HYPERBOLIC ACTIONS OF HIGHER-RANK LATTICES COME FROM RANK-ONE FACTORS

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Abstract. We study actions of higher rank lattices $\Gamma < G$ on hyperbolic spaces, and we show that all such actions satisfying mild properties come from the rank-one factors of G. In particular, all non-elementary actions on an unbounded hyperbolic space are of this type. Our results also apply to lattices in products of trees, so that for example Burger–Mozes groups have exactly two non-elementary actions on a hyperbolic space, up to a natural equivalence.

1. Introduction

How can a given group act by isometries on a hyperbolic space? The aim of this paper is to study this question for irreducible lattices in a semisimple group G of rank ≥ 2. Thomas Haettel [15] addresssed the case where all simple factors of the ambient product G have rank ≥ 2 and showed, in that case, that the isometric actions of the lattice on hyperbolic spaces are all degenerate (see below for a more precise formulation). In this paper, we allow G to have simple factors of rank 1. Since rank 1 simple groups have a natural geometric action on a proper hyperbolic space (namely, a symmetric space or a tree), the lattices in G do admit non-degenerate actions on hyperbolic spaces via their projections on the rank 1 simple factors of G. We show that, up to a natural equivalence, those are the only actions of lattices in G on hyperbolic spaces. Our results also cover some non-linear groups including, for example, lattices in products of trees.

1.A. Generalities on actions on hyperbolic spaces.

Before stating our main theorem, we now explain some general facts about actions on hyperbolic spaces. First of all, any group has actions on hyperbolic spaces that fix a bounded set, as well as actions that fix a point at infinity. Such actions can therefore not be used to deduce anything about the group: from our viewpoint, they are degenerate, and we will disregard them. Moreover, given an action on a hyperbolic space, one could make a larger hyperbolic space containing the first one as a quasiconvex subspace, maintaining the group action. This can be done, for example, by attaching equivariantly geodesic rays. To take this possibility into account, it is natural to also rule out actions that admit a quasi-convex invariant set that is not coarsely dense. In view of all this we define *coarsely minimal* actions (Definition 3.4) by, essentially, ruling out the pathological behaviours discussed above. Arguably, those are the most general actions that one might want to classify. Moreover, actions on hyperbolic spaces that admit an equivariant quasi-isometry should be considered

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equivalent, and we capture this in Definition 3.5, where there is a subtlety to deal with actions where given subgroups fix a bounded set rather than single points.

1.B. **Higher rank groups and generalizations.** As hinted at above we will cover more than higher rank Lie groups in our main result. For this, we include the following notion of a *standard rank one group* following [9, Theorem D]: a locally compact group G is a **standard rank one group** if it has no nontrivial compact normal subgroups and either

(1) G is the group of isometries or orientation-preserving isometries of a rank one symmetric space X of noncompact type, or
(2) G has a continuous, proper, faithful action by automorphisms on a locally finite non-elementary tree T, without inversions and with exactly two orbits of vertices, such that the action of G on the set of ends ∂T is 2-transitive.

The symmetric space X in case (1) and the tree T in case (2) is called the **model space** for the standard rank one group G. While standard rank one groups of type (1) correspond to real Lie groups of rank one, type (2) includes, but is not restricted to, simple algebraic groups over non-archimedean local fields of rank one.

Theorem 1.1. *Let $N \geq n \geq 0$ be integers. Let $G = \prod_{i=1}^{N} G_i$ be a product of N locally compact groups, where for all $i \in \{1, \dots, n\}$, G_i is a standard rank one group, and for all $j \in \{n+1, \dots, N\}$, G_j is a simple algebraic group defined over a local field k_j with $\mathrm{rk}_{k_j}(G_j) \geq 2$. Let $\Gamma < G$ be a lattice. Assume that $n \geq 2$ or that $N > n$. If $N > 1$, assume in addition that Γ has a dense projection to each proper sub-product.*

Then any coarsely minimal action of Γ on a geodesic hyperbolic space is equivalent to one of the actions

$$\Gamma \longrightarrow G \xrightarrow{\mathrm{pr}_i} G_i \longrightarrow \mathrm{Isom}(X_i, d_i) \qquad (1 \leq i \leq n)$$

where each X_i is a rank-one symmetric space or a tree, corresponding to the standard rank one factor G_i being of type (1) or (2).

As mentioned above, we refer to Definition 3.5 for the precise notion of *equivalence* appearing in the theorem.

In the special case where G consists only of a single higher rank factor, that is the case where $n = 0$ and $N = 1$, our considerations recover the main theorem of [15].

Corollary 1.2. *Let $G = \mathbf{G}(k)$ be a simple algebraic group defined over a local field k, with $\mathrm{rk}_k(G) \geq 2$, and $\Gamma < G$ a lattice. Then Γ does not admit any coarsely minimal action on a geodesic hyperbolic space.*

One can also view Theorem 1.1 as a generalization of Margulis' [17], where he studied possible amalgam decompositions of lattices in higher rank.

1.C. **Hyperbolic structures.** The setup adopted here is inspired by the notion of *hyperbolic structures*, defined in [1] to capture *cobounded* actions on hyperbolic spaces. Coarsely minimal actions provide a similar but broader setup (see [1, Proposition 3.12] for a comparison). In what follows, we regard a hyperbolic structure as an equivalence class (in the sense of Definition 3.5) of cobounded actions on hyperbolic spaces. Any such action is either coarsely minimal, or the hyperbolic space being acted on is bounded (giving rise to what is called the trivial structure).

That is, the number of hyperbolic structures up to equivalence is the number of coarsely minimal action up to equivalence plus one.

Therefore, in the language of [1], Theorem 1.1 implies that the lattices under consideration have exactly $n+1$ inequivalent hyperbolic structures. Note that in [1], for every integer $n \geq 1$ the authors construct a finitely generated group Γ admitting precisely n distinct hyperbolic structures; irreducible lattices in higher-rank semi-simple groups provide naturally occurring examples of that same phenomenon. Note that any lattice in a higher-rank simple Lie group has only the trivial hyperbolic structure by either [15] or Theorem 1.1.

Example 1.3 (Groups with $n \geq 2$ non-trivial hyperbolic structures)**.**

Choose $n-1$ distinct primes $p_2, \ldots, p_n$ and consider the group

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p_2}, \ldots, \frac{1}{p_n}])$$

that embeds as an irreducible lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_{p_2}) \times \cdots \times \mathrm{SL}_2(\mathbb{Q}_{p_n})$. Theorem 3.7 implies that it has precisely n non-trivial hyperbolic structures that arise from its actions on the hyperbolic plane $\mathbb{H}^2$ via pr_1, or on the (p_i+1)-regular tree T_i via pr_i for $i = 2, \ldots, n$. These structures can also be viewed as coming from the Cayley graphs $X(\Gamma, S_i)$, where $S_1 = \{\gamma \in \Gamma \ / \ \| \mathrm{pr}_i(\gamma) \| \leq 1 + \epsilon_1\}$ for an arbitrary fixed $\epsilon_1 > 0$, and for $i \geq 2$ the set S_i consists of those $\gamma \in \Gamma$ that contain the prime p_i in the denominators of the matrix elements in power not exceeding 1.

Example 1.4 (A group with a single non-trivial hyperbolic structure)**.**
Consider the quadratic form $q(x_1, \ldots, x_5) = x_1^2 + x_2^2 + x_3^2 + \sqrt{2}x_4^2 - x_5^2$, its orthogonal group $\mathrm{SO}(q) = \{g \in \mathrm{SL}_5 \ / \ q \circ g = q\}$, and let $\Gamma = \mathrm{SO}(q)_{\mathbb{Z}[\sqrt{2}]}$ be the group of its integer points. This group has only one non-trivial hyperbolic structure, because Γ is an irreducible lattice in the semi-simple real Lie group $\mathrm{SO}(4,1) \times \mathrm{SO}(3,2)$ that has a single rank-one factor $\mathrm{SO}(4,1) \simeq \mathrm{Isom}(\mathbb{H}^4)$ and another simple $\mathrm{SO}(3,2)$ factor of rank two.

Example 1.5 (Each Burger–Mozes group has two non-trivial hyperbolic structures)**.**
M. Burger and S. Mozes [8] have constructed irreducible lattices $\Gamma < G_1 \times G_2$ in a product of two standard rank one groups $G_i < \mathrm{Aut}(T_i)$, and proved many remarkable properties of these groups. Theorem 3.7 shows that each of these lattices Γ has precisely two distinct non-trivial hyperbolic structures, coming from their actions on the trees T_1 and T_2.

1.D. **Outline of proofs.** In the proofs we will use boundary theory as outlined in [4]. Roughly, given a group Γ one can associate to it a Lebesgue space B, called Γ-boundary, which on one hand has very strong ergodic properties, and on the other hand has the property that whenever Γ acts on a compact space Z, there is a Γ-equivariant map $B \longrightarrow \mathrm{Prob}(Z)$, called a **boundary map**. In Section 2 we will study a general group Γ acting nicely on a hyperbolic space X, and show that in this case the boundary map actually takes its values in the Gromov boundary ∂X of X, and enjoy various extra rigidity properties. The case where X is proper had been considered already in [4], and indeed the main result of the section is a direct generalization of [4, Theorem 3.2]. To deal with the case of potentially non-proper spaces, we make use of the horoboundary and its relation to the Gromov boundary.

Similar strategies were considered by Duchesne in [10] and by Maher and Tiozzo in [16]. Related ideas appeared already in the much earlier work [17] of Margulis, where he studied actions of higher rank lattices on trees.

In Section 3 we specialize to the case where Γ is a (generalized) higher rank lattice, as in our main theorem. In this case the Γ-boundary of Γ splits as a product, with factors corresponding to the factors of the ambient locally compact group G. Due to the ergodicity properties of Γ-boundaries, we see that when Γ acts nicely on a hyperbolic space X, the boundary map from the Γ-boundary to ∂X factors through one of the algebraic factors of the ambient group G. At this point, there are two cases to analyse. The first case is when the said factor is of rank ≥ 2: we have to show that this cannot occur. This is done in Subsection 3.E by adapting the Weyl group method of Bader–Furman [3]. The second case is that the factor as above corresponds to a rank-one factor G_i of G. In that case, we have to show that X is equivalent to the model space X_i for G_i (a symmetric space or a tree). This is done in Subsections 3.F and 3.G. By hypothesis, the group G has at least two factors in this case, the projection of Γ to G_i has dense image. To show the equivalence between X_i and X, metric properties are transferred from X_i to X via the boundary map. A key ingredient is that bounded subsets of X_i correspond to precompact subsets of G_i, and the latter property can be rephrased in terms of the boundedness of Radon–Nikodym derivatives for the action on the G_i-boundary.

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2. Boundary maps

In this section we fix the group Γ and discuss *boundary maps* associated to various hyperbolic structures on Γ. In particular we prove Theorem 2.3 below. The novel aspect of this theorem is that the Gromov hyperbolic spaces it deals with are not assumed to be proper. Recall that the Gromov boundary of a proper Gromov hyperbolic space is compact and the associated action is a *convergence group* action. Boundary maps associated with such actions were considered in [4, Theorem 3.2]. Thus Theorem 2.3 below is an extension of [4, Theorem 3.2]. Our main task in this section is to recall the setting of the latter and to explain the required adjustments in its proof.

Let us first recall some definitions. Let G be a locally compact second countable group. This includes the case of countable discrete group Γ. A *Lebesgue G-space* is a Lebesgue space (Ω, μ) with a measurable, measure class preserving action map $G \times \Omega \longrightarrow \Omega$. A *Borel G-space V* is a standard Borel space V with a Borel action map $G \times V \longrightarrow V$. Given a Lebesgue G-space Ω and a standard Borel G-space V, we denote by $\mathrm{Map}_G(\Omega, V)$ the space of equivalence classes of measurable maps $f : \Omega \to V$ that satisfy $f(g.\omega) = g.f(\omega)$ for a.e. $g \in G$ and a.e. $\omega \in \Omega$, where $f, f' : \Omega \to V$ are identified if $f(\omega) = f'(\omega)$ for a.e. $\omega \in \Omega$. Any such map f is equivalent to $f_0 : \Omega \to V$ such that for every $g \in G$ we have $f_0(g.\omega) = g.f_0(\omega)$ a.e. $\omega \in \Omega$ ([18, Proposition B.5]).

We say that a Lebesgue G-space Ω is *metrically ergodic* if given any separable metric space (S, d) and a continuous homomorphism $\pi : G \to \mathrm{Isom}(S, d)$, the only G-equivariant measurable maps $F : \Omega \longrightarrow S$ are essentially constant ones, i.e.

$$\mathrm{Map}_G(\Omega, S) = \mathrm{Map}_G(\{*\}, S).$$

Let $p : \Omega \longrightarrow \Sigma$ be a measurable, measure class preserving, G-equivariant map between Lebesgue G-spaces. We say that p is a *relatively metrically ergodic* map if for any measurable family $\{(S_y, d_y)\}_{y\in\Sigma}$ of separable metric spaces, with a measurable family

$$\{\pi_y(g) : (S_y, d_y) \longrightarrow (S_{gy}, d_{gy})\} \qquad (g \in G,\ y \in \Sigma)$$

of isometries with $\pi_y(gh) = \pi_{hy}(g) \circ \pi_y(h)$, the only G-equivariant measurable maps $\{F(x) \in S_{p(x)}\}_{x\in\Omega}$ are pull-backs $F = f \circ p$ of measurable G-equivariant family $\{f(y) \in S_y\}_{y\in\Sigma}$. In particular, for any fixed separable metric space (S, d) and any continuous homomorphism $\pi : G \longrightarrow \mathrm{Isom}(S, d)$ we have a natural isomorphism $\mathrm{Map}_G(\Omega, S) \simeq \mathrm{Map}_G(\Sigma, S)$.

Definition 2.1 ([4, Definition 2.3])**.** A pair (B_-, B_+) of Lebesgue G-spaces forms a *boundary pair* if the actions $G \curvearrowright B_-$ and $G \curvearrowright B_+$ are amenable, and the projections

$$\mathrm{pr}_- : B_- \times B_+ \longrightarrow B_-, \qquad \mathrm{pr}_+ : B_- \times B_+ \longrightarrow B_+$$

are relatively metrically ergodic. A Lebesgue G-space B for which (B, B) is a boundary pair will be called a *G-boundary.*

We recall the following facts (see [4, §2]).

Proposition 2.2. (a) *Any lcsc group G admits a boundary pair (and also G-boundaries), arising from Furstenberg–Poisson boundaries associated with a generating and admissible probability measure μ and its reflection $\check{\mu}$ on G (see* [4, Theorem 2.7]*).*

(b) *For a simple Lie group G the quotient $B = G/P$ by a minimal parabolic subgroup $P < G$, equipped with a G-invariant measure class, gives a G-boundary (see* [4, Theorem 2.5]*). More generally, for a local field k, (the k-points of) a k-simple group G and a k-minimal parabolic subgroup $P < G$, G/P is a G-boundary (see* [2, Example 2.14]*).*

(c) *For a locally compact group G which acts continuously, properly, by automorphisms on a locally finite tree, such that the boundary action is 2-transitive, G/P is a G-boundary, where $P < G$ the stabilizer of a boundary point, (see* [2, Example 2.15]*).*

(d) *Let $G = G_1 \times \cdots \times G_N$ be a product of lcsc groups, and let $(B_-^{(i)}, B_+^{(i)})$ be G_i-boundary pairs, $1 \le i \le N$. Then $B_- = B_-^{(1)} \times \cdots \times B_-^{(N)}$ and $B_+ = B_+^{(1)} \times \cdots \times B_+^{(N)}$ form a boundary pair for G.*

(e) *Let $\Gamma < G$ be a lattice. Then any G-boundary pair (B_-, B_+) is a boundary pair for Γ.*

In the following theorem (X, d) is a separable Gromov hyperbolic space with a group Γ acting by isometries on X. We denote by ∂X Gromov boundary, ∂X^2 its square, and

$$\partial X^{(2)} = \{(\xi, \eta) \ /\ \xi \neq \eta \in \partial X\}$$

the subset of distinct pairs of boundary points. Since X is not assumed to be proper, ∂X is not necessarily compact. Yet, ∂X is a standard Borel space, and so are $\partial X^{(2)} \subset \partial X^2$. The action of Γ on all these spaces is Borel.

Theorem 2.3 (cf. [4, Theorem 3.2])**.**
Let (B_+, B_-) be a boundary pair for Γ. Let (X, d) be a separable, Gromov hyperbolic (possibly non-proper), geodesic metric space and assume that Γ acts continuously and isometrically on X. Denote by ∂X the Gromov boundary of X and recall it is a Polish space (possibly non-compact) on which Γ acts continuously. Assume that Γ does not fix a bounded set in X and does not fix a point or a pair of points in ∂X.

Then there exist $\phi_- \in \mathrm{Map}_\Gamma(B_-, \partial X)$, $\phi_+ \in \mathrm{Map}_\Gamma(B_+, \partial X)$ such that the image of the map $\phi_{\bowtie} \in \mathrm{Map}_\Gamma(B_- \times B_+, \partial X^2)$ given by

$$\phi_{\bowtie}(x, y) = (\phi_-(x), \phi_+(y))$$

is essentially contained in the set of distinct pairs $\partial X^{(2)} \subset \partial X^2$. Moreover:

(i) $\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(\partial X)) = \{\delta \circ \phi_-\}$, *and* $\mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(\partial X)) = \{\delta \circ \phi_+\}$.
(ii) $\mathrm{Map}_\Gamma(B_- \times B_+, \partial X) = \{\phi_- \circ \mathrm{pr}_-, \phi_+ \circ \mathrm{pr}_+\}$,
(iii) $\mathrm{Map}_\Gamma(B_- \times B_+, \partial X^{(2)}) = \{\phi_{\bowtie},\ \tau \circ \phi_{\bowtie}\}$, *where* $\tau(\xi, \xi') = (\xi', \xi)$.

The rest of this section is devoted to the proof of this Theorem.

2.A. **The horoclosure of a separable metric space.** Let (X, d) be a separable metric space. We consider the space of functions from X to $\mathbb{R}$ endowed with the pointwise convergence topology, i.e the product space $\mathbb{R}^X$, and the constant function $\mathbf{1} \in \mathbb{R}^X$. We endow $\mathbb{R}^X/\mathbb{R} \cdot \mathbf{1}$ with the quotient topological vector space structure. We map X to $\mathbb{R}^X$ by $x \mapsto d(\cdot, x)$ and consider its image in $\mathbb{R}^X/\mathbb{R} \cdot \mathbf{1}$. We denote the closure of the image of X in $\mathbb{R}^X/\mathbb{R} \cdot \mathbf{1}$ by $\bar{X}$ and call it the *horoclosure* of X. We denote the obvious map $X \to \bar{X}$ by i, and the preimage of $\bar{X}$ in $\mathbb{R}^X$ by $\tilde{X}$. Elements of $\bar{X}$ (and by abuse of notations, also elements of $\tilde{X}$) are called *horofunctions*. It is a common practice to fix a base point $x \in X$ and to consider the subspace

$$\tilde{X} \supset \tilde{X}_x = \left\{ h \in \tilde{X} \ / \ h(x) = 0 \right\}.$$

Lemma 2.4. *$\bar{X}$ is a compact metrizable space and the map $i : X \to \bar{X}$ is an injective continuous map. For a fixed $x \in X$, the map $\tilde{X}_x \to \bar{X}$ is a homeomorphism.*

Proof. The fact that i is continuous is obvious. For $x \neq y$ in X, note that the difference function $d(x, \cdot) - d(y, \cdot)$ is not constant, as it attains different values at x and y. Thus i is injective. We now fix $x \in X$. First we note that $\tilde{X}_x$ is closed subset of

$$\prod_{y \in X} [-d(x, y), d(x, y)] \subset \prod_{y \in X} \mathbb{R} = \mathbb{R}^X,$$

thus it is compact. Fixing a countable dense subset X_0 in X, the obvious map $\tilde{X}_x \to \mathbb{R}^X \to \mathbb{R}^{X_0}$ is a continuous injection (as $\tilde{X}_x$ consists of continuous functions), hence a homeomorphism onto its image. The image is a Frechet space, thus metrizable. It follows that $\tilde{X}_x$ is metrizable. Since the natural map $\tilde{X}_x \to \bar{X}$ is also a continuous bijection, we conclude that it is a homeomorphism and deduce that $\bar{X}$ is compact and metrizable. $\square$

Loosely speaking, we identify many times X with $i(X) \subset \bar{X}$. Note however that the image of X is in general not open in $\bar{X}$ and the map i is not a homeomorphism onto its image.

We decompose $\tilde{X}$ as follows.

$$\tilde{X}^b = \left\{ h \in \tilde{X} \ / \ f \text{ is bounded from below} \right\},$$
$$\tilde{X}^u = \left\{ h \in \tilde{X} \ / \ f \text{ is unbounded from below} \right\}.$$

This decomposition is constant on the fibers of $\tilde{X} \to \bar{X}$, thus gives a corresponding decomposition $\bar{X} = \bar{X}^b \cup \bar{X}^u$. Clearly we have $i(X) \subseteq \bar{X}^b$, so that $\bar{X}^b$ is dense in $\bar{X}$.

Lemma 2.5. *The decompositions $\tilde{X} = \tilde{X}^b \cup \tilde{X}^u$ and $\bar{X} = \bar{X}^b \cup \bar{X}^u$ are measurable and* $\mathrm{Isom}(X)$*-equivariant.*

Proof. The equivariance of the decompositions is obvious. Fix a dense countable subset X_0 in X and use the fact that $\tilde{X}$ consists of continuous functions to note that

$$\tilde{X}^u = \bigcap_{n\in\mathbb{N}} \bigcup_{x\in X_0} \left\{ h \in \tilde{X} \ / \ h(x) \leq -n \right\},$$

thus $\tilde{X}^u \subset \tilde{X}$ is measurable. Fixing $x \in X$, using the measurability of $\tilde{X}^u_x$, we observe that $\bar{X}^u \subset \bar{X}$ is measurable. □

We denote by $\mathrm{Bdd}(X)$ the space of closed non-empty bounded subsets of X and endow it with the Hausdorff metric.

Lemma 2.6. *The Borel σ-algebra on* $\mathrm{Bdd}(X)$ *is generated by the collection*

$$\mathcal{C} = \{ K \in \mathrm{Bdd}(X) \ / \ K \subset U, \ U \ \textit{open in } X \}.$$

Proof. Denote by $\langle \mathcal{C} \rangle$ the σ-algebra generated by the collection $\mathcal{C}$. For every $x \in X$ and $r > 0$, the set $\{K \ / \ d(K,x) > r\}$ is in $\mathcal{C}$, thus $\{K \ / \ d(K,x) \leq r\}$ is in $\langle \mathcal{C} \rangle$. Given any subset $K_0 \in \mathrm{Bdd}(X)$, using a dense countable subset K_0' in K_0, we get that

$$\{K \ / \ \forall\, x \in K_0, \ d(K,x) \leq r\} = \bigcap_{x\in K_0'} \{K \ / \ d(K,x) \leq r\} \in \langle \mathcal{C} \rangle.$$

Note also that

$$\{K \ / \ \forall\, x \in K, \ d(K_0,x) \leq r\} = \bigcap_n \left\{ K \ / \ \forall\, x \in K, \ d(K_0,x) < r + \frac{1}{n} \right\} \in \langle \mathcal{C} \rangle$$

and that the intersection of the two sets above consists of the closed ball of radius r around K_0 in $\mathrm{Bdd}(X)$. As the open ball of radius r around K_0 is given by the union of the closed balls of radius $r - 1/n$ around K_0, we conclude that all open balls in $\mathrm{Bdd}(X)$ are in $\langle \mathcal{C} \rangle$. □

2.B. **The horoclosure of a hyperbolic metric space.** We now assume in addition that the separable metric space (X,d) is geodesic and Gromov hyperbolic (as before, it is possibly non-proper).

Lemma 2.7. *The function*

$$\inf : \tilde{X}^b \to \mathbb{R}, \quad h \mapsto \inf \{ h(x) \ / \ x \in X \}$$

is measurable and $\mathrm{Isom}(X)$*-invariant. For every* $h \in \tilde{X}_b$*, the set*

$$\tilde{I}(h) = \overline{\{x \in X \ / \ h(x) < \inf(h) + 1\}}$$

is bounded in X*. The obtained map* $\tilde{I} : \tilde{X}^b \to \mathrm{Bdd}(X)$ *is measurable and factors via* $\tilde{X}^b \to \bar{X}^b$*, defining a measurable map* $I : \bar{X}^b \to \mathrm{Bdd}(X)$*. The maps* $\tilde{I}$ *and* I *are* $\mathrm{Isom}(X)$*-equivariant.*

Remark 2.8. In fact, as the proof below shows, the sets in the image of I are uniformly bounded.

Proof. To see that inf is measurable, fix a dense countable subset X_0 in X and use the continuity of the functions in $\tilde{X}^b$ to observe that

$$\inf(h) = \inf \{h(x) \ / \ x \in X_0\}.$$

The invariance of this function is clear.

Fix $h \in \tilde{X}_b$. We argue to show that $\tilde{I}(h)$ is of diameter bounded by $C = 8 + 4\delta$, where δ is the hyperbolicity constant associated with the thin triangles property of X. Without loss of generality we assume that $\inf(h) = 0$. Assuming the negation, we fix two points x, x' satisfying $d(x, x') > C$ and $h(x), h(x') < 1$. We consider a finite sequence of points $x_0, x_1, \ldots, x_n$ on a geodesic segment from x to x' such that $x_0 = x$, $x_n = x'$ and $d(x_i, x_{i+1}) < 1$. We consider the image of h in $\bar{X}$ along with its neighborhood given by

$$U = \left\{ f + \mathbb{R} \cdot \mathbf{1} \ / \ f \in \mathbb{R}^X, \ \forall 0 \le i, j \le n, \ |\left(f(x_i) - f(x_j)\right) - \left(h(x_i) - h(x_j)\right)| < 1 \right\}.$$

We fix a point $y \in X$ whose image in $\bar{X}$ is in U. We thus have:

$$\forall 0 \le i, j \le n, \quad |d(y, x_i) - h(x_i) - d(y, x_j) + h(x_j)| < 1. \tag{2.1}$$

We consider geodesic segments from y to x and from y to x' and, using that x, x' and y are the vertices of a thin triangle, we fix i such that x_i lies at distance at most $1 + \delta$ from these segments. Thus

$$d(y, x_i) + d(x_i, x) \le d(y, x) + 2 + 2\delta,$$
$$d(y, x_i) + d(x_i, x') \le d(y, x') + 2 + 2\delta.$$

Note that $d(x, x_i) + d(x_i, x') = d(x, x')$. Upon possibly interchanging the roles of x and x', we will assume that $d(x, x_i) \ge d(x, x')/2$. In particular, $d(x, x_i) \ge C/2$. Taking $j = 0$ in Equation (2.1), we now have

$$\begin{aligned}
0 &= \inf(h) \\
&\le h(x_i) \\
&< 1 + d(y, x_i) + h(x) - d(y, x) \\
&\le 1 + (d(y, x) + 2 + 2\delta - d(x_i, x)) + h(x) - d(y, x) \\
&< (3 + h(x)) + 2\delta - d(x_i, x) \\
&\le 4 + 2\delta - C/2 = 0.
\end{aligned}$$

This is a contradiction, thus indeed the diameter of $\tilde{I}(h)$ is bounded by C.

We now turn to prove the measurability of $\tilde{I}$ and I. Fix an open set U in X. Observe that for a countable dense subset F_0 in $X - U$,

$$\tilde{I}^{-1}(\{K \mid K \subset U\}) = \bigcap_{x \in F_0} \left\{ h \in \tilde{X} \ / \ h(x) - \inf(h) \ge 1 \right\},$$

thus this is a measurable set. By Lemma 2.6, it follows that $\tilde{I}$ is measurable. The fact that $\tilde{I}$ factors via $\tilde{X}^b \to \bar{X}^b$ is clear. To see that the obtained map $I : \bar{X}^b \to \mathrm{Bdd}(X)$ is measurable, use Lemma 2.4 and the fact that, fixing $x \in X$, $\tilde{I}$ is measurable on $\tilde{X}^b_x$. The fact that $\tilde{I}$ and I are $\mathrm{Isom}(X)$-equivariant is straightforward. □

2.C. **Measurable barycenters.** We now describe for each $\epsilon \in (0, 1/2)$ a measurable "barycenter" $\mathrm{Isom}(X)$-map

$$\beta_\epsilon : \mathrm{Prob}(\mathrm{Bdd}(X)) \longrightarrow \mathrm{Bdd}(X)$$

to be the set of centers of balls of almost minimal radius containing $(1-\epsilon)$ mass of sets.

More precisely, given a probability measure m on $\mathrm{Bdd}(X)$, a point $x \in X$ and $R < \infty$, consider the ball $B(x, R) = \{y \in X \ / \ d(x, y) < R\}$ and the value

$$F_{m,x}(R) := m\,\{A \in \mathrm{Bdd}(X) \ / \ A \subset B(x, R)\}\,.$$

Clearly $F_{m,x}(R) \to 1$ as $R \to 1$. So for every $x \in X$ one has a well defined finite $R_{x,\epsilon,m} = \inf\{R \ / \ F_{m,x}(R) > 1-\epsilon\}$. Define $R^*_{\epsilon,m} := \inf\{R_{x,\epsilon,m} \ / \ x \in X\}$. We set

$$\beta_\epsilon(m) = \overline{\left\{x \in X \ / \ R_{x,\epsilon,m} < R^*_{\epsilon,m} + 1\right\}},$$

which is easily seen to be a bounded set, since if we have points $x_1, x_2 \in X$ so that $m\left\{A \in \mathrm{Bdd}(X) \ / \ A \subset B(x_i, R^*_{\epsilon,m} + 1)\right\} > 1/2$ for $i = 1, 2$, then we must have $B(x_1, R^*_{\epsilon,m} + 1) \cap B(x_2, R^*_{\epsilon,m} + 1) \neq \varnothing$.

From the definition, β_ϵ is $\mathrm{Isom}(X)$-equivariant.

Lemma 2.9. *Given $C > 0$ and $\epsilon \in (0, 1/2)$, if $m_1, m_2 \in \mathrm{Prob}(\mathrm{Bdd}(X))$ are such that for every measurable $E \in \mathrm{Bdd}(X)$*

$$C^{-1} m_1(E) \le m_2(E) \le C m_1(E)$$

then

$$\beta_{\epsilon/C}(m_2) \subseteq N_{R(\epsilon,m_1)}(\beta_\epsilon(m_1)),$$

where $R(\epsilon, m_1) = R^*_{\epsilon,m_1} + R^*_{\epsilon/C^2,m_1} + 3$.

Proof. This is similar to the reason why $\beta_\epsilon(m)$ is bounded. Suppose that x_1, x_2 are so that

$$m_1\left\{A \ / \ A \subset B(x_1, R^*_{\epsilon,m_1} + 1)\right\} > 1-\epsilon$$

and

$$m_2\left\{A \ / \ A \subset B(x_2, R^*_{\epsilon/C,m_2} + 1)\right\} > 1-\epsilon/C.$$

Then $m_1\left\{A \in \mathrm{Bdd}(X) \ / \ A \subset B(x_2, R^*_{\epsilon/C,m_2} + 1)\right\} > 1-\epsilon$, hence

$$B(x_1, R^*_{\epsilon,m_1} + 1) \cap B(x_2, R^*_{\epsilon/C,m_2} + 1) \neq \varnothing,$$

yielding $d(x_1, x_2) \le R^*_{\epsilon,m_1} + R^*_{\epsilon/C,m_2} + 2$.

Moreover, $R^*_{\epsilon/C,m_2} \le R^*_{\epsilon/C^2,m_1} + 1$, because given any x with the property that $m_1\left\{A \ / \ A \subset B(x, R^*_{\epsilon/C^2,m_1} + 1)\right\} > 1-\epsilon/C^2$, by comparability of the measure we also have that

$$m_2\left\{A \in \mathrm{Bdd}(X) \ / \ A \subset B(x, R^*_{\epsilon/C^2,m_1} + 1)\right\} > 1-\epsilon/C.$$

□

2.D. **The Gromov boundary.** While the construction of the Gromov boundary ∂X is fairly standard, it is commonly taken under a properness assumption on X. In preparation for our more general discussion we review this construction below. As common, we fix from now on a base point $o \in X$. For $x \in X$ we use the shorthand notation $|x| = d(o,x)$. Gromov products will be taken, unless otherwise stated, with respect to o. That is, for $x, y \in X$ we set

$$(x,y) = \frac{1}{2}\left(|x| + |y| - d(x,y)\right).$$

In our discussion below we fix $\delta > 0$ such that for every $x, y, z \in X$ we have

$$(x,z) \geq \min\{(x,y),(y,z)\} - \delta.$$

We recall that a sequence of points (x_n) in X is said to *converge to infinity* if (x_n, x_m) converges to infinity when both m and n do.

Lemma 2.10. *Assume (x_n) is a sequence of points in X which converges in $\bar{X}$ and denote $\bar{h} = \lim x_n$. Then (x_n) converges to infinity if and only if $\bar{h} \in \bar{X}^u$. In that case, if (x'_n) is another sequences in X satisfying $\lim x'_n = \bar{h}$ then $(x_n, x'_n) \to \infty$.*

Proof. We will denote the lift of $\bar{h}$ in $\tilde{X}_o$ by h and show that (x_n) converges to infinity if and only if $h \in \tilde{X}^u_o$. Note that for every $x \in X$, $d(x_n, x) - |x| \to h(x)$.

Assuming first (x_n) converges to infinity, we will show that $h \in \tilde{X}^u_o$. Fix $r > 0$. Fix N such that for $n, m > N$, $(x_n, x_m) > r$. Fix $m > N$, note that $|x_m| \geq r$ and let x be a point on a geodesic segment from o to x_m with $|x| = r$. Then by hyperbolicity,

$$(x_n, x) \geq \min\{(x_n, x_m), (x_m, x)\} - \delta = r - \delta,$$

Thus

$$h(x) = \lim_{n\to\infty}\left(d(x_n, x) - |x_n|\right) = \lim_{n\to\infty}\left(|x| - 2(x_n, x)\right) \leq 2\delta - r.$$

As r was arbitrary, indeed we get that $h \in \tilde{X}^u_o$.

Assuming now $h \in \tilde{X}^u$, we will show that (x_n) converges to infinity. Fix $r > 0$. Fix x such that $h(x) < -r$. Fix N such that for every $n > N$, $d(x_n, x) - |x_n| < -r$ and observe that for such n,

$$(x_n, x) = \frac{1}{2}\left(|x_n| + |x| - d(x_n, x)\right) \geq -\frac{1}{2}\left(d(x_n, x) - |x_n|\right) > \frac{1}{2}r.$$

Then by hyperbolicity, for $n, m > N$,

$$(x_n, x_m) \geq \min\{(x_n, x), (x, x_m)\} - \delta > \frac{1}{2}r - \delta.$$

As r was arbitrary, indeed we get that the sequence (x_n) converges to infinity.

In the setting of the former paragraph, if (x'_n) is another sequences in X satisfying $x'_n \to \bar{h}$, fixing $N' \geq N$ such that for every $n > N'$, $d(x'_n, x) - |x'_n| < -r$, the same computation shows that $(x_n, x'_n) > r/2 - \delta$. Thus indeed, $(x_n, x'_n) \to \infty$. □

Two sequences which converge to infinity, (x_n) and (y_n), are said to be equivalent if $(x_n, y_n) \to \infty$. We conclude that if (x_n) and (x'_n) are two sequences in X satisfying

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n \in \bar{X}^u$$

then (x_n) is equivalent to (x'_n).

A point in ∂X is, by definition, an equivalence class of sequences which converge to infinity. We denote by π the unique map $\pi : \bar{X}^u \to \partial X$ satisfying

$$\lim_{n\to\infty} x_n \in \bar{X}^u \qquad \Longrightarrow \qquad \pi(\lim_{n\to\infty} x_n) = [x_n].$$

For a point $\xi \in \partial X$ and $r > 0$ we set

$$U(\xi, r) = \left\{ \eta \in \partial X \ / \ \sup \left\{ \liminf_{n\to\infty} (x_n, x'_n) \ / \ (x_n) \in \xi, \ (x'_n) \in \eta \right\} \geq r \right\}.$$

We note that the collection of sets $U(\xi, r)$ forms a basis for a topology and endow ∂X with the corresponding topology.

Lemma 2.11. *The map π is continuous and* $\mathrm{Isom}(X)$*-equivariant.*

Proof. The equivariance of π is obvious. In order to show continuity, we fix a point $\bar{h} \in \bar{X}^u$ and show the continuity of π at $\bar{h}$. Thus we fix $r > 0$ and argue to show that there exists a neighborhood V of $\bar{h}$ in $\bar{X}^u$ such $\pi(V) \subset U(\pi(\bar{h}), r)$. We will denote the lift of $\bar{h}$ in $\tilde{X}^u_o$ by h, set $t = 2(r + \delta)$ and fix a point $x \in X$ such that $h(x) < -t$. We let $V \subset \bar{X}^u_o$ be the open neighborhood of $\bar{h}$ corresponding to the set $\{h' \in \tilde{X}^u_o \mid h'(x) < -t\}$. Fix $\bar{h}' \in V$ and denote its lift in $\tilde{X}^u_o$ by h'. Let (x_n) and (x'_n) be sequences in X converging to $\bar{h}$ and $\bar{h}'$ respectively. In particular,

$$h(x) = \lim_{n\to\infty} \left(d(x_n, x) - |x_n|\right), \qquad h'(x) = \lim_{n\to\infty} \left(d(x'_n, x) - |x'_n|\right).$$

Fix N such that for every $n > N$ both $(d(x_n, x) - |x_n|) < -t$ and $(d(x'_n, x) - |x'_n|) < -t$. Note that for $n > N$

$$(x_n, x) = \frac{1}{2}\left(|x_n| + |x| - d(x_n, x)\right) \geq -\frac{1}{2}\left(d(x_n, x) - |x_n|\right) > \frac{1}{2}t$$

and similarly $(x'_n, x) > t/2$. Thus

$$(x_n, x'_n) \geq \min\{(x_n, x), (x, x'_n)\} - \delta > \frac{1}{2}t - \delta = r.$$

It follows that $\liminf (x_n, x'_n) \geq r$ and in particular,

$$\sup \left\{ \liminf_{n\to\infty} (x_n, x'_n) \ / \ (x_n) \in \pi(\bar{h}), \ (x'_n) \in \pi(\bar{h}') \right\} \geq r.$$

Thus, $\pi(\bar{h}') \in U(\pi(\bar{h}), r)$. We conclude that indeed $\pi(V) \subset U(\pi(\bar{h}), r)$. $\square$

Lemma 2.12. *Assume $\Gamma < \mathrm{Isom}(X)$ is a countable group. Then there is a Borel Γ-map*

$$\tau : \partial X^{(3)} \longrightarrow \mathrm{Bdd}(X).$$

Proof. Of course, the idea is just that τ gives the coarse center of an ideal triangle, but some care is needed because X might not be proper, and because we want a Borel map. Let $\delta > 0$ be a hyperbolicity constant for X.

Fix a dense, countable, Γ-invariant subset $C \subseteq X$. For $x \in X$, let $T_x \subseteq X^3$ be the set of all triples (x_1, x_2, x_3) so that $d(x_i, x_j) > d(x_i, x) + d(x, x_j) - 10\delta$ for all distinct $i, j \in \{1, 2, 3\}$. We then define $\tau(a) = \overline{\{x \in C : a \in \overline{T_x}\}}$, where the closure of T_x is taken in $(X \cup \partial X)^3$. Notice that $\tau(a)$ is indeed a bounded, closed, non-empty subset of X, and that τ is Γ-equivariant.

Let us now show that τ is Borel. Let U be an arbitrary open set in X and let $B_U = \{K \in \mathrm{Bdd}(X) : K \subset U\}$. To show that τ is Borel, it suffices to show that $\tau^{-1}(B_U) = \{a \in \partial X^{(3)} : \tau(a) \subseteq U\}$ is a Borel set. Fix an exhaustion $\{U_n\}$ of U. Notice that $\tau(a) \subseteq U$ if and only if there exists n so that $\{x \in C : a \in \overline{T_x}\} \subseteq U_n$. In turn, we have $\{x \in C : a \in \overline{T_x}\} \subseteq U_n$ if and only if $a \notin \overline{T_x}$ for all $x \in C - U_n$. Hence, setting $A_n = \{a \in \partial X^{(3)} : a \notin \overline{T_x}\ \forall x \in C - U_n\}$, we have $\tau^{-1}(B_U) = \bigcup_{n\in\mathbb{N}} A_n$. Hence, it suffices to show that each A_n is Borel. We have $A_n = \bigcap_{x\in C-U_n}\{a \in \partial X^{(3)} : a \notin \overline{T_x}\}$, so that A_n is a countable intersection of closed sets, and we are done. □

2.E. **Digression: Atom-less measures.** We now show a result needed in the next section, but not needed for the proof of Theorem 2.3; we include it here since we established the setup for its proof.

Given a hyperbolic space X, denote by $\mathrm{Prob}_c(\partial X)$ be the set of all atom-less probability measures on the standard Borel space ∂X.

Lemma 2.13. *Given a countable group Γ acting on the hyperbolic space X. Then there is a Γ-map*

$$\Psi : \mathrm{Prob}_c(\partial X) \longrightarrow \mathrm{Prob}(\mathrm{Bdd}(X)).$$

Proof. We have a Γ-map

$$\mathrm{Prob}_c(\partial X) \longrightarrow \mathrm{Prob}(\partial X^3), \qquad \mu \mapsto \mu\times\mu\times\mu.$$

In fact, the assumption that μ has no atoms on a space ∂X implies that $\mu\times\mu\times\mu$ gives zero mass to the diagonals in $\partial X\times\partial X\times\partial X$, and so is fully supported on $\partial X^{(3)}$. We thus get a well defined map

$$\mathrm{Prob}_c(\partial X) \longrightarrow \mathrm{Prob}(\partial X^{(3)}), \qquad \mu \mapsto \mu\times\mu\times\mu.$$

By Lemma 2.12 there is a Borel Γ-map

$$\partial X^{(3)} \longrightarrow \mathrm{Bdd}(X).$$

We therefore obtain a Γ-map

$$\Psi : \mathrm{Prob}_c(\partial X) \longrightarrow \mathrm{Prob}(\mathrm{Bdd}(X)),$$

as required. □

2.F. **Proof of Theorem 2.3.** We start with a preliminary claim that we will use a few times.

Claim 2.14. *We have*

$$\mathrm{Map}_\Gamma(B_-\times B_+, \mathrm{Prob}(\mathrm{Bdd}(X))) = \varnothing$$

and therefore

$$\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(\mathrm{Bdd}(X))) = \mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(\mathrm{Bdd}(X))) = \varnothing.$$

Proof. It suffices to rule out Γ-maps $f : B_-\times B_+ \to \mathrm{Prob}(\mathrm{Bdd}(X))$. If we had such map, by composing with a map β_ϵ we would then also have a Γ-equivariant map $B_-\times B_+ \to \mathrm{Bdd}(X)$. By metric ergodicity (where we are thinking of $\mathrm{Bdd}(X)$ as a metric space), any such map is essentially constant, with value a fixed point of Γ in the space $\mathrm{Bdd}(X)$. Since we are assuming that Γ has unbounded orbits in X, this is impossible. □

By amenability of $\Gamma \curvearrowright B_\pm$, there exist Γ-equivariant maps $\phi''_\pm : B_\pm \to \mathrm{Prob}(\bar{X})$.

Claim 2.15. *We have:*

$$\mathrm{Map}_\Gamma(B_-, \mathrm{Prob}(\bar{X}^b)) = \mathrm{Map}_\Gamma(B_+, \mathrm{Prob}(\bar{X}^b)) = \varnothing.$$

Proof. Recall from Lemma 2.7 that we have a Γ-equivariant measurable map $I : \bar{X}^b \to \mathrm{Bdd}(X)$. Hence, if we had a map f as above, we would also have a Γ-equivariant map $B_\pm \to \mathrm{Prob}(\mathrm{Bdd}(X))$, contradicting Claim 2.14. $\square$

In view of the claim, for a.e. $b \in B_\pm$ we have that the support of $\phi''_\pm(b)$ must be contained in $\bar{X}^u$, so that we can think of $\phi''_\pm$ as a map $B_\pm \to \mathrm{Prob}(\bar{X}^u)$. We can then compose $\phi''_\pm$ with $\pi_* : \mathrm{Prob}(\bar{X}^u) \to \mathrm{Prob}(\partial X)$, to obtain the maps

$$\phi'_\pm : B_\pm \longrightarrow \mathrm{Prob}(\partial X).$$

Claim 2.16. *For a.e. $b \in B_\pm$, we have that $\phi'_\pm(b) = \delta_{\xi_\pm}$ for some $\xi_\pm \in \partial X$.*

For later purposes, we remark that the proof applies to any Γ-maps $\phi'_\pm : B_\pm \to \mathrm{Prob}(\partial X)$ (meaning not necessarily obtained in the way described above).

Proof. We consider the Γ-equivariant map $\psi : B_- \times B_+ \to \mathrm{Prob}((\partial X)^3)$ defined by

$$\psi(b_-, b_+) = \phi'_-(b_-) \times \phi'_+(b_+) \times \frac{1}{2}(\phi'_-(b_-) + \phi'_+(b_+)).$$

By ergodicity of $\Gamma \curvearrowright B_- \times B_+$, either the image of ψ is essentially contained in $\mathrm{Prob}((\partial X)^{(3)})$, or it is essentially contained in $\mathrm{Prob}(\Delta(\partial X))$, where $\Delta(\partial X)$ is the set of triples so that at least two entries coincide.

In the latter case, we see that $\psi(b_-, b_+)$ is atomic with at most two atoms for a.e. (b_-, b_+), which implies that $\phi'_\pm(b)$ is a Dirac measure for a.e. $b \in B_\pm$, as we wanted. Hence, we have to rule out the first case.

If, by contradiction, the image of ψ is essentially contained in $\mathrm{Prob}((\partial X)^{(3)})$, then in view of Lemma 2.12 we also have a Γ-map $B_- \times B_+ \to \mathrm{Prob}(\mathrm{Bdd}(X))$, which does not exist by Claim 2.14. This concludes the proof. $\square$

In view of the previous claim, we have Γ-equivariant maps

$$\phi_\pm : B_\pm \longrightarrow \partial X, \quad \text{defined by} \quad \phi'_\pm(b) = \delta_{\phi_\pm(b)}.$$

Now we have to show various properties.

First, we show that the image of $\phi_{\bowtie} = \phi_- \times \phi_+$ is essentially contained in $\partial X^{(2)}$. If not, by ergodicity the image would be essentially contained in the diagonal. By varying the coordinates in $B_- \times B_+$ separately, we see that this would imply that both maps $\phi_\pm$ are essentially constant, which is impossible because Γ does not fix any point in ∂X.

Next, we show that $\phi_\pm$ are essentially unique. This is because Claim 2.16 shows that any Γ-map $B_\pm \to \mathrm{Prob}(\partial X)$ has image essentially contained in the set of Dirac measures. However, if we had two essentially distinct Γ-maps $\phi^i_\pm : B_\pm \to \partial X$, then we would have the Γ-map $B_\pm \to \mathrm{Prob}(\partial X)$ given by $b \mapsto \frac{1}{2}(\delta_{\phi^1_\pm(b)} + \delta_{\phi^1_\pm(b)})$, which contradicts the said property.

Consider now $\psi \in \mathrm{Map}_\Gamma(B_- \times B_+, \partial X)$, and let $\Psi = \psi \times (\phi \circ \mathrm{pr}_-) \times (\phi \circ \mathrm{pr}_+)$, which is a Γ-map $B_- \times B_+ \to (\partial X)^3$. As in the proof of Claim 2.16, if the image was essentially contained in $(\partial X)^{(3)}$, then in view of Lemma 2.12 we would have a Γ-map $B_- \times B_+ \to \mathrm{Prob}(\mathrm{Bdd}(X))$, which does not exist by Claim 2.14. Hence, Ψ essentially takes values in $(\partial X)^3 \setminus (\partial X)^{(3)}$, and since the image of $\phi_{\bowtie}$ is essentially contained in $(\partial X)^{(2)}$, more precisely Ψ essentially takes value in either $\{(x, y, z) : x = y\}$ or $\{(x, y, z) : x = z\}$. This is equivalent to (ii).

Finally, notice that (iii) can be deduced from (ii) by looking at the coordinates in $(\partial X)^{(2)}$. $\square$

3. Classification of actions on hyperbolic spaces

3.A. **Rank-one groups.** In order to work with rank-one Lie groups and automorphisms of trees simultaneously, we fix the following setup:

Setup 3.1. Consider a locally compact group G that acts continuously, properly and cocompactly by isometries on a proper hyperbolic space X with $|\partial X| \geq 3$, and has a compact subgroup K that acts transitively on the boundary. We call X the **model space** for G.

Remark 3.2. Standard rank-one groups as defined in the introduction, that is, simple algebraic groups over a local field and groups of automorphisms of a tree acting 2-transitively on the boundary both fit the above setup (in the latter case, the model space is the tree being acted on). In fact, by [9, Theorem 8.1] a group G as in Setup 3.1 is a standard rank-one group up to modding out a compact kernel.

Lemma 3.3. *Any group G as in Setup 3.1 admits a G-boundary (B, ν) with the property that given any precompact subset $\{\gamma_i\} \subseteq G$, the corresponding Radon-Nikodym derivatives are uniformly bounded meaning:*

$$\sup \| \frac{d\gamma_i \nu}{d\nu} \|_\infty < +\infty.$$

Proof. One way to prove this involves appealing to Proposition 2.2 via [9, Theorem 8.1]; we also give another argument below as it might be of interest.

By [12, Theorem 1.4(1)], for suitable measures on G the Furstenberg-Poisson boundary can be realized by a measure on ∂X (to check that the theorem applies note that the action having bounded exponential growth follows from cocompactness of G and properness of X, and note also the Furstenberg-Poisson boundary is not trivial in our case since G is non-amenable in view of the assumption that ∂X has at least 3 points). By Proposition 2.2-(a) we can take $B = \partial X$. Furthermore, letting K be as in Setup 3.1, we can take the measure on G to be K-invariant, and hence ν will also be K-invariant. Note that K acts transitively on ∂X by the assumptions from Setup 3.1.

Endow ∂X with any visual metric ρ, that is, any metric bilipschitz equivalent to $e^{-\epsilon(\cdot,\cdot)_o}$ for a fixed $\epsilon > 0$ and basepoint o for the Gromov product. We will use the fact that $(\partial X, \rho)$ is a doubling metric space (see [5, Theorem 9.2]), that is, there exists a constant N such that all balls in ∂X can be covered by at most N balls of half the radius. We will also use the fact that precompact subsets of G act by uniformly bilipschitz homeomorphisms of ∂X. This follows from the fact the Gromov product changes a bounded amount when changing the basepoint, meaning that $|(\cdot,\cdot)_o - (\cdot,\cdot)_{o'}| \leq d(o, o')$. This in particular applies to K, and we denote by L_0 the corresponding bilipschitz constant.

Now, we claim that there exists a constant C_0 such that for any $R > 0$ and any balls B, B' in ∂X of radii R and $R/2$ respectively (possibly centered at a different point), we have $\nu(B)/\nu(B') \leq C_0$. This is because B can be covered by boundedly many (at most $N^{\lceil \log_2(2L_0) \rceil}$) balls of radius $R/(2L_0)$, and each such ball has measure at most that of B' since it can be mapped inside B' using an element of K.

Fix now any precompact subset $\{\gamma_i\} \subseteq G$, and let L be the corresponding bilipschitz constant for the action on the boundary. By the bound on ratios of measures of balls, this implies that there exists a constant C such that for any ball B in ∂X and any i we have $\nu(\gamma_i B)/\nu(B) \leq C$. This implies the desired bound on Radon-Nikodym derivatives in view of the Lebesgue differentiation theorem (for doubling metric measure spaces, see [11, Theorem 2.9.8, Theorem 2.8.17]). □

3.B. **Lattices acting on hyperbolic spaces.** Recall that a subset A of a metric space is coarsely dense if there exists a constant R such that X is the R-neighborhood of A. Also recall that given a group Γ acting on a hyperbolic space X, its limit set is the set of boundary points that are equivalence classes of sequence of points γx, for some fixed $x \in X$.

Definition 3.4. We say that an action $\Gamma \curvearrowright X$ on a hyperbolic space is *coarsely minimal* if X is unbounded, the limit set of Γ in ∂X is not a single point, and every quasi-convex Γ-invariant subset of X is coarsely dense.

Note that coarse minimality is a stronger requirement than asking that the orbits of Γ have full limit set (for example, start with the action of a hyperbolic group on its Cayley graph and attach arbitrarily long geodesics equivariantly).

Notice that if H is an infinite normal subgroup of infinite index of the hyperbolic G, then the action of H on a Cayley graph of G is coarsely minimal, but not cobounded.

Given a metric space X and $C \geq 0$, denote by $\mathrm{Bdd}_C(X)$ the set of all closed subsets of diameter at most C, endowed with the Hausdorff metric. Notice that $\mathrm{Bdd}_C(X)$ is quasi-isometric to X.

Definition 3.5. Two actions $\Gamma \curvearrowright X_1, X_2$ on metric spaces X_1, X_2 are *equivalent* if there exists an equivariant quasi-isometry $X_1 \to \mathrm{Bdd}_C(X_2)$ for some $C \geq 0$.

The reason for having $\mathrm{Bdd}_C(X_2)$ instead of X_2 is that we want to allow the situation where some group element has a fixed point in X_1 but merely a bounded orbit in X_2; for example, we want to declare all actions on bounded metric spaces to be equivalent.

Remark 3.6. Consider an action $\Gamma \curvearrowright X$ on a geodesic hyperbolic space.

(1) If the action is cobounded, then it is coarsely minimal.

The following two items follow from a construction well-known to experts, namely taking the coarse convex hull of an orbit and approximating it with a graph; this is explained for example in [14, Remark 4].

(2) If the limit set of Γ is not a single point, then there is a coarsely minimal action $\Gamma \curvearrowright Y$ on a geodesic hyperbolic space Y and an equivariant quasi-isometric embedding $Y \to X$.

(3) If Γ is countable and $\Gamma \curvearrowright X$ is coarsely minimal, then $\Gamma \curvearrowright X$ is equivalent to an action on a separable geodesic hyperbolic space (in fact, a graph).

Consider a locally compact group $G = G_1 \times \cdots \times G_N$ where each factor is either a simple algebraic group over a local field k_i of rank at least 2, or a standard rank-one group. Also, assume that either $N \geq 2$ or $N = 1$ and $G = G_1$ is a simple algebraic group as above of rank at least 2.

Re-order the factors in such a way that G_i is a standard rank-one group if and only if $1 \leq i \leq n$, for some $n \leq N$.

We now re-state our main theorem, in the context of standard rank-one groups.

Theorem 3.7. *Let Γ be an irreducible lattice in $G = G_1 \times \cdots \times G_N$ as above. Then every coarsely minimal action of Γ on a geodesic hyperbolic space is equivalent to* *the action*

$$\Gamma \longrightarrow G \xrightarrow{\mathrm{pr}_i} G_i \longrightarrow \mathrm{Isom}(X_i, d_i)$$

for some $i \in \{1, \ldots, n\}$*, where* X_i *is the model space for* G_i*.*

Let us extend the notation of the theorem by denoting X_j, for $j > n$, the symmetric space for G_j.

Let $\Gamma \curvearrowright (X, d)$ be a coarsely minimal action of Γ on the geodesic hyperbolic space X. Denote by d_i the pseudo-metric on Γ corresponding to $\Gamma \curvearrowright X_i$ (with respect to some basepoint x_i, for $i \leq n$), and d the pseudo-metric corresponding to $\Gamma \curvearrowright (X, d)$.

3.C. **Ruling out elementary actions.** In order to be able to apply Theorem 2.3, we have to rule out that Γ fixes a pair of points in ∂X (the case that it fixes one point being ruled out by hypothesis). If that were the case, the subgroup Γ' of index at most 2 of Γ that fixes a boundary point would admit the quasimorphism described in [9, Proposition 3.7]. According to [6, 7], Γ' does not admit unbounded quasimorphisms, so that according to [9, Lemma 3.8] the action $\Gamma \curvearrowright X$ has a single limit point, contradicting minimality of the action. From now on, we will assume that Γ does not fix a point or a pair of points in X.

Finally, we can assume that X is separable by Remark 3.6.

3.D. **Boundary map from one factor.** For $i \leq n$ (the rank one factors), we let (B_i, ν_i) be a boundary as in Lemma 3.3. For $i > n$ (the higher rank factors), we let $B_i = G_i/P_i$ and we let ν_i be a measure in the Haar class on B_i. By Proposition 2.2, (B_i, ν_i) is a G_i-boundary. Moreover, again by the Proposition, $B = B_1 \times \cdots \times B_N$ is a G-boundary, hence a Γ-boundary.

Theorem 2.3 affords now two Γ-maps $B \to \partial X$ satisfying various properties. The first of these properties implies that these two maps must coincide a.e. We assume henceforth that both maps are identical, and we denote it by $\phi\colon B \to \partial X$.

Claim 3.8. *The map* ϕ *factors through one of* B_i*: there is* $i \in \{1, \ldots, N\}$ *and a* Γ *map* $B_i \xrightarrow{\phi_i} \partial X$ *such that*

$$\phi : B \xrightarrow{\mathrm{pr}_i} B_i \xrightarrow{\phi_i} \partial X.$$

Proof. As ∂X contains no Γ-fixed points and by the ergodicity of $B \times B$, the map ϕ is not constant. It thus depend on B_i for some i, which we now fix. We let B' be the product of the other factors. We thus identify $B \simeq B_i \times B'$. Using this identification we consider the map

$$\Phi : B_i \times B' \times B_i \times B' \to \partial X^2, \quad (x, y, x', y') \mapsto (\phi(x, y), \phi(x, y')).$$

By Theorem 2.3(iii) we have three cases: $\Phi(B \times B)$ is contained in the diagonal $\Delta \subset \partial X^2$, $\Phi = \phi_{\bowtie}$, or $\Phi = \tau \circ \phi_{\bowtie}$, where $\phi_{\bowtie} = \phi \times \phi$ and $\tau(m, m') = (m', m)$. In the first case we see that $\phi(x, y)$ is independent of $y \in B'$, and therefore descends to a Γ-map $B_i \to \partial X$ as required. In the second and third cases, ϕ is independent of $x \in B_i$, contradicting our choice of i. $\square$

From now on we fix i to be such that ϕ factors via B_i.

3.E. **The factor B_i is associated with a rank one factor.**

We now explain that the factor B_i alluded to in Claim 3.8 is associated with a rank one factor, that is $i \leq n$. We argue by contradiction, assuming $i > n$, that is, G_i is of higher rank.

As before, we let $P_i < G_i$ be a minimal parabolic. In the sequel we will omit the index i and denote $P = P_i$. Let $A < P$ be a maximal split torus. We let $W = N_{G_i}(A)/Z_{G_i}(A)$ be the corresponding Weyl group and let $S \subset W$ be the standard Coxeter generators associated with the positivity defined by P. Letting $Z = Z_{G_i}(A)$ be the centralizer of A, we note that W acts naturally on G_i/Z by G_i-automorphisms.

As usual we identify the set S with the set of simple roots of G_i associated with the pair (A, P). Any subset T of S generates a subgroup $W_T < W$ and it corresponds to a standard parabolic $P_T < G_i$ containing $P = P_\varnothing$. All the subgroups of G_i containing P are of this form. Denoting by $\pi_T : G_i/Z \to G_i/P_T$ the standard map coming from the inclusion $Z < P < P_T$, we note that

$$W_T = \{w \in W \mid \pi_T \circ w = \pi_T\}.$$

We let $\pi = \pi_\varnothing : G_i/Z \to G_i/P$ be the standard map and let $w_0 \in W$ be the longest element (with respect to the word distance induced by S). It is a standard fact that map $\pi \times \pi \circ w_0 : G_i/Z \to G_i/P \times G_i/P$ is injective and its image is Zariski open (this image is the big cell in the Bruhat decomposition of $G_i/P \times G_i/P$).

Recall that $B_i = G_i/P$ is endowed with the Haar measure class. We also endow G_i/Z with the Haar measure class and identify it, as measured G_i spaces, with $B_i \times B_i = G_i/P \times G_i/P$ via the map $\pi \times \pi \circ w_0$. We set $\phi_i : B_i \to \partial X$ to be the map given in Claim 3.8. Note that ϕ_i is not essentially constant, as ∂X has no Γ fixed points. We thus may find a bounded measurable function $f_0 : \partial X \to \mathbb{C}$ such that $f_0 \circ \phi_i$ is not essentially constant. We fix such a function f_0.

We consider the map $\psi = \phi_i \circ p_1 : B_i \times B_i \longrightarrow \partial X$, where $p_1 : B_i \times B_i \longrightarrow B_i$ is the projection on the first factor, and let

$$U = \{w \in W \mid \psi \circ w \text{ agrees a.e with } \psi\} < W.$$

By Theorem 2.3(ii) we conclude that $U < W$ is of index at most 2.

Consider now the algebra $L^\infty(G_i/Z)$ and its subalgebra $\pi^*(L^\infty(G_i/P))$ consisting of functions pulled back from $L^\infty(G_i/P)$ under $\pi : G_i/Z \to G_i/P$ (which we identify with $p_1 : B_i \times B_i \to B_i$). Consider the subalgebra

$$\{f \in L^\infty(G_i/Z) \mid f \in L^\infty(G_i/P) \text{ and for every } u \in U,\ f \circ u \text{ agrees a.e with } f\}.$$

This is a weak*-closed G_i-invariant subalgebra of $L^\infty(G_i/Z)$. By Mackey's point realization theorem this algebra coincides with the subalgebra of functions pulled back from a G_i-factor of G_i/Z. As all functions in it are pulled back from G_i/P, this factor is of the form G_i/P_T for some $T \subset S$. As the algebra includes the non-constant function $f_0 \circ \psi$, we conclude that $P_T \neq G_i$, thus $T \neq S$ and $W_T \neq W$. We have that $\pi_T \circ u = \pi_T$ for every $u \in U$, thus $U < W_T$. It follows that W_T is of index 2 in W and in particular it is a normal subgroup.

As G_i is of higher rank, $|W| > 2$, thus W_T is non-trivial. Consider the standard reflection representation V of W. This is a faithful representation. By simplicity of G_i, the Coxeter system (W, S) is irreducible thus the representation V is irreducible. W_T has non-trivial invariant vectors in V. Indeed, it preserves a proper, non-trivial face of the Weyl chamber. As W_T is normal and V is irreducible, W_T is in the kernel of V. This contradicts the non-triviality of W_T, as V is faithful.

3.F. **Bounded in G_i is d-bounded.**

Next, we show that d is "smaller" than d_i, for i as in Claim 3.8.

Claim 3.9. *There exist L, C so that for all γ, γ' we have*

$$d(\gamma, \gamma') \leq L \cdot d_i(\gamma, \gamma') + C.$$

As G is of higher rank, we know that $N > 1$ and in particular, we get that $\mathrm{pr}_i(\Gamma)$ is dense in G_i. Hence, in the metric d_i any pair of points is connected by a $(1,1)$-quasi-geodesic, and to prove the claim it suffices to show that sequences that are bounded in G_i are bounded in (Γ, d). In other words, it suffices to show that any sequence $\{\gamma_j\}$ in Γ for which $\{\mathrm{pr}_i(\gamma_j)\}$ is precompact in G_i, one has

$$\sup_j d(\gamma_j, 1) < +\infty.$$

Recall that we denote the G_i-boundaries by (B_i, ν_i); let us drop the subscript i from ν_i and let $\mu = \phi_* \nu \in \mathrm{Prob}(\partial X)$ be its pushforward. By the metric ergodicity of B_i, μ has no atomic part. Indeed, if it had we would get a countable invariant subset of ∂X and upon endowing it with the discrete metric, in view of the metric ergodicity of B_i, we will conclude that this set contains a single point which is Γ invariant, contradicting our assumption that ∂X is fixed point free.

By Lemma 2.13, we have a Γ-map

$$\Psi : \mathrm{Prob}_c(\partial X) \longrightarrow \mathrm{Prob}(\mathrm{Bdd}(X)),$$

where recall that $\mathrm{Prob}_c(\partial X)$ is the set of all atom-less probability measures on ∂X.

We assume that $\{\gamma_j\}$ is such that $\{\mathrm{pr}_i(\gamma_j)\}$ is precompact in G_i.

By Lemma 3.3 we have:

$$\sup_{j \geq 1} \|\frac{d\gamma_j \nu}{d\nu}\|_\infty = C < +\infty.$$

We thus have the same bound on the Radon-Nikodym derivatives of μ

$$\sup_{j \geq 1} \|\frac{d\gamma_j \mu}{d\mu}\|_\infty \leq C$$

and also

$$\sup_{j \geq 1} \|\frac{d\Psi(\gamma_j \mu)}{d\Psi(\mu)}\|_\infty \leq C.$$

Then for $\epsilon \in (0, 1/2)$ as in Lemma 2.9, we have that for all n the bounded sets

$$\gamma_j(\beta_{\epsilon/C} \circ \Psi(\mu)) = \beta_{\epsilon/C} \circ \Psi(\gamma_j \mu)$$

all lie in a neighborhood of finite radius of $\beta_\epsilon \circ \Psi(\mu)$. We thus have that the sequence $\{\gamma_j\}$ is bounded in (Γ, d).

3.G. **Unbounded in G_i is d-unbounded.**

Claim 3.10. *There exist L, C so that for all γ, γ' we have*
$$d_i(\gamma, \gamma') \leq L \cdot d(\gamma, \gamma') + C.$$

Let $\gamma_0 \in \Gamma$ be a loxodromic element for $\Gamma \curvearrowright X$, which exists by Gromov's classification of actions on hyperbolic spaces [13, Section 8]. Since the identity map $(\Gamma, d_i) \to (\Gamma, d)$ is coarsely Lipschitz, γ_0 is loxodromic for $\Gamma \curvearrowright X_i$ as well. Let us choose a constants A so that for each $j \geq 0$ we have $j/A \leq d_i(1, \gamma_0^j) \leq Aj$ and $j/A \leq d(1, \gamma_0^j) \leq Aj$. Note that, up to increasing A, since G_i is rank one and γ_0 is loxodromic, the set
$$\{\kappa\gamma_0^j \mid \kappa \in K_i,\ j \geq 0\}$$
is A-dense in X_i, where we denote by K_i the compact subgroup as in Setup 3.1. This is because, since K_i acts transitively on ∂X, there exists a constant C such that given any two points on a sphere around x_i, there is an element of K_i that moves the first point C-close to the second one.

Since the rank of G_i equals one, we know by hypothesis that $N > 1$ and in particular, we get that $\mathrm{pr}_i(\Gamma)$ is dense in G_i. Approximating elements of K_i by elements of Γ we get that the set
$$\{\kappa\gamma_0^j \mid d_i(1, \kappa) \leq 1,\ j \geq 0\}$$
is $A+1$-dense in Γ for the metric d_i. Enlarging A if necessary, let us further assume that $d(1, \gamma) \leq A d_i(1, \gamma) + A$ for all $\gamma \in \Gamma$.

Consider an arbitrary $\gamma \in \Gamma$. Using the above we find $\kappa \in \Gamma$ such that $d_i(1, \kappa) \leq 1$ and $d_i(\kappa\gamma, \gamma_0^j) \leq A + 1$ for some $j \geq 0$. Therefore, we obtain
$$\begin{aligned}
d_i(1, \gamma) &= d_i(\kappa, \kappa\gamma) \\
&\leq d_i(1, \gamma_0^j) + A + 2 \\
&\leq Aj + A + 2 \\
&\leq A^2 d(1, \gamma_0^j) + A + 2 \\
&\leq A^2(d(1, \kappa) + d(\kappa, \kappa\gamma) + d(\kappa\gamma, \gamma_0^j)) + A + 2 \\
&\leq A^2 d(1, \gamma) + A^3 + 2A^2 + 3A + 2,
\end{aligned}$$
as required.

3.H. **Conclusion.** We have seen in Section 3.E that the rank of G_i is 1. Claims 3.9 and 3.10 imply that there is a Γ-equivariant quasi-isometric embedding $f : X_i \to \mathrm{Bdd}_C(X)$ for some $C \geq 0$ (since (Γ, d_i) is Γ-equivariantly quasi-isometric to X_i, and there is a Γ-equivariant quasi-isometric embedding $(\Gamma, d_i) \to X$). Since X_i is hyperbolic, the image of that quasi-isometric embedding is quasi-convex. Since the Γ-action on X is coarsely minimal, it follows that the Γ-action on X is cobounded, so that the quasi-isometric embedding $(\Gamma, d_i) \to X$ is in fact a quasi-isometry. This proves that $\Gamma \curvearrowright X_i$ is indeed equivalent to $\Gamma \curvearrowright X$, as required. □

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