On the structure and arithmeticity of lattice envelopes

Uri Bader [a], Alex Furman [b], Roman Sauer [c]

[a] *Technion, Haifa, Israel*

[b] *University of Illinois at Chicago, Chicago, USA*

[c] *Karlsruhe Institute of Technology, Karlsruhe, Germany*

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Abstract

We announce results about the structure and arithmeticity of all possible lattice embeddings of a class of countable groups which encompasses all linear groups with simple Zariski closure, all groups with non-vanishing first ℓ^2-Betti number, non-elementary acylindrically hyperbolic groups, and non-elementary convergence groups.

Résumé

Sur la structure et l'arithméticité des groupes enveloppant un réseau. Nous nous intéressons à l'ensemble des plongements possibles d'un groupe dénombrable comme réseau dans un groupe localement compact. Pour une grande classe de groupes dénombrables, nous annonçons des résultats de structure et d'arithméticité de tels plongements. Cette classe contient tous les groupes linéaires dont l'adhérence de Zariski est simple, les groupes dont le premier nombre de Betti ℓ^2 est non nul, les groupes hyperboliques acylindriques, et les groupes de convergence.

1. Introduction

Let Γ be a countable group. We are concerned with the study of its lattice envelopes, i.e. the locally compact groups containing Γ as a lattice. We aim at structural results that impose no restrictions on the ambient locally compact group and only abstract group-theoretic conditions on Γ. We say that Γ satisfies (†) if every finite index subgroup of a quotient of Γ by a finite normal subgroup

(†1) is not isomorphic to a product of two infinite groups, and

Email addresses: uri.bader@gmail.com (Uri Bader), furman@math.uic.ed (Alex Furman), roman.sauer@kit.edu (Roman Sauer).

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(†2) does not possess infinite amenable commensurated subgroups, and

(†3) satisfies: For a normal subgroup N and a commensurated subgroup M with $N \cap M = \{1\}$ there exists a finite index subgroup $M' < M$ such that N and M' commute.

The relevance of (†1) should be clear, the relevance of (†2) is that it yields an information about all possible lattice envelopes of Γ [1]:

Proposition 1.1 *Let Γ be a lattice in a locally compact group G. If Γ has no infinite amenable commensurated subgroups, then the amenable radical $R(G)$ of G is compact.*

The role of (†3) is less transparent, but be aware of the obvious observation: If M, N are both normal with $N \cap M = \{1\}$, then M and N commute. There are lattices Γ in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{Aut}(T)$, where T is the universal cover of the 1-skeleton $B^{(1)}$ of the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$ (see [4, 6.C] and [3, Prop. 1.8]). They are built in such a way that $N := \pi_1(B^{(1)})$, which is a free group of infinite rank, is a normal subgroup of Γ. Let $U < \mathrm{Aut}(T)$ be the stabilizer of a vertex. Then $M := \Gamma \cap (\mathrm{SL}_n(\mathbb{R}) \times U)$ is commensurated, but $M, N < \Gamma$ violate (†3). Moreover, this group Γ satisfies (†1) and (†2).

Proposition 1.2 *Linear groups with semi-simple Zariski closure satisfy conditions (†2) and (†3). Groups with some positive ℓ^2-Betti number satisfy condition (†2). All the (†) conditions are satisfied by all linear groups with simple Zariski closure, by all groups with positive first ℓ^2-Betti number, and by all non-elementary acylindrically hyperbolic groups and convergence groups.*

For a concise formulation of our main result, we introduce the following notion of S-arithmetic lattice embeddings up to tree extension: Let K be a number field. Let $\mathbf{H}$ be a connected, absolutely simple adjoint K-group, and let S be a set of places of K that contains every infinite place for which $\mathbf{H}$ is isotropic and at least one finite place for which $\mathbf{H}$ is isotropic. Let $\mathcal{O}_S \subset K$ denote the S-integers. The (diagonal) inclusion of $\mathbf{H}(\mathcal{O}_S)$ into $\prod_{\nu \in S} \mathbf{H}(K_\nu)$ is the prototype example of an S-arithmetic lattice. Let H be a group obtained from $\prod_{\nu \in S} \mathbf{H}(K_\nu)^+$ by possibly replacing each factor $\mathbf{H}(K_\nu)^+$ with K_ν-rank 1 by an intermediate closed subgroup $\mathbf{H}(K_\nu)^+ < D < \mathrm{Aut}(T)$ where T is the Bruhat-Tits tree of $\mathbf{H}(K_\nu)$. The lattice embedding of $\mathbf{H}(\mathcal{O}_S) \cap H$ into H is called an *S-arithmetic lattice embedding up to tree extension.*

A typical example is $\mathrm{SL}_2(\mathbb{Z}[1/p])$ embedded diagonally as a lattice into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$. The latter is a closed cocompact subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{Aut}(T_{p+1})$, where T_{p+1} is the Bruhat-Tits tree of $\mathrm{SL}_2(\mathbb{Q}_p)$, i.e. a $(p+1)$-regular tree. So $\mathrm{SL}_2(\mathbb{Z}[1/p]) < \mathrm{SL}_2(\mathbb{R}) \times \mathrm{Aut}(T_{p+1})$ is an S-arithmetic lattice embedding up to tree extension. We now state the main result [1]:

Theorem 1.3 *Let Γ be a finitely generated group satisfying (†), e.g. one of the groups considered in Proposition 1.2. Then every embedding of Γ as a lattice into a locally compact group G is, up to passage to finite index subgroups and dividing out a normal compact subgroup of G, isomorphic to one of the following cases:*

(i) an irreducible lattice in a center-free, semi-simple Lie group without compact factors;

(ii) an S-arithmetic lattice embedding up to tree extension, where S is a finite set of places;

(iii) a lattice in a totally disconnected group with trivial amenable radical.

The same conclusion holds true if one replaces the assumption that Γ is finitely generated by the assumption that G is compactly generated.

Finite generation of Γ implies compact generation of any locally compact group containing Γ as a lattice. The examples above for $n \geq 3$ show that condition (†3) in Theorem 1.3 is indispensable. Since non-uniform lattices with a uniform upper bound on the order of finite subgroups do not exist in totally disconnected groups, our main theorem yields the following classification of non-uniform lattice embeddings.

Corollary 1.4 *Let Γ be a group that satisfies (†) and admits a uniform upper bound on the order of all finite subgroups. Then every non-uniform lattice embedding of Γ into a compactly generated locally compact group G is, up to passage to finite index subgroups and dividing out a normal compact subgroup of G, either a lattice in a center-free, semi-simple Lie group without compact factors or an S-arithmetic lattice embedding up to tree extension.*

For groups Γ that are not lattices in Lie groups (classical or S-arithmetic) but satisfy (†) and are torsion-free, possible lattice envelopes are uniform and totally disconnected (Theorem 1.3 (iii)). This can be used, for example, to show that Gromov-Thurston groups $\Gamma = \pi_1(M)$, where M is a compact manifold that admits an almost hyperbolic Riemannian metric, but not a hyperbolic one, have only the trivial lattice embeddings $\Gamma < \Gamma \ltimes K$, where K is a compact group.

The following arithmeticity theorem [1] is at the core of the proof of Theorem 1.3. Actually, it is a more general version that is used in which we drop condition (†1) (see the comment in Step 3 of Section 2).

In the proof of Theorem 1.3 we only need Theorem 1.5 in the case where D, thus $L \times D$, is compactly generated which means that the set S of primes is finite. Caprace-Monod [4, Theorem 5.20] show Theorem 1.5 for compactly generated D and under the hypothesis that L is the k-points of a simple k-group (where k is a local field) but the latter hypothesis is too restrictive for our purposes. Morevover, our proof of Theorem 1.5 does not become much easier if we assume compact generation from the beginning. Regardless of its role in Theorem 1.3 we consider the following result as a first step in the classification of lattices in locally compact groups that are not necessarily compactly generated.

Theorem 1.5 *Let L be a connected center-free semi-simple Lie group without compact factors, and let D be a totally disconnected locally compact group without compact normal subgroups. Let $\Gamma < L \times D$ be a lattice such that the projections of Γ to both L and D are dense and the projection of Γ to L is injective and Γ satisfies (†1).*

Then there exists a number field K, a (possibly infinite) set S of places of K, and a connected adjoint, absolutely simple K-group $\mathbf{H}$ such that the following holds:

Let $\tilde{\mathbf{H}} \to \mathbf{H}$ be the simply connected cover of $\mathbf{H}$ in the algebraic sense. Let $\mathcal{O}(S)$ be the S-integers of K. The group $L \times D$ embeds as a closed subgroup into the restricted (adelic) product $\prod'_{\nu \in S} \mathbf{H}(K_\nu)$. Under this embedding and under passing to a finite index subgroup, Γ is contained in $\mathbf{H}(\mathcal{O}(S))$ and the intersection of Γ with the image of $\prod'_{\nu \in S} \tilde{\mathbf{H}}(K_\nu)$ is commensurable to the image of $\tilde{\mathbf{H}}(\mathcal{O}(S))$.

The above theorem states essentially that all lattice inclusions $\Gamma < L \times D$ satisfying some natural conditions could be constructed from arithmetic data. The following example describes, up to passing to finite index subgroups, all pairs of groups Γ, D and embeddings $\Gamma < L \times D$ satisfying the condition of the theorem, for the special case $L = \mathrm{PGL}_2(\mathbb{R})$. These are obtained by the choice of the set S and the subgroups A and B described below. Similar classifications for other semi-simple groups L could be achieved using Galois cohomology.

Example 1.6 *Fix a possibly infinite set of primes $S \neq \emptyset$ and consider the localization $\mathbb{Z}_S < \mathbb{Q}$. Fix a closed subgroup A in the compact group $\prod_S \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ and a subgroup B in the discrete group $\bigoplus_S \mathbb{Z}/2\mathbb{Z}$ (note that $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \simeq \mathbb{Z}/2\mathbb{Z}$ for $p > 2$ and $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). The determinant homomorphism from $\prod'_S \mathrm{GL}_2(\mathbb{Q}_p)$ to the corresponding idele group, which we naturally identify with $\prod_S \mathbb{Z}_p^\times \times \bigoplus_S \mathbb{Z}$, restricts to $\prod'_S \mathrm{GL}_2(\mathbb{Q}_p) \supset \mathrm{GL}_2(\mathbb{Z}_S) \to \bigoplus_S \mathbb{Z} < \prod_S \mathbb{Z}_p^\times \times \bigoplus_S \mathbb{Z}$. The determinant is well defined on PGL_2 modulo squares, thus we obtain a map $\prod'_S \mathrm{PGL}_2(\mathbb{Q}_p) \to \prod_S \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \times \bigoplus_S \mathbb{Z}/2\mathbb{Z}$ which restricts to $\mathrm{PGL}_2(\mathbb{Z}_S) \to \bigoplus_S \mathbb{Z}/2\mathbb{Z}$. Let D be the preimage of $A \times B$ under the first map and Γ be the preimage of B under the second. Denote $L = \mathrm{PGL}_2(\mathbb{R})$ and embed $\mathrm{PGL}_2(\mathbb{Z}_S)$ in L via $\mathbb{Z}_S \hookrightarrow \mathbb{R}$. Thus Γ embeds diagonally in $L \times D$ and its image is a lattice, whose projections to both L and D are dense and the projection of Γ to L is injective while Γ satisfies (†1).*

2. Sketch of proof of Theorem 1.3

Step 1: Using (†2) to reduce to products.

Burger and Monod [2, Theorem 3.3.3] observed that one obtains as a consequence of the positive solution of Hilbert's 5th problem: Every locally compact group has a finite index subgroup that modulo its amenable radical splits as a product of a connected center-free semi-simple real Lie group L without compact factors and a totally disconnected group D with trivial amenable radical. By Proposition 1.1 we conclude that the amenable radical of G, thus that of any of its finite index subgroups, is compact. Therefore we may assume (up to passage to finite index subgroups and by dividing out a normal compact subgroup) that G is of the form $G = L \times D$ with L and D as above.

If we assume that not all ℓ^2-Betti numbers of Γ vanish, we can reach the same conclusion without appealing to Proposition 1.1 but by using ℓ^2-Betti numbers of locally compact groups [9] instead. Since Γ has a positive ℓ^2-Betti number in some degree, the same is true for G [7, Theorem B], thus G has a compact amenable radical [7, Theorem C].

Step 2: Separating according to discrete and dense projections to the connected factor

The connected Lie group factor L splits as a product of simple Lie groups $L = \prod_{i \in I} L_i$. The projection of Γ to L might not have dense image. It is easy to see that there is a maximal subset $J \subset I$ such that the projection pr_J of Γ to $L_J := \prod_{j \in J} L_j$ has discrete image. Then $\Gamma_J = \mathrm{pr}_J(\Gamma)$ and $\Gamma' = \ker(\mathrm{pr}_J) \cap \Gamma$ are lattices in L_J and $L_{J^c} \times D$, respectively. So we obtain an extension of groups

$$\Gamma' \hookrightarrow \Gamma \to \Gamma_J \tag{1}$$

which are lattices in the corresponding (split) extension of locally compact groups $L_{J^c} \times D \hookrightarrow L \times D \to L_J$. The projection of Γ' to L_{J^c} turns out to be dense.

Notice that finite generation of Γ does not guarantee that Γ_{J^c} is finitely generated. However $L_{J^c} \times D$ is still compactly generated if G is so.

Step 3: Distinguishing cases of the theorem

Let $U < D$ be a compact open subgroup. Let $M := \Gamma \cap (L \times U)$ and $M' := \Gamma \cap (L_{J^c} \times U)$. We prove that G is totally disconnected if M is finite and that $G = L$ if M is infinite, but M' is finite. The latter step involves condition (†1). Hence if M or M' is finite, the proof is finished. In the remainder we discuss the case that M' is infinite. For simplicity let us first assume that the projection of Γ' to D is dense; we return to this issue in the last step.

Consider $N' := \Gamma' \cap (\{1\} \times D) \triangleleft \Gamma'$ which can also be regarded as a subgroup of D. As such it is also normal by denseness of the projection $\Gamma' \to D$. The assumptions of Theorem 1.5 apart from condition (†1) are satisfied for the lattice embedding $\Gamma'/N' < L_{J^c} \times D/N'$. From a more general version of Theorem 1.5 one concludes a posteriori that Γ'/N' satisfies (†1), so the conclusion of Theorem 1.5 holds true for $\Gamma'/N' < L_{J^c} \times D/N'$. Because of compact generation of $L_{J^c} \times D/N'$ we can exclude the adelic case and conclude that Γ'/N' is an S-arithmetic lattice for a finite set S of primes.

Step 4: Using (†3) to identify group extensions

In this step we show that N' is finite, thus trivial (since D has no compact normal subgroups). The proof involves the use of (†3) for N' and M' and Margulis' normal subgroup theorem. Hence Γ' is an S-arithmetic lattice. As such it has a finite outer automorphism group which implies that, after passing to finite index subgroups, the extension (1) splits. By (†1) $\Gamma < G$ is an S-arithmetic lattice embedding.

Step 5: Quasi-isometric rigidity results

We have previously assumed that the projection $\Gamma' \to D$ is dense. If it is not we have to identify the difference between the closure D' of the image of the projection and D. The subgroup $D' < D$ is cocompact, thus $D' \hookrightarrow D$ is a quasi-isometry. By the argument before we know that D' is a product of algebraic groups over non-Archimedean fields and thus acts by isometries on a product B of Bruhat-Tits buildings. By conjugating with $D' \hookrightarrow D$ we obtain a homomorphism of D' to the quasi-isometry group of B. We finally appeal to the quasi-isometric rigidity results of Kleiner-Leeb [6] and Mosher-Sageev-White [8] to conclude that $\Gamma < G$ is an S-arithmetic lattice up to tree extension.

3. Special cases

It is instructive to investigate the consequences of Theorem 1.3 for specific groups. Rather than just applying Theorem 1.3 we sketch a blend of ad hoc arguments and techniques of the proof of Theorem 1.3 to most easily classify all lattice embeddings in the following three cases.

(a) Γ *is a free group.*

Let Γ be a non-commutative finitely generated free group. Let $\Gamma < G$ be a lattice embedding. We show that, up to finite index and dividing out a normal compact subgroup, $F < G$ is $\mathrm{PSL}_2(\mathbb{Z}) < \mathrm{PSL}_2(\mathbb{R})$ or G embeds as a closed cocompact subgroup in the automorphism group of a tree.

As explained in the first step of Subsection 2 one can avoid the use of Proposition 1.1 by using the positivity of the first ℓ^2-Betti number of Γ to conclude that G has a compact amenable radical. Up to passage to a finite index subgroup and dividing out a compact amenable radical we may assume that G is a product $G \cong L \times D$. By the Künneth formula $L \times D$ can have positive first ℓ^2-Betti number only if one of the factors is compact. Since G has trivial amenable radical, this implies that G is either L or D. In the first case G must be $\mathrm{PSL}_2(\mathbb{R})$. In the second case G is totally disconnected, and $\Gamma < G$ as a torsion-free lattice must be cocompact. By [8, Theorem 9] G embeds as a closed cocompact subgroup of the automorphism group of a tree.

(b) Γ *is a surface group.*

Let Γ be the fundamental group of a closed oriented surface Σ_g of genus $g \geq 2$. Let $\Gamma < G$ be a lattice embedding. Similarly as for free groups, by using the positivity of the first ℓ^2-Betti number, we conclude that G, up to passage to a finite index subgroup and dividing out a compact amenable radical, is either $\mathrm{PSL}_2(\mathbb{R})$ or a totally disconnected group with trivial amenable radical. In the latter case Γ is cocompact.

We argue that the totally disconnected case cannot happen unless G is discrete and so $\Gamma < G$ is the trivial lattice embedding: The inclusion $\Gamma \to G$ is a quasi-isometry in that case. So we obtain a homomorphism $G \to \mathrm{QI}(G) \cong \mathrm{QI}(\Gamma)$. Each quasi-isometry induces a homeomorphism of the boundary $\partial\Gamma \cong S^1$, so we obtain a homomorphism $f : G \to \mathrm{QI}(\Gamma) \to \mathrm{Homeo}_+(S^1)$. One can verify that f is continuous [5, Theorem 3.5] and $\ker(f)$ is compact, thus trivial by the triviality of the amenable radical. Let $U < G$ be a compact-open subgroup. Then $f(U) < \mathrm{Homeo}_+(S^1)$ is a compact subgroup, hence $f(U)$ is either finite or isomorphic to $\mathrm{SO}(2)$ [5, Lemma 3.6]. But it cannot be isomorphic to a connected group. Therefore $f(U)$ is finite, which implies that G is discrete.

(c) $\Gamma = \mathrm{PSL}_n(\mathbb{Z}[1/p])$*,* $n \geq 3$*.*

Recall that Γ embeds as a non-uniform lattice in $\mathrm{PSL}_n(\mathbb{R}) \times \mathrm{PSL}_n(\mathbb{Q}_p)$ via $\mathbb{Z}[1/p] \to \mathbb{R} \times \mathbb{Q}_p$; we denote by $\mathrm{pr}_1 : \Gamma \to \mathrm{PSL}_n(\mathbb{R})$ and $\mathrm{pr}_2 : \Gamma \to \mathrm{PSL}_n(\mathbb{Q}_p)$ the injective projections. Let us verify (†2): For any commensurated amenable subgroup $A < \Gamma$, the connected component H^0 of the Zariski closure $H = \overline{\mathrm{pr}_1(A)}$ is amenable and normal in $\mathrm{PSL}_n(\mathbb{R})$ because replacing A by a finite index subgroup does not change H^0. Hence H^0 is trivial, and so H and A are finite.

Let Γ be embedded as a lattice in some locally compact group G. Using (†2) as in the first step of Subsection 2 we replace G by $L \times D$, where L is a (possibly trivial) connected real Lie group, D is totally disconnected, and both have trivial amenable radicals.

The case $L = \{1\}$ corresponds to the trivial lattice embedding $\Gamma < \Gamma$. Indeed, in this case Γ is a lattice in a totally disconnected D, and having bounded torsion, it is cocompact. This allows us to use the results on quasi-isometric rigidity [10] and obtain a homomorphism $D \to \mathrm{QI}(D) = \mathrm{QI}(\Gamma) \simeq \mathrm{PSL}_n(\mathbb{Q})$ (hereafter $\simeq$ stands for commensurability) that can be further shown to have an image commensurable to Γ.

If L is non-trivial, then it is a center-free, semi-simple Lie group without compact factors. By Borel's density theorem the projection map $\mathrm{pr}_L : \Gamma \to L$ has Zariski dense image, and Margulis' superrigidity implies that $L = \mathrm{PSL}_n(\mathbb{R})$ and $\mathrm{pr}_L = \mathrm{pr}_1$ (this can also be shown by elementary means by conjugating unipotent matrices). Let $E < D$ denote the closure of $\mathrm{pr}_D(\Gamma)$; we get a lattice embedding $\Gamma < L \times E$ where E is totally disconnected, and E has finite covolume in D. If $U < E$ is a compact open subgroup, $\Gamma_U = \Gamma \cap (\mathrm{PSL}_n(\mathbb{R}) \times U)$ is a lattice in $\mathrm{PSL}_n(\mathbb{R}) \times U$, that projects to a lattice $\Delta < \mathrm{PSL}_n(\mathbb{R})$. We claim that $\Delta \simeq \mathrm{PSL}_n(\mathbb{Z})$. Indeed, it follows from Margulis' superrigidity (recall that $n \geq 3$) applied to $\mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : \Delta \to \Gamma_U \to \mathrm{PSL}_n(\mathbb{Q}_p)$ that $\mathrm{pr}_2(\Gamma_U)$ is contained in a maximal compact subgroup commensurated to $\mathrm{PSL}_n(\mathbb{Z}_p)$, yielding $\Gamma_U \simeq \mathrm{PSL}_n(\mathbb{Z})$.

Let H be the closure in $E \times \mathrm{PSL}_n(\mathbb{Q}_p)$ of $\Lambda = \{(\mathrm{pr}_E(\gamma), \mathrm{pr}_2(\gamma)) \mid \gamma \in \Gamma\}$. Then $H \cap (U \times \mathrm{PSL}_n(\mathbb{Q}_p))$ is the closure of $\Lambda \cap (U \times \mathrm{PSL}_n(\mathbb{Q}_p))$, thus compact because $\Gamma_U \simeq \mathrm{PSL}_n(\mathbb{Z})$. Also $\mathrm{Ker}(\mathrm{pr}_E : H \to E)$ is compact. The group $\mathrm{pr}_E(H \cap (U \times \mathrm{PSL}_n(\mathbb{Q}_p)))$ is compact, hence closed and equals $\overline{\mathrm{pr}_E(\Lambda \cap (U \times \mathrm{PSL}_n(\mathbb{Q}_p))} = U$. Since $\mathrm{pr}_E(H)$ is dense in E and contains the open subgroup U, we obtain that $\mathrm{pr}_E(H) = E$. Similarly, $\mathrm{pr}_2(H) = \mathrm{PSL}_n(\mathbb{Q}_p)$. This implies that H is a graph of a continuous surjective homomorphism $E \to \mathrm{PSL}_n(\mathbb{Q}_p)$ whose kernel K is contained in U, thus compact.

Finally, to recover the original group D, use compactness of D/E and quasi-isometric rigidity [6] of the Bruhat-Tits building X_n of $\mathrm{PSL}_n(\mathbb{Q}_p) \cong E/K$ to get $D \to \mathrm{QI}(D) \cong \mathrm{QI}(E) \cong \mathrm{QI}(X_n) \simeq \mathrm{PSL}_n(\mathbb{Q}_p)$.

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