MOSTOW-MARGULIS RIGIDITY WITH LOCALLY COMPACT TARGETS

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Abstract. Let Γ be a lattice in a simple higher rank Lie group G. We describe all locally compact (not necessarily Lie) groups H in which Γ can be embedded as a lattice. For lattices Γ in rank one groups G (with the only exception of non-uniform lattices in $G \simeq \mathrm{SL}_2(\mathbb{R})$, which are virtually free groups) we give a similar description of all possible locally compact groups H, in which Γ can be embedded as a uniform lattice.

1. Introduction and Statement of the Main Results

Throughout this paper we use the following terminology: *(semi)simple Lie group* stands for (semi)simple, connected real Lie group with finite center and no non-trivial compact factors. If (semi)simple Lie groups G, G' are locally isomorphic we write $G \simeq G'$. *Locally compact groups* are assumed to be second countable, but otherwise may be very general. A countable subgroup Γ in a locally compact group G is said to form a *lattice* if it is discrete and G/Γ carries a finite G-invariant measure (note that any locally compact group which has a lattice is necessarily unimodular). A lattice is said to be *uniform* if G/Γ is compact. An embedding $\pi : \Gamma \longrightarrow G$ of a countable group in a locally compact group G is said to be a *lattice embedding* (resp. *uniform lattice embedding*) if $\pi(\Gamma)$ forms a lattice (resp. uniform lattice) in G. A lattice Γ in a semisimple Lie group $G = \Pi G_i$ is called *irreducible* if its projection on each of the simple factors G_i is dense.

The starting point of our discussion is Mostow's rigidity:

Strong Rigidity Theorem (Mostow, Prasad, Margulis)**.** *Let G and H be semisimple Lie groups, where $G \not\simeq \mathrm{SL}_2(\mathbb{R})$. Let $\Gamma \subset G$ be an irreducible lattice and $\pi : \Gamma \longrightarrow H$ be a lattice embedding. Then $G \simeq H$ and there exists an isomorphism $\bar{\pi} : \mathrm{Ad}\, G \longrightarrow \mathrm{Ad}\, H$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \pi(\Gamma) \\ \big\downarrow{\scriptstyle \mathrm{Ad}} & & \big\downarrow{\scriptstyle \mathrm{Ad}} \\ \mathrm{Ad}\, G & \xrightarrow{\bar{\pi}} & \mathrm{Ad}\, H \end{array}$$

Originally, Mostow proved this remarkable theorem for uniform lattices [Mo2]. Mostow's approach was extended by Prasad [Pr] to encompass non-uniform lattices

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in rank one groups (and $\mathbb{Q}$-rank one lattices). Finally, the remaining cases of non-uniform (irreducible) lattices in higher rank (semi)simple Lie groups were obtained as one of the corollaries of Margulis' superrigidity [Ma].

Motivated by the strong rigidity theorem above, we consider the following general

Problem. Given an (irreducible) lattice Γ in a (semi)simple Lie group G, classify all locally compact groups H which admit lattice embedding, or uniform lattice embedding, of Γ.

Consider the following examples:

Example 1.1. Let Γ be a torsion free subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$. The following locally compact groups admit (non-uniform) lattice embeddings of Γ: $G = \mathrm{SL}_n(\mathbb{R})$; $\mathrm{Ad}\, G = \mathrm{PSL}_n(\mathbb{R})$; $\mathrm{Aut}\, G$; groups of the form $G' \times K$ where K is a compact group and G' is as above; skew-products $\mathrm{Aut}\, G \ltimes_{\mathrm{Out}\, G} K$ where $\mathrm{Aut}\, G$ acts through the finite group $\mathrm{Out}\, G$ on K; "almost direct" products: $H = (G' \times K)/C$, where C is a finite abelian group embedded diagonally in the centers of G' as above and the center of a compact K. An example of the latter type is $H = (\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SO}_n)/\pm I \times I$, for even n.

Example 1.2. Let Γ be as above. Obviously it has a uniform lattice embedding (the identity map) in itself; in any discrete $\Gamma' \supset \Gamma$ with $[\Gamma' : \Gamma] < \infty$ (for example in $\Gamma' = \mathrm{SL}_n(\mathbb{Z})$ or in $\mathrm{PSL}_n(\mathbb{Z})$); as well as in direct products $\Gamma' \times K$ where K is any compact group; or in skew products with compact groups, such as $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the torus. Γ also has uniform lattice embeddings in "almost skew-products", described by an exact sequence

$$1 \longrightarrow F \longrightarrow \Gamma' \ltimes K \longrightarrow H \longrightarrow 1$$

where $\Gamma' \supset \Gamma$ (with $[\Gamma' : \Gamma] < \infty$) acts by automorphisms on a compact group K, and F is a finite abelian group diagonally embedded in the center of Γ' and as a normal subgroup in K which is pointwise fixed by the Γ'-action.

Roughly speaking, the locally compact groups H in Example 1.1 are built from the ambient Lie group G, while the groups H in Example 1.2 are built from the lattice itself. Non uniform lattices in $\mathrm{SL}_2(\mathbb{R})$ (i.e. virtually free groups) admit uniform lattice embeddings of a completely different nature:

Example 1.3. Finitely generated non-abelian free groups $\Gamma = F_k$ form (non-uniform) lattices in $\mathrm{SL}_2(\mathbb{R})$. These groups have uniform lattice embeddings in $\mathrm{SL}_2(\mathbb{Q}_p)$ and in $\mathrm{Aut}\,(T)$ - the group of automorphisms of the regular $2k$-tree T. Taking direct and skew-products with compact groups one obtains additional examples.

The last example suggests the following general

Construction 1.4. Let Γ be a finitely generated group and let $X_{\Gamma,\Sigma}$ denote the (unlabeled) Cayley graph of Γ with respect to some finite set Σ of generators. Then

Γ has a uniform lattice embedding in the totally disconnected locally compact group $\mathrm{Aut}\,(X_{\Gamma,\Sigma})$ of all automorphisms of this graph. (Indeed, Γ acts simply transitively on the vertices of $X_{\Gamma,\Sigma}$, while the stabilizer of a vertex is a compact subgroup in $\mathrm{Aut}\,(X_{\Gamma,\Sigma})$).

We assert that for lattices in higher rank simple Lie groups the only lattice embeddings are such as the ones described in Examples 1.1 and 1.2:

Theorem A. *Let Γ be a lattice in a simple higher rank Lie group G, let H be some locally compact group, which admits a lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then $\Lambda = \pi(\Gamma)$ is contained in a closed subgroup H_0 of finite index in H, where H_0 has one of the following forms:*

(1) *H_0 is a central extension of locally compact groups*
$$1 \longrightarrow C \longrightarrow H_0 \xrightarrow{p} \mathrm{Ad}\,G \times K \longrightarrow 1$$
where C is a compact abelian group and K is a compact group. More precisely, $H_0 \cong (G' \times K')/C$, where the compact abelian group C is diagonally embedded in the centers of a connected locally compact G' with $G'/C \cong \mathrm{Ad}\,G$, and a compact group K' with $K'/C \cong K$. If the universal covering $\tilde{G}$ of G has finite center, one can take C to be a finite abelian group, in which case G' is a simple Lie group locally isomorphic to G.

(2) *H_0 admits an exact sequence*
$$1 \longrightarrow F \longrightarrow \Gamma \ltimes K \longrightarrow H_0 \longrightarrow 1$$
where Γ acts by automorphisms on a compact group K, and F is a finite abelian group diagonally embedded in the center of Γ and as a normal subgroup of K, which is elementwise fixed by the Γ-action. If Γ has trivial center, then $H_0 \cong \Gamma \ltimes K$.

Moreover, denoting by $\Phi : H_0 \longrightarrow \mathrm{Ad}\,G$ the (continuous) homomorphism

(1) $\Phi : H_0 \xrightarrow{p} \mathrm{Ad}\,G \times K \xrightarrow{pr_1} \mathrm{Ad}\,G$, *or*

(2) $\Phi : H_0 \longrightarrow \mathrm{Ad}\,\Gamma \times (K/F) \xrightarrow{pr_1} \mathrm{Ad}\,\Gamma \subset \mathrm{Ad}\,G$

corresponding to the above cases (here pr_1 is the projection on the first factor) the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \Lambda \\ \Big\downarrow \mathrm{Ad} & & \Big\downarrow \subset \\ \mathrm{Ad}\,G & \xleftarrow{\Phi} & H_0 \end{array} \tag{1.1}$$

For uniform lattice embeddings, we have a similar result for lattices in other semisimple Lie groups:

Theorem B. *Let Γ be an irreducible lattice in a semisimple Lie group $G \not\cong \mathrm{SL}_2(\mathbb{R})$ and H be some locally compact group which admits a uniform lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then the conclusions of Theorem A hold.*

For uniform lattices in $G \simeq \mathrm{SL}_2(\mathbb{R})$ similar results hold, except for the commutativity of the diagram (1.1) in case (1). More precisely:

Theorem C. *Let Γ be a uniform lattice in $G \simeq \mathrm{SL}_2(\mathbb{R})$ and let H be a locally compact group, which admits a uniform lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then $\Lambda = \pi(\Gamma)$ is contained in a closed subgroup H_0 of finite index in H, where H_0 has one of the forms (1) or (2) as in Theorem A. Furthermore, if $\Phi : H_0 \longrightarrow \mathrm{Ad}\, G$ denotes the (continuous) homomorphism as in Theorem A, then $\Phi(\Lambda) \cong \mathrm{Ad}\,\Gamma$.*

In view of the Construction 1.4, one can deduce from Theorems B and C the following

Corollary 1.5. *Let Γ be an irreducible lattice in a semisimple Lie group G. In case of $G \simeq \mathrm{SL}_2(\mathbb{R})$ assume that Γ is uniform in G. Let Σ be some finite generating set for Γ, and let $X_{\Gamma,\Sigma}$ denote the corresponding undirected unlabeled Cayley graph. Then $X_{\Gamma,\Sigma}$ admits at most finite number of outer automorphisms, i.e.*

$$[\mathrm{Aut}\,(X_{\Gamma,\Sigma})\ :\ \Gamma] < \infty$$

Moreover, the above index is bounded by some constant $i(\Gamma)$, which does not depend on the chosen generating set Σ for a given Γ. If $\Gamma \subset G \subset \mathrm{Aut}\,(G)$ is not contained in a larger lattice $\Gamma_* \subset \mathrm{Aut}\,(G)$ then $i(\Gamma) = 1$.

We refer to the phenomenon described in Theorems A and B as Mostow-Margulis rigidity with locally compact targets. The remaining problem (corresponding to Prasad's result), of describing locally compact groups H which admit a non-uniform lattice embedding of a rank one lattice Γ, remains open.

Remarks 1.6. (a) Another related problem, posed by Zimmer, is "superrigidity with locally compact targets", namely classification of locally compact groups H, for which there exists a *homomorphism* $\pi : \Gamma \longrightarrow H$ with the image $\Lambda = \pi(\Gamma)$ being sufficiently "dense" in H, where the notion of "density" should replace Zariski density in semisimple Lie groups. Theorem A applies to the situation where $\pi(\Gamma)$ forms a lattice in H. Indeed, Margulis' Normal Subgroup Theorem states that either $\pi(\Gamma)$ is finite (so that H is compact), or π factors through an *embedding* $\pi' : \Gamma' \longrightarrow H$ where Γ' is a lattice with $\mathrm{Ad}\,\Gamma' \cong \mathrm{Ad}\,\Gamma$, to which Theorem A applies.

(b) In principle, using structure Theorems A (respectively B), one can also classify not only the targets H of (uniform) lattice embeddings of Γ, but rather (uniform) lattice embeddings $\pi : \Gamma \longrightarrow H$ themselves, up to $\mathrm{Aut}\, H$. We have not addressed this question here.

(c) The proofs of Theorems A and B give effective bounds on the index $[H : H_0]$ in terms of $|\mathrm{Out}\,(G)| < \infty$ in case (1), and the index of $\mathrm{Ad}\,\Gamma$ in a maximal lattice Γ_* in $\mathrm{Aut}\,(\mathrm{Ad}\, G)$, containing $\mathrm{Ad}\,\Gamma$ in case (2). The latter gives the bound $i(\Gamma)$ on the index $[\mathrm{Aut}\,(X_{\Gamma,\Sigma}) : \Gamma]$ in Corollary 1.5.

(d) In Section 3.6 we show that the description of H_0 in case (1) of Theorem C, as an almost direct product $H_0 \cong (G' \times K)/C$ over a *compact* abelian center C, cannot be replaced by an almost direct product over a *finite* abelian center C (as it is claimed in Theorems A and B for the case where the universal covering $\tilde{G}$ has a finite center). However, it is unclear to the author, whether *compact* center C can be replaced by a *finite* center in case (1) of Theorems A, B.

(e) Theorem C leaves open the problem of the description of locally compact groups H which contain a *uniform* lattice Γ, where $\Gamma \subset G \simeq \mathrm{SL}_2(\mathbb{R})$ is a *non-uniform* lattice. Such groups Γ contain a finitely generated non-abelian free group F_k as a finite index subgroup. So the question reduces to a description of locally compact groups H which contain F_k is a cocompact lattice. As one of the corollaries of their study of quasi-actions on trees Lee Mosher, Michah Sageev and Kevin Whyte [MSW] recently proved that such a group H admits a short exact sequence $\{1\} \to K \to H \to L \to \{1\}$, where K is a compact group and $L \subseteq \mathrm{Aut}\,(T)$ is a closed cocompact subgroup of isometries of a tree T which is quasi-isometric to a regular tree (and to F_k).

About the proofs. Having very similar appearance, Theorems A and B, C have quite different proofs. The proof of Theorem A uses measure-theoretic aspects of (semi)simple Lie group actions, such as Zimmer's superrigidity for cocycles (subsection 2.1) which is a generalization of Margulis' superrigidity, and an argument (subsection 2.2), which has a lot in common with the smoothness of algebraic actions on spaces of measures. On the other hand, the proof of Theorem B (and that of C) is based on, by now well developed, theory of quasi-isometries, which has Mostow's proof of strong rigidity as one of its origins. The proof of Theorem C also uses some special features of the group of homeomorphisms of the circle.

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2. Proof of Theorem A

Outline of the proof. The essential idea of the proof is to establish a homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$, for which the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \Lambda \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \subset} \\ \mathrm{Aut}\,(\mathrm{Ad}\, G) & \xleftarrow{\Phi} & H \end{array} \tag{2.1}$$

This is done in two steps:

- Using superrigidity for measurable cocycles, a measurable bi-Γ-equivariant map $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$, satisfying (2.1), is constructed (subsection 2.1).
- It is proved that any such bi-Γ-equivariant measurable map coincides a.e. with a continuous homomorphism (subsection 2.2).

The image $\Phi(H) \subseteq \mathrm{Aut}\,(\mathrm{Ad}\,G)$ is shown either to contain $\mathrm{Ad}\,G$ or to form a lattice $\Gamma_* \supseteq \mathrm{Ad}\,\Gamma$ in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, in which case $[\Gamma_* : \Gamma] < \infty$ (subsection 2.3), while the kernel $K = \mathrm{Ker}(\Phi)$ is a compact normal subgroup in H. Passing to the finite index subgroup $H_0 := \Phi^{-1}(\mathrm{Ad}\,G)$ or $H_0 := \Phi^{-1}(\mathrm{Ad}\,\Gamma)$, descriptions (1) or (2) are obtained in subsection 2.4.

2.1. Construction of a measurable equivariant map $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\,G)$.

Theorem 2.1. *There exists a unique measurable map $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\,G)$ such that*

$$\Phi(\pi(\gamma_1)\, h\, \pi(\gamma_2)) = (\mathrm{Ad}\,\gamma_1)\,(\Phi(h))\,(\mathrm{Ad}\,\gamma_2) \tag{2.2}$$

for every $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $h \in H$.

The group structure of H is immaterial for this statement. It holds whenever H is an (infinite) measure space with two commuting, measure preserving, free, finite covolume actions of a higher rank lattice Γ (here the actions are from the left and from the right). This was proved in [F1] Theorem 4.1. For the sake of completeness we give a

Sketch of the proof. Consider the finite measure Lebesgue space H/Λ with the measure preserving (transitive) left H-action on it. Choose a (Lebesgue) measurable cross section $s : H/\Lambda \longrightarrow H$ for the natural projection $H \longrightarrow H/\Lambda$, and define a measurable cocycle

$$\alpha_0 : H \times (H/\Lambda) \longrightarrow \Lambda, \qquad \text{by} \qquad \alpha_0(h_1, h\Lambda) := s(h_1 h\Lambda)^{-1} h_1\, s(h\Lambda)$$

Let $\alpha : \Gamma \times (H/\Lambda) \longrightarrow \mathrm{Ad}\,\Gamma \subset \mathrm{Ad}\,G$ be the measurable cocycle, defined by

$$\alpha(\gamma, h\Lambda) := \mathrm{Ad} \circ \pi^{-1}\,(\alpha_0(\pi(\gamma), h\Lambda))$$

Working separately on each of the Γ-ergodic components, one checks that α is Zariski dense in $\mathrm{Ad}\,G$ ([F1] Lemma 4.2, see also [Zi] p. 99). At this point we use the assumption that G is a simple Lie group of higher rank to apply the superrigidity for cocycles theorem ([Zi] 5.2.5), which gives an existence of a measurable $\phi : H/\Lambda \longrightarrow \mathrm{Ad}\,G$ and a homomorphism $\rho : \Gamma \to \mathrm{Ad}\,G$ such that

$$\alpha(\gamma, h\Lambda) = \phi(\pi(\gamma)h\Lambda)^{-1}\rho(\gamma)\phi(h\Lambda)$$

In the above formula, $\rho(\gamma)$ and $\phi(h\Lambda)$ can be replaced by $g^{-1}\rho(\gamma)\,g$ and $g^{-1}\,\phi(h\Lambda)$ for any $g \in \mathrm{Ad}\,G$. Allowing ϕ to take values in $\mathrm{Aut}\,(\mathrm{Ad}\,G) \supset \mathrm{Ad}\,G$ (which enables to use $g \in \mathrm{Aut}\,g$), and using Margulis' superrigidity ([Ma]), one can always assume that $\rho(\gamma) = \mathrm{Ad}\,\gamma$. Fixing ρ this way, the map ϕ turns out to be uniquely determined

(Remark 2.4.(a)). Now one reassembles the definition of ϕ on the ergodic components to a single, still measurable, map $\phi : H/\Lambda \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$. Extending ϕ to $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ in a Γ-equivariant (from the right) way, namely by

$$\Phi(h) := \phi(h\Lambda)\,\mathrm{Ad}\,(s(h\Lambda)^{-1}h)$$

one verifies that Φ satisfies the relation (2.2). □

Remark 2.2. The above argument is the only place in the proof, where the assumption that G is a *higher rank simple* Lie group, is used. To apply this construction to an *irreducible* lattice Γ in a higher rank *semisimple* $G = \Pi G_i$, one needs to know that each of the simple factors G_i acts ergodically on the skew-product $G/\Gamma \times X$, where X is a Λ-ergodic component of H/Λ. This would follow, for example, if one knew that Λ is mixing on each of the Λ-ergodic components of H/Λ.

2.2. **Equivariant measurable map Φ is a continuous homomorphism.** The following Theorem is formulated for our particular setup, but it might have an independent interest:

Theorem 2.3. *Let G be a semisimple Lie group, H a locally compact group, and $\Lambda \subseteq H$ a lattice in H. Assume that there exists a homomorphism $\rho : \Lambda \longrightarrow \mathrm{Ad}\,G$ with Zariski dense image, and a measurable map $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ which satisfies*

$$\Phi(\lambda_1\,h\,\lambda_2) = \rho(\lambda_1)\,\Phi(h)\,\rho(\lambda_2)$$

for all $\lambda_1, \lambda_2 \in \Lambda$ and m_H-a.e. $h \in H$. Then the map Φ coincides m_H-a.e. with a continuous homomorphism $\Phi_0 : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ with $\Phi_0|_\Lambda = \rho$, the image $L = \Phi_0(H)$ is a closed subgroup of Aut (Ad G) *for which the pushforward measure $\Phi_*(m_H)$ gives the Haar measure m_L on L.*

Proof. Consider the measurable function $F : H \times H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ defined by

$$F(h_1, h_2) := \Phi(h_1)\,\Phi(h_1^{-1}h_2)\,\Phi(h_2)^{-1}$$

Observe that for any $\lambda \in \Lambda$ and a.e. h_1, h_2 one has

$$\begin{array}{rcl} F(h_1\lambda,\, h_2) & = & F(h_1, h_2) \\ F(h_1,\, h_2\lambda) & = & F(h_1, h_2) \\ F(\lambda\, h_1, \lambda\, h_2) & = & \rho(\lambda)\, F(h_1, h_2)\, \rho(\lambda)^{-1} \end{array} \tag{2.3}$$

Hence F descends to a measurable function f on a probability space $X := H/\Lambda \times H/\Lambda$. This f satisfies

$$f(\lambda \cdot x) = \rho(\lambda)\, f(x) \rho(\lambda)^{-1}$$

where $\lambda : x \mapsto \lambda \cdot x$ denotes the measure preserving diagonal left Λ-action on $H/\Lambda \times H/\Lambda$.

Let $\mu := f_*(m_{H/\Lambda} \times m_{H/\Lambda})$ denote the push forward measure on Aut (Ad G). Then μ is a probability measure which is invariant under conjugation by elements of the Zariski dense subgroup $\rho(\Lambda)$. We claim that such μ has to be concentrated on the identity: $\mu = \delta_e$. Indeed, it is well known that if g is a regular semisimple element of $\operatorname{Ad} G \subset \operatorname{Aut}(\operatorname{Ad} G)$, and $g' \in \operatorname{Aut}(\operatorname{Ad} G)$ does not commute with g, then $g^n g' g^{-n} \to \infty$ for $n \to \infty$ or $n \to -\infty$. Poincaré recurrence (for μ under conjugation by g) implies that μ is supported on the centralizer of g. Since regular elements of $\rho(\Lambda)$ are Zariski dense in $\operatorname{Ad} G$, we conclude that μ is supported on the center of $\operatorname{Aut}(\operatorname{Ad} G)$, which is trivial.

Hence almost everywhere $F(h_1, h_2) = e$, so that

$$\Phi(h_1^{-1} h_2) = \Phi(h_1)^{-1} \Phi(h_2), \qquad \text{a.e. on } H \times H$$

This means that Φ is an a.e. homomorphism and $\Phi(H)$ is an a.e. group with respect to the pushforward measure $m_* = \Phi_*(m_H)$. This is known to imply (cf. [Zi] Appendix B) that m_* is the Haar measure m_L of a closed subgroup $L \subseteq \operatorname{Aut}(\operatorname{Ad} G)$ and $\Phi(h) = \Phi_0(h)$ a.e. where $\Phi_0 : H \longrightarrow L \subseteq \operatorname{Aut}(\operatorname{Ad} G)$ is a continuous homomorphism. In particular, for a fixed $\lambda \in \Lambda$ and a.e. $h \in H$:

$$\rho(\lambda)\, \Phi(h) = \Phi(\lambda h) = \Phi_0(\lambda h) = \Phi_0(\lambda)\, \Phi_0(h) = \Phi_0(\lambda)\, \Phi(h)$$

so that $\Phi_0(\lambda) = \rho(\lambda)$ for $\lambda \in \Lambda$. □

Remarks 2.4. (a) Under the conditions of Theorem 2.3 with a fixed $\rho : \Lambda \longrightarrow \Gamma$ there exists *at most* one, up to m_H-null sets, bi-equivariant measurable map Φ, and in particular a unique extension Φ_0 of ρ. The proof is similar to the above one: if Φ_1, Φ_2 are two bi-equivariant measurable maps, then $F(h) := \Phi_1(h)\, \Phi_2(h)^{-1}$ has properties similar to (2.3), so that $F_*(m_{H/\Lambda})$ is a probability measure on $\operatorname{Aut}(\operatorname{Ad} G)$ which is invariant under the conjugation by $g \in \rho(\Lambda)$, and hence is δ_e.

(b) Consider the following example: let $G = H = \mathrm{PSL}_2(\mathbb{R})$, Λ_1 be an infinite cyclic subgroup of the diagonal group, $\Lambda_2 \subset G$ be a uniform lattice (i.e. a surface group) and $\rho = \mathrm{Id}$. The $\Lambda_1 \cong \mathbb{Z}$-action on G/Λ_2 is a (discrete time) map of the geodesic flow on the unit tangent bundle SM to the Riemann surface $M = K \backslash G / \Lambda_2$ equipped with the standard measure. Such an action is measurably isomorphic to a Bernoulli shift, which is known to have a vast family of measurable automorphisms. These automorphisms give rise to many measurable maps $\Phi : G \longrightarrow G$ which satisfy a.e. on G:

$$\Phi(\lambda_1 g \lambda_2) = \lambda_1\, \Phi(g)\, \lambda_2, \qquad (\lambda_1 \in \Lambda_1,\ \lambda_2 \in \Lambda_2)$$

This example shows that the conclusion of Theorem 2.3 does not follow just from the *ergodicity* of the $\rho(\Lambda_1) \times \rho(\Lambda_2)$-action on G.

Applying Theorem 2.3 with $\rho = \pi^{-1}$ to the measurable map $\Phi : H \longrightarrow \operatorname{Aut}(\operatorname{Ad} G)$, constructed in Theorem 2.1, we conclude that Φ (possibly, after an adjustment on a null set) is a continuous homomorphism satisfying

$$\Phi \circ \pi = \operatorname{Ad} \ : \Gamma \longrightarrow \operatorname{Ad} \Gamma \subset \operatorname{Ad} G \subset \operatorname{Aut}(\operatorname{Ad} G) \tag{2.4}$$

Hereafter Φ denotes this continuous homomorphism.

2.3. **The Image and the Kernel of $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\, G)$.** Since $\mathrm{Ad}\,\Gamma$ is Zariski dense in $\mathrm{Ad}\, G$ the connected component of the closed subgroup $\Phi(H) \subseteq \mathrm{Aut}\,(\mathrm{Ad}\, G)$, which contains $\mathrm{Ad}\,\Gamma$ is either $\mathrm{Ad}\, G$, or trivial. In the former case $\mathrm{Ad}\, G$ has finite index in $\Phi(H)$ (recall that $[\mathrm{Aut}\,(\mathrm{Ad}\, G) : \mathrm{Ad}\, G] < \infty$), and in the latter case $\Phi(H)$ is a discrete group containing $\mathrm{Ad}\,\Gamma$, necessarily as a finite index subgroup. We shall now restrict our attention to the closed subgroup $H_0 \subseteq H$ of finite index, defined by $H_0 := \Phi^{-1}(\mathrm{Ad}\, G)$ or $H_0 := \Phi^{-1}(\mathrm{Ad}\,\Gamma)$, corresponding to the above cases. Note also that (2.4) shows that $\Lambda \subseteq H_0$.

Next we claim that the kernel $K = \mathrm{Ker}(\Phi) \lhd H_0$ is compact. First consider the case of $\Phi(H_0) = \mathrm{Ad}\, G$. Let $F \subset \mathrm{Ad}\, G$ be a measurable subset which forms an $\mathrm{Ad}\,\Gamma$-fundamental domain. Observe that the equivariance of Φ implies that $E := \Phi^{-1}(F)$ forms a Λ-fundamental domain in H_0, and we can normalize the Haar measure m_{H_0} (and hence m_*) so that $m_{H_0}(E) = m_*(F) = 1$. Disintegration of $m_{H_0}|_E$ (and then all of m_{H_0}) with respect to $m_*|_F$ (respectively m_*) gives a family of *probability* measures $\{\mu_g\}_{g\in \mathrm{Ad}\, G}$ supported on the fibers $K_g := \Phi^{-1}(\{g\})$ which are K-cosets. The uniqueness of the Haar measures (normalized as above) as (left) invariant measures, together with the uniqueness of the disintegration procedure, give for every $h \in H$ and m_*-a.e. $g \in \mathrm{Ad}\, G$:

$$h\,\mu_g = \mu_{\Phi(h)g}$$

In particular for each $k \in K$ and m_*-a.e. g, one has $k\,\mu_g = \mu_g$, and, by a standard Fubini argument, one concludes that μ_e is a left K-invariant *probability* measure on K, i.e. K is compact.

The case of $\Phi(H_0) = \mathrm{Ad}\,\Gamma$ is even simpler: the measure on the fiver $K = \Phi^{-1}(\{e_{\mathrm{Ad}\,\Gamma}\})$ is finite and K-invariant, so K is compact.

2.4. **The structure of H_0.** By now we have shown that, given $\pi : \Gamma \longrightarrow \Lambda \subseteq H$, there exists a (unique) continuous homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$ satisfying (2.1). Moreover, H_0 is described either by the exact sequence

$$1 \longrightarrow K \longrightarrow H_0 \stackrel{\Phi}{\longrightarrow} \mathrm{Ad}\, G \longrightarrow 1 \tag{2.5}$$

or by the exact sequence

$$1 \longrightarrow K \longrightarrow H_0 \stackrel{\Phi}{\longrightarrow} \mathrm{Ad}\,\Gamma \longrightarrow 1 \tag{2.6}$$

In both cases K denotes a compact group.

Case (1): $\Phi(H_0) = \mathrm{Ad}\, G$.

Theorem 2.5. *Suppose that a locally compact H admits an exact sequence of continuous homomorphisms*

$$1 \longrightarrow K \longrightarrow H \stackrel{\Phi}{\longrightarrow} L \longrightarrow 1$$

where K is a compact normal subgroup of H and L is a connected group. Then H contains a closed normal connected subgroup L', which centralizes K, and forms a central extension of L by a compact abelian group $A := L' \cap K$:

$$1 \longrightarrow A \longrightarrow L' \stackrel{\Phi}{\longrightarrow} L \longrightarrow 1$$

The group H is then described by the exact sequence

$$1 \longrightarrow A \stackrel{j}{\longrightarrow} L' \times K \longrightarrow H \longrightarrow 1$$

where $j(a) := (a, -a)$ is a diagonal embedding of the compact abelian A in the centers of L' and K.

Proof. Let us denote by $C_H(K)$ the centralizer of K in H, i.e.

$$C_H(K) := \{h \in H \mid [h,k] = 1 \ \forall k \in K\}$$

$C_H(K)$ is a closed normal subgroup of H. We claim that $H = K \cdot C_H(K)$.

Let us assume first that K has a faithful finite dimensional representation, i.e. it is a subgroup of some $U(n)$. In this case $\operatorname{Out} K$ is known to be discrete. The conjugations $h : k \mapsto h^{-1} k\, h$ gives a continuous homomorphism

$$i : H \longrightarrow \operatorname{Aut} K \quad \text{which descends to} \quad j : L \longrightarrow \operatorname{Out} K$$

Since L is connected, this j must be trivial, so $i(H) = i(K) = \operatorname{Inn} K \subseteq \operatorname{Aut} K$. Note that $i(h) = 1$ iff h lies in the centralizer $C_H(K)$ of K in H. Hence $H = K \cdot C_H(K)$ as asserted.

Now consider the general case of a compact K. Let $\pi \in \hat{K}$ be an irreducible (unitary) K-representation. The H-action on K moves π to $\pi^h \in \hat{K}$ defined by

$$\pi^h(k) := \pi(h^{-1} k\, h) \qquad (k \in K,\ h \in H,\ \pi \in \hat{K})$$

The corresponding continuous H-action on the characters $\chi_\pi(k) = \operatorname{Tr} \pi(k)$ is trivial on K, and therefore descends to an action of L. Since $\{\chi_\pi\}_{\pi \in \hat{K}}$ are orthogonal, the latter action of the connected group L is trivial. Thus $\chi_{\pi^h} = \chi_\pi$, implying $\pi^h \sim \pi$. Hence $\operatorname{Ker}(\pi)$ is normal not only in K but also in H.

Let us enumerate by π_i, $i = 0, 1, \ldots$ all the irreducible K-representations and form the increasing sequence of finite-dimensional unitary K-representations

$$\sigma_n = \oplus_{i=0}^n \pi_i$$

Since $\operatorname{Ker}(\sigma_n) = \bigcap_0^n \operatorname{Ker}(\pi_i)$ is a closed normal subgroup of H, the exact sequence of (continuous homomorphisms)

$$1 \longrightarrow K \longrightarrow H \longrightarrow L \longrightarrow 1$$

descends to the exact sequence

$$1 \longrightarrow K_n \longrightarrow H_n \longrightarrow L \longrightarrow 1$$

where $K_n = K/\mathrm{Ker}(\sigma_n)$ and $H_n = H/\mathrm{Ker}(\sigma_n)$. The discussion of the finite dimensional case above yields that $H_n = K_n \cdot C_{H_n}(K_n)$. This means that given a fixed element $h \in H$ we can find $k_n \in K$ and $z_n \in H$ so that

$$h = k_n\, z_n \qquad \text{and} \qquad \forall k \in K\,:\,[z_n, k] \in \mathrm{Ker}(\sigma_n)$$

Passing to a subsequence n_i, we can assume that $k_{n_i} \to k_* \in K$ and hence $z_{n_i} \to z_* = h\, k_*^{-1} \in H$. Since $\mathrm{Ker}(\sigma_{n_i})$ form a decreasing sequence of compact subsets, we have

$$h = k_*\, z_* \qquad \text{and} \qquad \forall k \in K\,:\,[z_*, k] \in \mathrm{Ker}(\sigma_{n_i}),\ \forall i$$

Recall that $\{\pi_n\}_n$ separate points of K, and so do $\{\sigma_{n_i}\}_i$, hence $\bigcap_i \mathrm{Ker}(\sigma_{n_i}) = \{1\}$ and therefore $z_* \in C_H(K)$. This shows that $H = K \cdot C_H(K)$.

Now let C denote the center of K, i.e. $C = K \cap C_H(K)$. Observe that $C_H(K)$ is a central extension of L by C:

$$L = H/K = K \cdot C_H(K)/K \cong C_H(K)/K \cap C_H(K) = C_H(K)/C$$

Taking L' to be the connected component of $C_H(K)$ and $A := L' \cap K$ one obtains the assertion of the Theorem. $\square$

Applying Theorem 2.5 to the sequence (2.5) one obtains the first assertion of Theorem A, case (1). In the case of G with a *finite* universal covering $\tilde{G}$, the description of H_0 as $(G' \times K')/C$ where a compact C is diagonally embedded in the centers of G' and K', can be reduced to a *finite* C, using the following:

Lemma 2.6. *Let G' be a locally compact group which is a (topological) central extension*

$$1 \longrightarrow C \longrightarrow L \stackrel{p}{\longrightarrow} G \longrightarrow 1 \tag{2.7}$$

of a semisimple Lie group G by a compact abelian group C. Assume that the center of G is trivial, and the center Z of its universal covering $\tilde{G}$ is finite. Then there exists a closed connected subgroup $G' \subseteq L$ with a finite center $Z' = G' \cap C$, so that the finite covering

$$1 \longrightarrow Z' \longrightarrow G' \stackrel{p}{\longrightarrow} G \longrightarrow 1$$

is a local isomorphism of the semisimple Lie groups $G' \simeq G$.

Proof. First consider the situation, where C is totally disconnected. Then (2.7) is a fiber bundle with a totally disconnected fiber C, over the base G which is a connected manifold. Hence any point of G can be connected by a path in G, starting at e_G; such path has a unique lifting to a path in L, starting at e_L; while homotopic paths in G lift to homotopic paths in L. Define G' to be the path-connected component of the identity e_L in L. Then $G' \cap C$ is a homomorphic image of $\pi_1(G, e_G) = Z$ which is finite. Hence G' is a finite covering of G. It is closed in L, since the covering map $p|_{G'} : G' \longrightarrow G$ is finite to one.

Next consider the general case of a compact C. The connected component C_0 of the identity in C, is a closed (compact) subgroup of the center of L. Dividing (2.7) by C_0, one arrives in the situation discussed above, in which case L/C_0 contains a closed connected finite covering G'_0 of G. Let $L' \subseteq L$ be the preimage of $G'_0 \subseteq L/C_0$. It is a closed connected subgroup in L, which is a topological central extension

$$1 \longrightarrow C_0 \longrightarrow L' \xrightarrow{p'} G'_0 \longrightarrow 1$$

of the semisimple Lie group G'_0 by a connected compact group C_0. This central extension is an inverse limit of central extensions of Lie groups

$$1 \longrightarrow C_{0,n} \longrightarrow L'_n \xrightarrow{p'_n} G'_0 \longrightarrow 1$$

where $C_{0,n}$ are finite dimensional tori. Locally, these extensions are trivial, since the second cohomology $H^2(Lie(G'_0), \mathbb{R}^d)$ vanishes. Hence L'_n contains a closed subgroup G'_n for which $p'_n : G'_n \longrightarrow G'_0$ is a finite covering. Since the degree of this covering is bounded by $|Z| < \infty$, the subgroups G'_n stabilize (in the sense that $G'_{n+1} \longrightarrow G'_n$ is one-to-one for large n), so that the limit group L' contains a closed subgroup G' for which $p' : G' \longrightarrow G'_0$ is a finite covering. □

This completes the proof of Theorem A for Case (1).

Case (2): $\Phi(H_0) = \mathrm{Ad}\,\Gamma$**.** The compact group K is normal in $H_0 \supset \Lambda \cong \Gamma$, so Γ acts on K by automorphisms:

$$\gamma : k \mapsto k^\gamma, \quad \text{where} \quad k^\gamma := \pi(\gamma)^{-1}\, k\, \pi(\gamma)$$

The corresponding skew product $\Gamma \ltimes K$ maps homomorphically into H_0 by

$$p : \Gamma \ltimes K \longrightarrow H_0 \quad \text{defined by} \quad p(\gamma, k) := \pi(\gamma) \cdot k$$

Since the homomorphism

$$\Gamma \ltimes K \xrightarrow{p} H_0 \xrightarrow{\Phi} \mathrm{Ad}\,\Gamma$$

coincides with the natural homomorphism

$$\Gamma \ltimes K \longrightarrow \Gamma \xrightarrow{\mathrm{Ad}} \mathrm{Ad}\,\Gamma$$

it follows that p is onto, and that $Z = \mathrm{Ker}(p)$ consists of elements $z = (\gamma_z, k_z)$ with $\gamma_z \in \mathrm{Ker}(\mathrm{Ad} : \Gamma \longrightarrow \mathrm{Ad}\,\Gamma)$ - the finite abelian center of Γ, while $\pi(\gamma_z) = k_z^{-1} \in \Lambda \cap K$. Observe that

- $K_Z := \{k_z \in K \mid z \in Z\}$ is a finite abelian group, isomorphic to Z;
- K_Z is normal in K, because $\{1\} \times K$ and Z are normal in $\Gamma \ltimes K$, so that

$$\{1\} \times K_Z = (\{1\} \times K) \cap Z \lhd \Gamma \ltimes K$$

- for each $\gamma \in \Gamma$, one has $\gamma^{-1}\gamma_z\gamma = \gamma_z$, so that $k_z^\gamma = k_z$, for each $z \in Z$.

Hence H_0 is described by the exact sequence

$$1 \longrightarrow Z \xrightarrow{j} \Gamma \ltimes K \xrightarrow{p} H_0 \longrightarrow 1$$

with $j(z) = (\gamma_z, k_z)$, where $z \mapsto \gamma_z$ is an embedding of Z in the finite abelian center of Γ; and $K_Z = \{k_z\}_{z\in Z}$ forms a normal subgroup in K whose elements are fixed by the action of Γ.

Clearly, if Γ has trivial center, then H_0 is just isomorphic to the skew-product $H_0 \cong \Gamma \ltimes K$.

The proof of Theorem A is completed. $\square$

3. Proof of Theorems B and C

Outline of the proof. As in the proof of Theorem A, the essential step is to construct a continuous homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$. If $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ the constructed Φ will satisfy (2.1); while in the case $G \simeq \mathrm{SL}_2(\mathbb{R})$ we shall only obtain that

$$\Phi \circ \pi : \Gamma \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$$

takes Γ to a lattice $\Gamma' \subset \mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Aut}\,(\mathrm{Ad}\, G)$, isomorphic to $\mathrm{Ad}\,\Gamma$.

The homomorphism Φ is constructed using the quasi-isometry groups. For any finitely generated group Γ, there is an associated quasi-isometry group $\mathrm{QI}\,(\Gamma)$, which we shall consider as an abstract group with no topology. The quasi-isometry group has the property that any uniform lattice embedding π of Γ in a locally compact group H, gives rise to a homomorphism (of abstract groups) $\Phi_\pi : H \longrightarrow \mathrm{QI}\,(\Gamma)$ (see subsection 3.2).

Consider the case where Γ is a uniform lattice in a simple Lie group G of rank one. Then Γ is quasi-isometric to the symmetric space $X = G/K$, which has strictly negative Riemannian curvature (X is real, complex, quaternionic hyperbolic space or the Cayley plane), in which case there exists an embedding

$$\beta : \mathrm{QI}\,(\Gamma) \longrightarrow \mathrm{Homeo}\,(\partial X)$$

in the group of homeomorphisms of the boundary ∂X of X. This gives an abstract homomorphism of groups $\Psi_\pi : H \longrightarrow \mathrm{Homeo}\,(\partial X)$

$$\Psi_\pi : H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial X)$$

which turns out to be *continuous* (Theorem 3.5). The given uniform lattice embedding $\sigma : \Gamma \longrightarrow G$ gives rise to the homomorphism of abstract groups

$$\Phi_\sigma : \mathrm{Aut}\,(\mathrm{Ad}\, G) \longrightarrow \mathrm{QI}\,(\mathrm{Ad}\,\Gamma) \cong \mathrm{QI}\,(\Gamma)$$

and to another continuous homomorphism

$$\Psi_\sigma : \mathrm{Aut}\,(\mathrm{Ad}\, G) \xrightarrow{\Phi_\sigma} \mathrm{QI}\,(\mathrm{Ad}\,\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial X)$$

which is known to form a *continuous isomorphic embedding* with a *continuous* inverse Ψ_σ^{-1}. The following is crucial

Claim 3.1. *The images of* $\Psi_\pi(H)$ *and* $\Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$ *in* $\mathrm{Homeo}\,(\partial X)$*, are locally compact groups, which contain* $\Psi_\pi(\Lambda)$ *and* $\Psi_\sigma(\mathrm{Ad}\,\Gamma)$ *as uniform lattices. From the construction one has*

$$\Psi_\pi(\pi(\gamma)) = \Psi_\sigma(\mathrm{Ad}\,\gamma), \qquad (\gamma \in \Gamma)$$

Moreover, if $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ *then* $\Psi_\pi(H) \subseteq \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$ *and*

$$\Phi := \Psi_\sigma^{-1} \circ \Psi_\pi \;:\; H \longrightarrow \mathrm{Homeo}\,(\partial X) \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$$

is a continuous homomorphism, satisfying (1.1).

In the case of $G \simeq \mathrm{SL}_2(\mathbb{R})$*, there exists an* $f \in \mathrm{Homeo}\,(\partial X)$ *such that*

$$f^{-1}\,\Psi_\pi(H)\,f \subseteq \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$$

and the continuous homomorphisms

$$\Phi := \Psi_\sigma \circ f^{-1}\Psi_\pi\, f : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$$

maps $\Lambda = \pi(\Gamma)$ *onto a lattice* $\Phi(\Lambda) \subset \mathrm{Aut}\,(\mathrm{Ad}\,G)$*, which is isomorphic (but not necessarily conjugate) to* $\mathrm{Ad}\,\Gamma$*.*

From this claim the proof of Theorems B and C can be completed as in subsections 2.3 and 2.4.

3.1. **Quasi-isometries.** Recall the notion of quasi-isometries (see [Gr], [GP] for detailed discussions): a map $q : X_1 \longrightarrow X_2$ between proper metric spaces (X_1, d_1) and (X_2, d_2) is said to be a *quasi-isometric embedding* if there exist constants M, A such that for all $x, y \in X_1$:

$$\frac{1}{M} \cdot d_1(x,y) - A \le d_2(q(x),\, q(y)) \le M \cdot d_1(x,y) + A \tag{3.1}$$

Such q can be called an (M, A)-quasi-isometric embedding to emphasize that (3.1) is satisfied for specific M and A.

A quasi-isometric embedding $q : X_1 \longrightarrow X_2$ with an image $q(X_1)$ which is within bounded distance from all of X_2, is called a *quasi-isometry*, and the spaces (X_1, d_1), (X_2, d_2) are said to be *quasi-isometric*. Two quasi-isometries (or quasi-isometric embeddings) $q, q' : X_1 \longrightarrow X_2$ are said to be *equivalent* (notation: $q \sim q'$) if the distance

$$D(q, q') := \sup\{d_2(q(x), q'(x)) \mid x \in X_1\}$$

is finite. Now consider the collection of all self quasi-isometries of a fixed proper metric space (X, d). It forms a semi-group with respect to composition (which respects the equivalence relation $\sim$). Modulo this relation, the semi-group of actual quasi-isometries becomes a group, called the *quasi-isometry group* of (X, d) and is denoted by $\mathrm{QI}\,(X, d)$. Quasi-isometric spaces have isomorphic quasi-isometry groups. Therefore, for a finitely generated group Γ it makes sense to talk about its quasi-isometry group $\mathrm{QI}\,(\Gamma)$ without specifying any particular left invariant word metric. Note also, that the isometric left Γ-action on itself (with respect to some/any left-invariant word metric) defines a natural homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{QI}\,(\Gamma)$$

3.2. **Uniform lattice embeddings and quasi-isometries.** We claim that the notions of quasi-isometries appear naturally in our situation, where a given finitely generated group Γ has an embedding $\pi : \Gamma \to H$ as a *uniform lattice* in an (unknown) locally compact group H. There are two different natural constructions both giving rise to a homomorphism (of abstract groups)

$$H \longrightarrow \mathrm{QI}(\Gamma)$$

the restriction of which to Γ coincides with the standard homomorphism $\rho : \Gamma \to \mathrm{QI}(\Gamma)$. The first of the two is the following:

Construction 3.2. Choose some open subset $E \subset H$ with compact closure, such that $H = \cup_{\gamma\in\Gamma}\pi(\gamma)E$, and fix some (typically discontinuous) function $p : H \longrightarrow \Gamma$, satisfying $h \in \pi(p(h))\,E$ for $h \in H$. With this choice of E and p define a family of maps $\{q_h : \Gamma \longrightarrow \Gamma\}_{h\in H}$ by the rule $q_h(\gamma) := p(h\pi(\gamma))$. If E is a neighborhood of the unit e, we can choose p with $p(\pi(\gamma)) = \gamma$, so that

$$q_\gamma(\gamma') = \gamma\gamma' \tag{3.2}$$

Lemma 3.3. *Let E, p and $\{q_h\}_{h\in H}$ be as above. Then*

(a) *Each $q_h : \Gamma \longrightarrow \Gamma$ is a quasi-isometry of Γ and its equivalence class $[q_h] \in \mathrm{QI}(\Gamma)$ depends only on $h \in H$ (and not on the choice of E and p).*

(b) *The map $\Phi_\pi : H \longrightarrow \mathrm{QI}(\Gamma)$, given by $\Phi_\pi(h) = [q_h]$, is a homomorphism of (abstract) groups, such that the composition*

$$\Gamma \xrightarrow{\pi} H \xrightarrow{\Phi_\pi} \mathrm{QI}(\Gamma)$$

coincides with the standard homomorphism $\rho : \Gamma \longrightarrow \mathrm{QI}(\Gamma)$.

(c) *$\{q_h\}_{h\in H}$ are (M, A)-quasi-isometries for some fixed M and A, depending just on E, p, and independent of $h \in H$.*

(d) *There exists a constant B with the following property: given any finite set $F \subset \Gamma$ there is a neighborhood of the identity $V \subset H$ such that $d(q_h(\gamma), \gamma) \le B$ for all $\gamma \in F$ and $h \in V$.*

Remarks 3.4. (a) The above construction is a particular case of Gromov's *topological equivalence*, where one considers two finitely generated groups Γ and Γ' with commuting, proper, cocompact actions on a locally compact space X (in our special case $\Gamma = \Gamma'$, $X = H$ and the commuting actions are from the left and from the right). In this general setup, each point $x \in X$ defines a quasi-isometry $q_x : \Gamma \longrightarrow \Gamma'$ so that properties (a) and (c) of the Lemma still hold. If H forms a locally compact group, one obtains in addition the homomorphism (b) satisfying (d).

(b) In general the collection of actual (M, A)-quasi-isometries $\{q_h\}_{h\in H}$ is not closed under composition: $q_{h_1} \circ q_{h_2}$ is equivalent, but typically is not equal, to $q_{h_1 h_2}$.

An alternative construction of a homomorphism $H \longrightarrow \mathrm{QI}(\Gamma)$ uses just the left Γ-action on H and a choice of a left invariant metric on H, as follows: fix a proper left invariant metric d on H (here "proper" means that the closed balls with respect to d

are compact subsets of H). Actually it suffices to use a proper pseudo-metric such as $d(h_1,h_2)=\min\{n\mid h_2^{-1}h_1\in B^n\}$ where $B\subseteq H$ is a symmetric compact neighborhood of the identity which generates H (such sets exists because H is compactly generated, since it contains a finitely generated cocompact lattice). One can show that the restriction of such d to $\pi(\Gamma)$ is quasi-isometric to a word metric d_w on Γ, and (H,d) becomes quasi-isometric to (Γ,d_w). This gives rise to a homomorphism

$$H\longrightarrow \mathrm{QI}\,(H)\cong \mathrm{QI}\,(\Gamma)$$

which has the properties (b), (c), (d) of the first construction.

However we shall work with the construction 3.2 which is conceptually closer to the ideas in Section 2.1.

Proof of Lemma 3.3 Fix some finite set S of generators for Γ, and let d denote the corresponding left invariant word metric on Γ. Observe that, for any two compact sets $Q,Q'\subset H$ there is at most finite number of $\gamma\in\Gamma$, so that $Q\,\pi(\gamma)\cap Q'\neq\emptyset$. This implies that for any sets $E,E'\subset H$ with compact closures, there exists an integer $A=A(E,E')$, so that

$$\pi(\gamma)\,E\,\cap\,\pi(\gamma')\,E'\neq\emptyset\quad\Longrightarrow\quad d(\gamma,\gamma')\leq A \tag{3.3}$$

In particular, if q_h and q'_h are constructed from E,p and E',p' respectively, then for some $A=A(E,E')$ any fixed $h\in H$ and all $\gamma\in\Gamma$:

$$h\,\pi(\gamma)\in\pi(q_h(\gamma))\,E\,\cap\,\pi(q'_h(\gamma))E'\quad\Longrightarrow\quad d(q_h(\gamma),q'_h(\gamma))\leq A \tag{3.4}$$

Denote by B_n the n-ball in Γ centered at the origin. The set $E\,B_1=\cup_{\gamma\in B_1}E\,\pi(\gamma)$ has a compact closure in H, and therefore $E\,B_1\subset B_M\,E$ for some sufficiently large integer $M=M(E)$. Then for all $n>1$, we have:

$$E\,\pi(B_n)\subset\pi(B_M)\,E\,\pi(B_{n-1})\subset\cdots\subset\pi(B_{M\cdot n})E$$

Take some fixed $h\in H$ and any $\gamma_1,\gamma_2\in\Gamma$. Denoting $n=d(\gamma_1,\gamma_2)$, we have

$$\begin{aligned} h\,\pi(\gamma_1)\ &\in\ q_h(\gamma_1)\,E \qquad \text{and}\qquad h\,\pi(\gamma_2)\in q_h(\gamma_2)\,E\\ h\,\pi(\gamma_2)\ &=\ h\,\pi(\gamma_1\,(\gamma_1^{-1}\gamma_2))\in\pi(q_h(\gamma_1))\,E\,\pi(\gamma_1^{-1}\gamma_2)\\ &\subset\pi(q_h(\gamma_1))\,E\,\pi(B_n)\subset\pi(q_h(\gamma_1)B_{M\cdot n})E \end{aligned}$$

Thus $\pi(q_h(\gamma_2))\,E$ intersects $\pi(q_h(\gamma_1)B_{M\cdot n})E$, and using (3.3), we deduce that

$$d(q_h(\gamma_1),q_h(\gamma_2))\leq M\cdot d(\gamma_1,\gamma_2)+A$$

Similar arguments show the other properties of quasi-isometries. Hence q_h is a quasi-isometry of Γ. Its class $[q_h]\in\mathrm{QI}\,(\Gamma)$, which we shall denote by $\Phi_\pi(h)$, does not depend on the choice of E, p as (3.4) shows. This proves (a).

Next take $h_1,h_2\in H$ and observe that for each $\gamma\in\Gamma$, $h_i\,\pi(\gamma)\in\pi(q_{h_i}(\gamma))\,E$ for $i=1,2$, and therefore

$$(h_1\,h_2)\,\pi(\gamma)\in h_1\,\pi(q_{h_2}(\gamma))\,E\subset\pi(q_{h_1}(q_{h_2}(\gamma)))\,E\cdot E$$

Since $E':=E\cdot E$ has a compact closure in H, (3.3) implies that

$$q_{h_1}\circ q_{h_2}\sim q_{h_1h_2}\qquad\text{i.e.}\qquad \Phi_\pi(h_1)\cdot\Phi_\pi(h_2)=\Phi(h_1h_2)$$

Similarly $q_h \circ q_{h^{-1}} \sim \mathrm{Id}_\Gamma$, so that Φ_π is a homomorphism of abstract groups. Note also, that since Φ_π does not depend on the choice of E, p, we could choose them so that (3.2) holds. This shows that $\Phi_\pi \circ \pi = \rho$. Hence (b) is proved.

Statement (c) follows from the proof of (a) above, where M and A were constructed independently of $h \in H$.

Next fix some neighborhood U of the identity e_H which has a compact closure. Then $E \cup U$ has a compact closure Q, and there is a constant $B = A(Q)$ such that

$$\gamma Q \cap \gamma' Q \neq \emptyset \quad \Longrightarrow \quad d(\gamma, \gamma') \leq B$$

Given any finite set $F \subset \Gamma$ there is an open neighborhood V of e_H such that $\pi(\gamma)^{-1} V \pi(\gamma) \subset U$ for all $\gamma \in F$, and therefore $V\pi(\gamma) \subseteq \pi(\gamma) U$ for $\gamma \in F$. For each $h \in V$ one also has $h\pi(\gamma) \in \pi(q_h(\gamma))\, E$, which yields $\pi(q_h(\gamma))Q \cap \pi(\gamma)Q$ is not empty for $\gamma \in F$, $h \in V$, so that $d(q_h(\gamma), \gamma) \leq B$ for all $\gamma \in F$, $h \in V$. Since B is independent of F, property (d) is proved. □

In our particular situation Γ is a lattice in a (semi)simple Lie group G. Quasi-isometries of lattices have been thoroughly analyzed by several people over a series of many papers, and by now a comprehensive understanding of these objects has been achieved. The reader is referred to the survey papers by Farb [Fa] and by Gromov and Pansu [GP] for details and further references for this beautiful theory. In what follows we shall only briefly mention some of the facts needed for the proof of Theorems B and C. We need to consider several cases.

3.3. **Γ is a uniform lattice in a simple G of rank one.** A uniform lattice Γ in a rank one simple Lie group G forms a hyperbolic group in the sense of Gromov. Given a general Gromov hyperbolic group Γ there is an associated compact metric space $\partial\Gamma$ - the boundary of Γ - equipped with a continuous (essentially faithful) Γ-action by homeomorphisms. The group $\mathrm{QI}\,(\Gamma)$ has a homomorphism (in fact an embedding)

$$\beta : \mathrm{QI}\,(\Gamma) \longrightarrow \mathrm{Homeo}\,(\partial\Gamma)$$

such that the composition $\Gamma \xrightarrow{\rho} \mathrm{QI}\,(\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial\Gamma)$ corresponds to the standard Γ-action on its boundary $\partial\Gamma$. We claim that if $\pi : \Gamma \longrightarrow H$ is a uniform lattice embedding into a locally compact group H, and $\Phi_\pi : H \longrightarrow \mathrm{QI}\,(\Gamma)$ is a homomorphism (of abstract groups) satisfying (b) and (c) of Lemma 3.3, then

Theorem 3.5. *The homomorphism*

$$\Psi_\pi : H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial\Gamma) \tag{3.5}$$

is continuous with respect to the uniform convergence in $\mathrm{Homeo}\,(\partial\Gamma)$. *The composition homomorphism* $\Psi_\pi \circ \pi : \Gamma \longrightarrow \mathrm{Homeo}\,(\partial\Gamma)$ *is the standard* Γ*-action on its boundary. The image* $\Psi_\pi(H)$ *is a locally compact subgroup of* $\mathrm{Homeo}\,(\partial\Gamma)$, *containing* $\Psi_\pi(\pi(\Gamma))$ *as a uniform lattice. The kernel* $\mathrm{Ker}(\Psi_\pi)$ *is a compact normal subgroup of* H, *which has at most finite intersection with* $\pi(\Gamma)$.

Proof. Let us first recall some general facts about Gromov hyperbolic groups. Let d be some fixed left invariant proper metric d on Γ (say a word metric). The (ideal) boundary $\partial\Gamma$ consists of equivalence classes of quasi-geodesic rays, which are by definitions quasi-isometric embeddings $r:\mathbb{R}_+\to(\Gamma,d)$, modulo the equivalence relation: $r\sim r'$ if the distance $D(r,r')$ is finite. Recall that $D(r,r')$ is defined as

$$D(r,r'):=\sup\{d(r(t),r'(t))\mid t\geq 0\}$$

This gives rise to the action of $\mathrm{QI}(\Gamma)$ on $\partial\Gamma$ since for any quasi-geodesic ray $r:\mathbb{R}_+\to\Gamma$ and any quasi-isometry q of Γ, the composition $q\circ r:\mathbb{R}_+\to\Gamma$ is a quasi-geodesic ray, and this definition respects the "bounded distance" equivalence relations on the rays and on the quasi-isometries. The resulting action of $\mathrm{QI}(\Gamma)$ on $\partial\Gamma$ turns out to be continuous, and the corresponding homomorphism $\mathrm{QI}(\Gamma)\longrightarrow\mathrm{Homeo}(\partial\Gamma)$ was denoted by β.

In a Gromov hyperbolic group points ξ of the boundary $\partial\Gamma$ (i.e. classes of quasi-geodesic rays) can be represented by quasi-geodesic rays $r:\mathbb{R}_+\to\Gamma$ with $r(0)=e_\Gamma$ and such that the quasi-geodesic constants are *uniformly bounded*, say by M_0 and A_0 (in general, such a representation of ξ by r_ξ with the above properties is not unique, but any two such rays are within finite distance which is uniformly bounded). The topology on $\partial\Gamma$ admits a uniform structure given by neighborhoods U_n of the diagonal in $\partial\Gamma\times\partial\Gamma$, defined as follows: fix a sufficiently large constant C and define a decreasing sequence of neighborhoods U_n of the diagonal in $\partial\Gamma\times\partial\Gamma$ by declaring (ξ_1,ξ_2) to be in U_n if ξ_1, ξ_2 can represented by (M_0,A_0)-quasi-geodesic rays $r_1,r_2:\mathbb{R}_+\to\Gamma$ with $r_1(0)=r_2(0)=e_\Gamma$ and

$$d(r_1(t),r_2(t))\leq C \qquad \text{for} \qquad t\in[0,n]$$

We shall also need the following general fact, often referred to as the Morse Lemma: given any (M,A)-quasi-geodesic ray $r:\mathbb{R}_+\to\Gamma$, there exists a (M_0,A_0)-quasi-geodesic ray $r_0:\mathbb{R}_+\to\Gamma$ with $r_0(0)=e_\Gamma$ which is within bounded distance $D(r,r_0)<\infty$ from r, and moreover the distance $D(r,r_0)$ can be bounded in terms of $d(r(0),e_\Gamma)$, M, A (and M_0, A_0).

With these preliminary remarks, let us prove the continuity of the homomorphism

$$\Psi_\pi:H\xrightarrow{\Phi_\pi}\mathrm{QI}(\Gamma)\xrightarrow{\beta}\mathrm{Homeo}(\partial\Gamma)$$

Let M, A and B be constants as in Lemma 3.3.(c) and (d). For any n let

$$F_n=\{\gamma\in\Gamma\mid d(\gamma,e)\leq M_0\cdot n+A_0\}$$

It is a finite subset of Γ. By property (d) of Lemma 3.3 there is a neighborhood V_n of e_H in H such that

$$d(q_h(\gamma),\gamma)\leq B \qquad \text{for} \qquad \gamma\in F_n,\ h\in V \tag{3.6}$$

Take an arbitrary $\xi\in\partial\Gamma$ represented by a (M_0,A_0)-quasi-geodesic ray $r_\xi:\mathbb{R}_+\to\Gamma$ with $r_\xi(0)=e_\Gamma$. Note that $r_\xi(t)\in F_n$ for $t\in[0,n]$. For any $h\in H$ the map $r'=q_h\circ r_\xi:\mathbb{R}_+\to\Gamma$ is a quasi-geodesic ray (representing $\Psi_\pi(h)\cdot\xi\in\partial\Gamma$) which has quasi-isometric constants (M_1,A_1) being bounded in terms of M, M_0, A, A_0.

For $h \in V_n$ one also has $d(r'(0), e_\Gamma) = d(q_h(e_\Gamma), e_\Gamma) \le B$. For any such $h \in V_n$ let $r_{h\cdot\xi}$ denote an (M_0, A_0)-quasi-geodesic ray corresponding to r' as in the "Morse Lemma" above. Then $D(r_{h\cdot\xi}, r')$ is bounded in terms of M_1, A_1 and B. Letting $C = D(r_{h\cdot\xi}, r') + B$, we obtain for all $t \in [0, n]$ and $h \in V_n$ the following estimate

$$d(r_{h\cdot\xi}(t), r_\xi(t)) \le D(r_{h\cdot\xi}, r') + d(q_h \circ r_\xi(t), r_\xi(t)) \le D(r_{h\cdot\xi}, r') + B = C$$

because $r_\xi(t) \in F_n$ for $t \in [0, n]$. Note that the bound C is independent of n, and the above inequality proves that for all $h \in V_n$

$$(\Psi_\pi(h) \cdot \xi, \xi) \in U_n \qquad (h \in V_n,\ \xi \in \partial\Gamma)$$

where U_n is defined using the uniform constant C, independent of n. This proves the continuity of $\Psi_\pi : H \longrightarrow \mathrm{Homeo}\,(\partial\Gamma)$.

Since $\Phi_\pi \circ \pi : \Gamma \longrightarrow \mathrm{QI}\,(\Gamma)$ coincides with the natural homomorphism $\rho : \Gamma \longrightarrow \mathrm{QI}\,(\Gamma)$, the homomorphism

$$\Gamma \xrightarrow{\pi} H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial\Gamma)$$

corresponds to the standard Γ-action on its boundary $\partial\Gamma$. The standard homomorphism $\Gamma \longrightarrow \mathrm{Homeo}\,(\partial\Gamma)$ is known to have finite kernel and a discrete image, which we denote by $\bar{\Gamma}$. Thus $\Psi_\pi(\Lambda)$ forms a discrete subgroup of $\Psi_\pi(H)$ and the kernel $K = \mathrm{Ker}(\Psi_\pi) \lhd H$ has at most finite intersection with $\Lambda = \pi(\Gamma)$.

Now take a compact subset $Q \subset H$ with $Q\,\Lambda = H$. By continuity, $\Psi_\pi(Q)$ is compact and its translations by the discrete group $\Psi_\pi(\Lambda)$ cover all of $\Psi_\pi(H)$. This easily implies that (i) $\Psi_\pi(H)$ is locally compact (in the topology of the uniform convergence), (ii) $\Psi_\pi(\Lambda)$ is a uniform lattice in $\Psi_\pi(H)$, and (iii) the kernel $K = \mathrm{Ker}(\Psi_\pi)$ is compact. □

Returning to the situation where Γ is a uniform lattice in a simple Lie group G of rank one, denote by ∂X the visual boundary of the symmetric space $X = G/K$. Then $\partial\Gamma$ is naturally identified with ∂X (as topological spaces with a Γ-action), and $\partial\Gamma = \partial X$ is homeomorphic to the sphere S^{n-1} where n is the real dimension of X. The image $\bar{\Gamma} \subset \mathrm{Homeo}\,(\partial X)$ of Γ is isomorphic to $\Gamma/Z(\Gamma) \cong \mathrm{Ad}\,\Gamma$. We shall also denote by $L = \Psi_\pi(H)$ the locally compact subgroup of $\mathrm{Homeo}\,(\partial X)$ constructed above. Our goal is to prove Claim 3.1.

There are several cases to be considered.

Case $G \simeq \mathrm{SL}_2(\mathbb{R})$ (Theorem C). In this case the symmetric space $X = G/K$ is the real hyperbolic plane $\mathbf{H}^2$ and ∂X is the circle S^1. We claim that, up to conjugation in $\mathrm{Homeo}\,(S^1)$, the locally compact group $L = \Psi_\pi(H) \subset \mathrm{Homeo}\,(S^1)$ is either $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Ad}\,G$, or $\mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{Aut}\,(\mathrm{Ad}\,G)$, or is a a discrete group Γ_* which forms a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$ or in $\mathrm{PGL}_2(\mathbb{R})$, containing $\bar{\Gamma} \cong \mathrm{Ad}\,\Gamma$ as a finite index subgroup.

At this point one should point out that it is possible to describe quite explicitly all *locally compact* subgroups of $\mathrm{Homeo}\,(S^1)$. Here we shall outline the arguments required in our particular situation. It is convenient to consider the orientation preserving subgroup $L_+ = L \cap \mathrm{Homeo}_+(S^1)$, which has index at most two in $L = \Psi_\pi(H)$.

The first main observation (pointed out to me by Etienne Ghys, whom I would like to thank) is that locally compact subgroups, such as L_+, of $\mathrm{Homeo}_+(S^1)$ have No Small Subgroups, i.e. one can find a neighborhood U of the identity in L_+ which contains no non-trivial subgroups. It suffices to show that there are no small *compact* subgroups, and this follows from the following easy

Lemma 3.6. *Any compact subgroup* $K \subset \mathrm{Homeo}_+(S^1)$ *is conjugate in* $\mathrm{Homeo}_+(S^1)$ *to a closed subgroup of rotations* $\mathrm{SO}(2)$*, and therefore* K *is either finite or coincides with* $\mathrm{SO}(2)$*.*

Using Montgomery-Zippin's results (see [MZ]) a locally compact group L_+ with No Small Subgroups has a real connected Lie group as its connected component L_+^0 of the identity.

Next recall that the action of $\bar{\Gamma}$ on S^1 is *minimal* and *strongly proximal* (this general fact applies to the actions of hyperbolic groups Γ on their boundaries $\partial\Gamma$, as well as for actions of lattices Γ on the Furstenberg's boundary $B(G)$ of the ambient semisimple Lie group G). Recall that minimality means that *all* orbits $\bar{\Gamma} \cdot x$ are dense in $\partial\Gamma$, while strong proximality means that $\bar{\Gamma}$-orbit of any probability measure μ has Dirac measures in its closure with respect to weak topology. Since L_+ contains $\bar{\Gamma}$ its action on S^1 is also minimal and strongly proximal.

Lemma 3.7. *Let* B *be a metric compact space and* $L \subset \mathrm{Homeo}(B)$ *be a locally compact subgroup which acts minimally and strongly proximally on* B*. Then* L *has no non-trivial closed normal amenable subgroups.*

Proof. Let $N \lhd L$ be a closed normal subgroup, and let $V_N \subset \mathcal{P}(B)$ be the set of all N-invariant probability measures on B. Then V_N is a non-empty (because A is amenable), closed and convex subset of $\mathcal{P}(B)$ which is L-invariant. Strong proximality of the L-action implies that V_N contains some Dirac measures, and the minimality implies that all the Dirac measures $\{\delta_\xi\}_{\xi \in B}$ are in V_N, which means that N acts trivially on B. □

Applying this to $B = S^1$ and our group L_+, one concludes that the solvable part of the Levi decomposition of the Lie group L_+^0 is trivial, so L_+^0 is reductive, and moreover it is a semisimple Lie groups with trivial center and no compact factors. By Lemma 3.6 the maximal compact of L_+^0 has to be isomorphic to $\mathrm{SO}(2)$ or to be trivial. Thus L_+^0 is either isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or is trivial.

One can show that in the first case, an isomorphism of L_+^0 with $\mathrm{PSL}_2(\mathbb{R})$ can be realized by a conjugation in $\mathrm{Homeo}_+(S^1)$ (the conjugation map coming from Lemma 3.6). Since $L_+^0 \cong \mathrm{PSL}_2(\mathbb{R})$ is normal in L_+, the latter acts by automorphisms on $\mathrm{PSL}_2(\mathbb{R})$, which has only one outer automorphism. This outer automorphism cannot be realized as an orientation preserving homeomorphism of S^1. This implies that L_+ centralizers L_+^0, and one can check that the centralizer of $\mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{Homeo}_+(S^1)$

is trivial (in fact this is true already for the maximal compact subgroup SO(2) of $\mathrm{PSL}_2(\mathbb{R})$). Therefore $L_+ = L_+^0$ is conjugate in $\mathrm{Homeo}_+(S^1)$ to $\mathrm{PSL}_2(\mathbb{R})$. Considering all of L in all of $\mathrm{Homeo}\,(S^1)$, one finds that if L^0 is non-trivial, then L is conjugate to either $\Psi_\pi(H)$ or to $\mathrm{PGL}_2(\mathbb{R})$.

If L_+^0 is trivial, then L_+ and $L = \Psi_\pi(H)$ are discrete subgroups of $\mathrm{Homeo}\,(S^1)$, because by Lemma 3.6 L_+ has No Small Subgroups. Since L and L_+ contain $\bar{\Gamma}$ as a cocompact lattice, the index $[L : \bar{\Gamma}]$ is finite. Finally, we claim that $L \subset \mathrm{Homeo}\,(S^1)$ lies in fact in $\mathrm{PSL}_2(\mathbb{R})$ and therefore forms a lattice there. This follows, for example, from Gabai's result [Ga] stating that *convergence* groups are Fuchsian. Indeed, since $\bar{\Gamma}$ is a convergence group, so is L. □

This proves Claim 3.1 and Theorem C follows.

Case $G \simeq \mathrm{SO}(n,1)$, $n > 2$. For this and the following cases of Γ being uniform in a rank one group $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, we shall use some known facts about the conformal structures on ∂X (following ideas which are going back to Mostow).

For $G \simeq \mathrm{SO}(n,1)$, $n > 2$, the boundary $\partial X \cong S^{n-1}$ of the real hyperbolic space $X = \mathbf{H}^n$, $n > 2$, carries a natural conformal structure (e.g. see [GP] 3.6). Any quasi-isometry class $\phi \in \mathrm{QI}\,(X)$ defines a quasi-conformal map $\beta(\phi)$ of S^{n-1} with the quasi-conformal constant bounded in terms of the quasi-isometric constant M of ϕ. In fact $\beta : \mathrm{QI}\,(X) \longrightarrow \mathcal{QC}(\partial X)$ is an isomorphism (of abstract groups). The subgroup of isometries $\mathrm{Isom}\,(X)$ is mapped isomorphically by β onto the group $\mathcal{C}(\partial X)$ of conformal maps. Hence, we have

$$\Psi_\pi(H) \subseteq \mathcal{QC}(\partial X) \subset \mathrm{Homeo}\,(\partial X), \qquad \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G)) = \mathcal{C}(\partial X)$$

The fact that all $\{\Phi_\pi(h)\}_{h\in H}$ can be represented by quasi-isometries $\{q_h\}_{h\in H}$ with a uniform bound on the quasi-isometric constant M ((c) of Lemma 3.3) means that $\Psi_\pi(H)$ is *uniformly quasi-conformal* subgroup of $\mathcal{QC}(S^{n-1})$. This group has an additional important property: its action on the *space of distinct triples*

$$\partial^3 X := \{(\xi_1, \xi_2, \xi_3) \in \partial X \times \partial X \times \partial X \mid \xi_i \neq \xi_j,\ i \neq j\}$$

is *cocompact*. This follows from the fact that already the action of $\Psi_\sigma(\mathrm{Ad}\,\Gamma) = \Psi_\pi(\Lambda) \subset \Psi_\pi(H)$ on the triple space is cocompact. Tukia [Tu] proved, that any uniformly quasi-conformal subgroup of $\mathcal{QC}(S^{n-1})$, $n > 2$, whose action on the triple space is cocompact, is conjugate into $\mathcal{C}(S^{n-1})$. Hence, there exists an $f \in \mathcal{QC}(\partial X)$, so that

$$f^{-1}\, \Psi_\pi(H)\, f \subset \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$$

However, no conjugation is really needed in our situation. Since both $\Psi_\pi(\Lambda) = \Psi_\sigma(\mathrm{Ad}\,\Gamma)$ and $f^{-1}\Psi_\pi(\Lambda)f$ are in $\mathcal{C}(\partial X)$, Mostow's rigidity implies that for some $g \in \mathcal{C}(\partial X)$, one has

$$(g \circ f)^{-1}\, \Psi_\pi(\lambda)\, (g \circ f) = \Psi(\lambda), \qquad (\lambda \in \Lambda)$$

In other words, $g \circ f$ is a Λ-equivariant continuous (actually quasi-conformal) map of ∂X to itself. However, the Λ-action (i.e. the $\mathrm{Ad}\,\Gamma$-action) on ∂X is known to be minimal and proximal. It is easy to see that such actions have no non-trivial equivariant continuous maps. Hence $f = g^{-1} \in \mathcal{C}(\partial X)$, and one has $\Psi_\pi(H) \subseteq \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$. This proves Claim 3.1 for uniform lattices Γ in $G \simeq \mathrm{SO}(n,1)$, $n > 2$.

Case $G \simeq \mathrm{SU}(n,1)$, $n > 1$. Here the symmetric space X is the complex hyperbolic spaces $\mathbf{CH}^n$ (of real dimension $2n$), and the appropriate conformal structure on ∂X corresponds to the Carnot-Caratheodory metric. As in the previous case, the group $\Psi_\pi(H)$ is uniformly quasi-conformal and its action on the triple space is cocompact. Following Tukia, Chow [Ch] proved that any such subgroup is conjugate into $\mathcal{C}(\partial X)$. As in the previous case, the fact that $\Psi_\pi(\Lambda)$ is already in $\mathcal{C}(\partial X)$, together with Mostow Rigidity, imply that no conjugation is needed. Thus

$$\Psi_\pi(H) \subseteq \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$$

and Claim 3.1 is proved.

Case $G \simeq \mathrm{Sp}(n,1)$ or F_4^{-20}. The symmetric spaces in these cases are the quaternionic hyperbolic space $X = \mathbf{QH}^n$ (corresponding to $G \simeq \mathrm{Sp}(n,1)$) and the Cayley plane $X = \mathbf{CaH}^2$ (corresponding to $G \simeq \mathrm{F}_4^{-20}$). These spaces are very rigid: Pansu [Pa] proved that for these spaces $\mathcal{QC}(\partial X) = \mathcal{C}(\partial X)$, while the latter again coincides with $\Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$, and Claim 3.1 follows.

Remark 3.8. Marc Bourdon suggested the following elegant argument for the above cases of rank-one $G \not\simeq \mathrm{PSL}_2(\mathbb{R})$. Let ν_0 denote the K-invariant measure on the boundary ∂X. It is well known that there is a function (closely related to the Busemann cocycle) $f : \partial^2 X \to \mathbb{R}_+$ such that the infinite Radon measure $dm_0(x,y) = d\nu_0(x)d\nu_0(y)/f(x,y)$ on the spaces of distinct pairs $\partial^2 X$ of points on the boundary is invariant and ergodic for the diagonal action of $\bar{\Gamma}$. Moreover, the conformal group can be identified in $\mathrm{Homeo}\,(\partial X)$ as the stabilizer of m_0 (see Sullivan [Su] for $\mathrm{SO}(n,1)$):

$$\Psi_\sigma(G) = \mathcal{C}(\partial X) = \{h \in \mathrm{Homeo}\,(\partial X) \mid (h \times h)_* m_0 = m_0\} \tag{3.7}$$

A fundamental step in Mostow's proof of strong rigidity implies that quasi-conformal mappings of ∂X preserve the measure class of ν_0. Hence m_0 is quasi-invariant for the diagonal $\Psi_\pi(H)$-action on $\partial^2 X$ and therefore the Radon measure

$$m_1 = \int_{H/\Gamma} (\Psi_\pi(h) \times \Psi_\pi(h))_* m_0 \, dm_{H/\Gamma}(h)$$

is $\Psi_\pi(H)$-invariant and is in the same measure class as m_0. Since m_0 is $\bar{\Gamma}$-ergodic, $m_1 = m_0$ and (3.7) yields $\Psi_\pi(H) \subseteq \Psi_\sigma(G)$.

Hence Claim 3.1 is proven and the proof of Theorem B can be completed exactly as in the Subsections 2.3 and 2.4 (the compactness of the kernel $\mathrm{Ker}(\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ in these cases can be deduced from the last assertion of Theorem 3.5).

3.4. Γ **is a uniform irreducible lattice in a semisimple** G **of higher rank.** For an irreducible higher rank symmetric space of non compact type X (G is simple of higher rank), it was proved by Kleiner and Leeb [KL], and independently by Eskin and Farb [EF], that $\mathrm{QI}(X) \cong \mathrm{Isom}(X)$. Hence, one has a homomorphism

$$\Phi_\pi : H \longrightarrow \mathrm{QI}(\Gamma) \cong \mathrm{QI}(X) \cong \mathrm{Isom}(X) \cong \mathrm{Aut}(\mathrm{Ad}\, G)$$

In the case of a semisimple G, the symmetric space $X = X_1 \times \cdots \times X_r$ is a product of symmetric spaces. For this case Kleiner and Leeb [KL] proved that $\mathrm{QI}(X)$ is equivalent (up to permutation of the factors) to $\mathrm{QI}(X_1) \times \cdots \times \mathrm{QI}(X_r)$. Applying the above arguments to each of the factors one realizes that $\Psi_\pi(H)$ is contained, up to finite index, in $\mathrm{Homeo}(\partial X_1) \times \cdots \times \mathrm{Homeo}(\partial X_r)$, and in fact $\Psi_\pi(H) \subseteq \Psi_\sigma(\mathrm{Aut}(\mathrm{Ad}\, G))$ (without finite index reduction in this statement), so that $\Phi : \Psi_\sigma^{-1} \circ \Psi_\pi : H \longrightarrow \mathrm{Aut}(\mathrm{Ad}\, G)$ is a continuous homomorphism satisfying (2.1). We should remark that the continuity of Ψ_π follows from the fact that the above factorization of $\mathrm{QI}(\Pi_1^r X_i)$ into the product $\Pi_1^r \mathrm{QI}(X_i)$ is effective in terms of the quasi-isometric constants (so an analogue of Theorem 3.5 holds). Special care should be taken of the case where one/some of the factors X_i are hyperbolic planes $\mathbf{H}^2$ (corresponding to $\mathrm{PSL}_2(\mathbb{R})$ factors of G). In this case $L = \Psi_\pi(H)$ is a locally compact subgroup of $\Pi_1^r \mathrm{Homeo}(\partial X_i)$ which contains copy/ies of $\mathrm{Homeo}(S^1)$, and one can verify that the arguments, similar to the case of a single $\mathrm{Homeo}(S^1)$, still apply (in particular such L has no small subgroups).

Remark 3.9. The case of lattices Γ in higher rank *semi-simple* Lie groups should probably be treated in the measure-theoretical framework of Theorem A, by resolving the difficulty pointed out in Remark 2.2.

3.5. Γ **is non-uniform in** $G \not\simeq \mathrm{SL}_2(\mathbb{R})$**.** Non-uniform irreducible lattices Γ in semisimple Lie groups $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ have only "algebraic" quasi-isometries, i.e. the quasi-isometry group $\mathrm{QI}(\Gamma)$ coincides with the commensurator group $\mathrm{Comm}(\Gamma)$ of Γ in $\mathrm{Isom}(X) \cong \mathrm{Aut}(\mathrm{Ad}\, G)$, defined by:

$$\mathrm{Comm}(\Gamma) := \{g \in \mathrm{Aut}(\mathrm{Ad}\, G) \mid [\Gamma : g^{-1}\Gamma g \cap \Gamma] < \infty\}$$

This remarkable fact was first discovered by Schwartz [S1] for non-uniform lattices in rank one simple Lie groups; and then proved for some higher rank cases by Farb-Schwartz [FS] and Schwartz [S2]. Using different ideas Eskin [Es] proved the above result when G has no rank-one factors. The combination of these methods enables to deduce the result in its full generality, as it is described by Farb ([Fa], pp. 710–711).

The commensurator $\mathrm{Comm}(\Gamma)$ is a countable (dense) subgroup of $\mathrm{Aut}(\mathrm{Ad}\, G)$. Clearly, $\Phi_\pi^{-1}(\{e\})$ is a measurable subset of H with a finite but positive Haar measure:

$$0 < c := m_H(\Phi_\pi^{-1}(\{e\})) \leq m_H(H/\Lambda) < \infty$$

Hence, the push forward measure $(\Phi_\pi) * m_H$ on $\mathrm{Aut}(\mathrm{Ad}\, G)$ is an atomic measure, equally distributed on some subgroup $\Gamma_* \supset \mathrm{Ad}\,\Gamma$. Moreover, $\mathrm{Ad}\,\Gamma$ has finite index in Γ_*, for

$$c \cdot [\Gamma_* : \mathrm{Ad}\,\Gamma] = m_H(H/\Gamma) < \infty$$

Hence Γ_* is a lattice in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, and taking $H_0 := \Phi_\pi^{-1}(\mathrm{Ad}\,\Gamma)$ we conclude the proof as in Subsection 2.4, Case (2).

The proof of Theorem B is now completed. $\square$

3.6. **An Exotic Example.** In this section we shall describe an example which shows that in Theorem C case (1) one cannot replace the "almost direct product" structure of H over a *compact abelian* group C, by a similar one over a *finite* abelian C.

Let Γ be a surface group of a compact Riemann surface of genus g. As an abstract group Γ has the presentation

$$\Gamma = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle$$

Such Γ can be embedded as a uniform lattice in $G = \mathrm{SL}_2(\mathbb{R})$. Let $\tilde{G}$ be the universal covering of G

$$1 \longrightarrow Z \longrightarrow \tilde{G} \stackrel{p}{\longrightarrow} G \longrightarrow 1$$

with $Z \cong \mathbb{Z}$. The preimage $\tilde{\Gamma} = p^{-1}(\Gamma)$ is a non-trivial central extension of Γ by Z, which can be presented as

$$\tilde{\Gamma} = \langle a_1, \dots, a_g, b_1, \dots, b_g, z \mid [a_i, z] = [b_i, z] = e,\ [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = z \rangle$$

where z generates the center Z. The crucial observation is that $\tilde{\Gamma}$ is linear. Indeed, if ρ is a the $\tilde{\Gamma}$-representation in the Heisenberg group

$$\rho(a_i) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(b_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(z) = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then $\mathrm{Ad} \times \rho$ is a faithful linear $\tilde{\Gamma}$-representation. (I would like to thank Gregory Soifer for pointing this out to me).

Hence $\tilde{\Gamma}$ is residually finite. Let K' be a profinite completion of $\tilde{\Gamma}$. The center $\overline{Z}$ of K' is a profinite completion of $Z \cong \mathbb{Z}$.

Now let G' be defined as the quotient of the direct product $\overline{Z} \times \tilde{G}$ by the diagonal embedding of Z. This embedding is discrete (due to the $\tilde{G}$-coordinate), so that G' becomes a locally compact group, which contains $\tilde{G}$ as a dense subgroup. Hence G' is connected, although it is neither locally connected, nor path-connected. The center of G' is a compact (profinite) abelian group, isomorphic to $\overline{Z}$. Observe, that $\tilde{\Gamma}$ embeds in G', via $\tilde{\Gamma} \subset \tilde{G} \subset G'$, however its image is no longer discrete.

Next, consider the *locally compact* group $H := (G' \times K')/\overline{Z}$, where $\overline{Z}$ is embedded diagonally in the centers of G' and K'. Denoting by π' the diagonal embedding of $\tilde{\Gamma}$ into $G' \times K'$, one observes that the homomorphism

$$\tilde{\Gamma} \stackrel{\pi'}{\longrightarrow} G' \times K' \longrightarrow H$$

has Z as its kernel, and therefore, factors through Γ as

$$\pi' : \tilde{\Gamma} \stackrel{p}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} H$$

where π is *injective.* We claim that $\pi(\Gamma)$ forms a *lattice* in H. Indeed, H contains two closed commuting subgroups, isomorphic to G' and K' (these are the projections of $G' \times \{e_{K'}\}$ and $\{e_{G'}\} \times K'$, respectively), which we shall again denote by G' and K', and $G', K' \lhd H$ intersect along their common center, which is isomorphic to $\overline{Z}$. Note that if $\Phi : H \longrightarrow H/K' \cong G$ denotes the projection, then

$$\Gamma \overset{\pi}{\longrightarrow} H \overset{\Phi}{\longrightarrow} G$$

coincides with the embedding $\Gamma \subset G$. Since $K' = \mathrm{Ker}(\Phi : H \longrightarrow G)$ is compact, we conclude that π is an embedding of Γ as a uniform lattice in H.

Any *connected* subgroup of H, which projects onto G, has to contain $\tilde{G} \subset G'$, and any *closed connected* one contains G', which has a compact, but not finite, abelian center $\overline{Z}$.

Remark 3.10. The above construction does not work for other simple Lie groups G, which have universal covering with an infinite center. It is known that for such G, there does not exist a residually finite $\Gamma' \subset \tilde{\Gamma} \subset \tilde{G}$ which projects onto $\Gamma \subset G$. However, it is not clear to the author whether a residually finite Γ' projecting on Γ can be found in $G' = \tilde{G} \times \overline{Z(G)}/Z(G)$.

Remark 3.11. A construction similar to (but somewhat simpler than) the above one gives an embedding of a surface group Γ in a locally compact group $H \cong G' \times K'/Z$, where G' is a simple Lie group with a finite center Z, with G' being locally isomorphic but not isomorphic to G.

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