{r}} &= \iint_D (Q_x - P_y)\, dA, \\ &= \int_0^{\pi} \int_0^1 3r^2 \cdot r\,dr\,d\theta, \\ &= \int_0^{\pi} \left[\frac{3}{4} r^4\right]_0^1 d\theta, \\ &= \int_0^{\pi} \frac{3}{4}\, d\theta, \\ &= \frac{3\pi}{4}\end{aligned}$$

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Problem 8 Solution

8. Compute the volume of the spherical wedge given in spherical coordinates by

$$W = \left\{1 \le \rho \le 2,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\right\}$$

Solution: Using spherical coordinates, the volume of the wedge is computed as follows

$$\begin{aligned}
V &= \iiint_W 1\, dV, \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta, \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{1}{3}\rho^3 \sin\phi\right]_1^2 d\phi\, d\theta, \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \frac{7}{3} \sin\phi\, d\phi\, d\theta, \\
&= \int_0^{\pi/2} \left[-\frac{7}{3}\cos\phi\right]_0^{\pi/2} d\theta, \\
&= \int_0^{\pi/2} \frac{7}{3}\, d\theta, \\
&= \frac{7\pi}{6}
\end{aligned}$$