Convolution kernels of 2D Fourier multipliers based on real analytic functions

Michael Greenblatt

May 27, 2016

1. Background and Theorem Statements.

In this paper, estimates are proven for convolution kernels associated to multipliers from a reasonably general class of compactly supported two-dimensional functions constructed out of real analytic functions. These estimates are both for overall decay rate and decay rate in specific directions. The estimates are sharp for a certain range of exponents appearing in the theorems. In a separate paper [G3], a class of "well-behaved" functions is described that contains a number of relevant examples and for which, after a little more work, these estimates can be explicitly described in terms of the Newton polygon of the function.

The compactly supported Fourier multipliers $m(x,y)$ we consider are as follows. For each (x_0, y_0) in the support of $m(x,y)$ we assume that on a neighborhood of $(0,0)$ the function $m(x_0+x, y_0+y)$ can be written in the form

$$m(x_0+x, y_0+y) = \alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^{n}|f_i(x,y)|^{\gamma_i} \tag{1.1}$$

Here $\alpha(x,y)$ is C^1 except at $(0,0)$ and for some constant A one has

$$|\alpha(x,y)| \leq A \tag{1.2}$$

$$|\nabla\alpha(x,y)| \leq A(x^2+y^2)^{-\frac{1}{2}} \tag{1.3}$$

The functions $f_i(x,y)$ are real analytic and not identically zero on a neighborhood of the origin. The set E is assumed to be either a disk $\{(x,y) : x^2+y^2 < r^2\}$ or to be expressible as a disjoint union of open sets $\cup_{i=1}^m E_i$, where each E_i is a region bounded by curves C_1, C_2 connecting the origin to a circle $x^2+y^2=r^2$, and the circle $x^2+y^2=r^2$ itself. The curves C_1 and C_2 are assumed to be either half of the graph of the form $y = h(|x|^{\frac{1}{N}})$ or $x = h(|y|^{\frac{1}{N}})$ for a real analytic h with $h(0) = 0$. There are two regions formed by the curves C_1, C_2, and the circle and we allow a given E_i to be either of them. We assume

This research was supported in part by NSF grant DMS-1001070.

that all the curves C_1 and C_2 are disjoint in the disk $\{(x,y) : x^2 + y^2 \leq r^2\}$ so that the E_i are wedge or sliver-shaped regions whose closures only intersect at the origin.

The above form of E is a convenient way to describe a general domain defined through real analytic functions. In fact any such curve is part of the zero-set of a real analytic function. For example, in the case of the graph of $y = h(x^{\frac{1}{N}})$ one can take $\prod_{j=0}^{N-1}(y - h(e^{\frac{2\pi ij}{N}} x^{\frac{1}{N}}))$. Conversely, by Puiseux's theorem the zero set of a real analytic function is locally the finite union of curves of the form used here.

The form of E used here allows us for example to define the multiplier in several ways on several regions. If the different regions can be defined via real analytic functions, then one can write the multiplier as the sum of several multipliers of the form used here, and then add the kernel estimates obtained by our theorems. Another reason to use this form is if instead of wanting $|f_i(x,y)|^{\gamma_i}$ in the multiplier, you wanted a factor to reflect the sign of $f_i(x,y)$, then you could write the multiplier as the sum of two terms depending on the sign of $f_i(x,y)$; the curves where $f_i(x,y) = 0$ can be incorporated into the boundary of E.

The only restriction we assume on the exponents γ_i is that $\chi_E(x,y)\prod_{i=1}^n f_i(x,y)^{\gamma_i}$ is integrable on a neighborhood of the origin; otherwise even taking the Fourier transform of $m(x,y)$ would involve delicate distribution theory issues.

Using a partition of unity we can write $m(x,y) = \sum_{j=1}^K m_j(x,y)$, where each $m_j(x_j + x, y_j + y)$ satisfies (1.1) for some (x_j, y_j). The convolution kernel of $m(x,y)$ can then be written in the form

$$K(t,u) = \sum_{j=1}^{M} \int m_j(x,y) e^{itx+iuy} \phi(x,y)\, dx\, dy \tag{1.4}$$

Although the multipliers of this paper do not appear to have been extensively studied before, they are related to damped scalar oscillatory integrals of the form

$$G(s,t,u) = \int_{\mathbf{R}^2} |f(x,y)|^{\alpha} e^{isS(x,y)+itx+iuy} \phi(x,y)\, dx\, dy$$

Here $S(x,y)$ is a real analytic function near the origin with $S(0,0) = 0$ and $\nabla S(0,0) = (0,0)$ and one seeks estimates of the form $|G(s,t,u)| \leq C(1 + |(s,t,u)|)^{-\epsilon}$. By taking $s = 0$ one is reduced to situations studied in this paper. Such oscillatory integrals come up frequenly when using the damping function techniques initiatied in [SoS] when studying maximal averages over surfaces, such as in the papers [CMa1] [IM] [IoSa1] [IoSa2] [G4]. On their own, such oscillatory integrals can be viewed as surface measure Fourier transforms for surfaces with damping functions, possibly singular, such as those considered in [CDMaM] [CMa2] [G1] [Gr].

In order to state the main theorems of this paper, we will need a couple of facts following from resolution of singularities which we will prove at the end of section 2.

Lemma 1.1. Let $g(x,y) = \chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$, where E, f_i, and γ_i are as before. There exist $c_1, c_2, c_3 > 0$, an $\epsilon > 0$, and $d = 0$ or 1 such that if $0 < r < c_3$ one has

$$c_1 r^\epsilon |\ln r|^d \leq \int_{x^2+y^2<r^2} g(x,y)\,dx\,dy \leq c_2 r^\epsilon |\ln r|^d \tag{1.5}$$

Lemma 1.2. Let $g(x,y)$ be as in Lemma 1.1. Suppose $v = (v_1, v_2)$ is a unit vector in $\mathbf{R}^2$, and let $v^\perp = (v_2, -v_1)$ be the orthogonal unit vector. There exist $\delta_v, c_v > 0$ and a $e_v = 0$ or 1 such that if $c < c_v$ then there are $a_{v,c}, b_{v,c} > 0$ such that for $0 < r < c$ one has

$$a_{v,c} r^{\delta_v} |\ln r|^{e_v} \leq \int_{\{(x,y):|(x,y)\cdot v^\perp|<r,\ |(x,y)\cdot v|<c\}} g(x,y)\,dx\,dy \leq b_{v,c} r^{\delta_v} |\ln r|^{e_v} \tag{1.6}$$

Note that for any direction v, the rate of decrease in (1.5) is at least as fast as the decrease rate in (1.6) since the domain of integration in (1.6) contains the disk of radius r centered at the origin, which is the domain of integration in (1.5).

We now give the local theorems for the (inverse) Fourier transform of $m(x,y)$ which will sum to give the overall kernel estimates. We use the following notation. Let $\phi(x,y)$ be a nonnegative bump function which is one on a neighborhood of (x_0, y_0), and let $m_{\phi,x_0,y_0}(x,y) = \phi(x_0+x, y_0+y)m(x_0+x, y_0+y)$. We assume that the support of $\phi(x,y)$ is small enough so that $\phi(x_0+x, y_0+y)m(x_0+x, y_0+y)$ can be written in the form (1.1). We then define $K_{\phi,x_0,y_0}(t,u)$ by

$$K_{\phi,x_0,y_0}(t,u) = \int_{\mathbf{R}^2} \phi(x,y)m(x,y)e^{itx+iuy}\,dx\,dy$$

$$= e^{itx_0+iuy_0}\int_{\mathbf{R}^2} m_{\phi,x_0,y_0}(x,y)e^{itx+iuy}\,dx\,dy$$

Thus $K_{\phi,x_0,y_0}(t,u)$ can be viewed as the contribution to the convolution kernel of $m(x,y)$ coming from the region near (x_0, y_0).

For each $f_i(x,y)$ appearing in (1.1), let $F_i(x,y)$ be the sum of the terms of $f_i(x,y)$'s Taylor expansion at $(0,0)$ of lowest total degree. The zeroes of a given $F_i(x,y)$ are either a finite union of lines through the origin, just the origin, or the empty set (in the case when $f_i(0,0) \neq 0$). We let $l_1, ... l_{p'}$ be the list of all such lines over all i (if there are any). We add to this list any lines that are tangent at the origin to the boundary curves C_1 and C_2 of the E_i as described after (1.3). We denote the combined list of lines by $l_1, ..., l_p$, with the understanding that the combined list might be empty.

We get the strongest results when the ϵ in Lemma 1.1 is less than $\frac{1}{2}$:

Theorem 1.3. Suppose (x_0, y_0) is in the support of $m(x,y)$ and let ϵ and d be as in Lemma 1.1 as applied to the $g(x,y)$ associated with $m(x_0+x, y_0+y)$. If $\epsilon < \frac{1}{2}$, then the following hold, where $|(t,u)|$ denotes the magnitude $(t^2+u^2)^{\frac{1}{2}}$ of the vector (t,u).

a) For a given line l through the origin, let l_H denote the points in $\mathbf{R}^2$ within distance H of l. Let δ_v and e_v be as in Lemma 1.2, where v is in the direction of l. If v is perpendicular to one of the lines $l_1, ..., l_p$, then if the support of $\phi(x, y)$ is sufficiently small, depending on v, there is a constant C depending $g(x, y)$, $\phi(x, y)$, H, l, and the constant A of $(1.2) - (1.3)$ such that for (t, u) in the strip l_H one has

$$|K_{\phi,x_0,y_0}(t, u)| \leq C(2 + |(t, u)|)^{-\delta_v}(\ln(2 + |(t, u)|))^{e_v} \tag{1.7a}$$

If v is not perpendicular to one of $l_1, ..., l_p$ (or there are no l_i at all), then $\delta_v = \epsilon$ and instead of $(1.7a)$ we have the estimate

$$|K_{\phi,x_0,y_0}(t, u)| \leq C(2 + |(t, u)|)^{-\epsilon}(\ln(2 + |(t, u)|))^{d} \tag{1.7b}$$

b) Let (δ, e) denote the slowest decay rate in part a) over all lines. If the support of $\phi(x, y)$ is sufficiently small there is a constant C' depending on $g(x, y)$, $\phi(x, y)$, and A such that for any (t, u) one has the estimate

$$|K_{\phi,x_0,y_0}(t, u)| \leq C'(2 + |(t, u)|)^{-\delta}(\ln(2 + |(t, u)|))^{e} \tag{1.8}$$

c) If there exists a $c > 0$ such that $\alpha(x, y)$ in (1.1) satisfies $\alpha(x, y) > c$ on a neighborhood of the origin, then parts a) and b) of this theorem are sharp in the sense that the exponents δ_v, $\delta_v = \epsilon$, and δ cannot be improved in $(1.7a)$, $(1.7b)$, and (1.8) respectively.

When $\epsilon \geq \frac{1}{2}$ but some $\delta_v < \frac{1}{2}$, we have the following weaker version of Theorem 1.3, which still gives the optimal overall decay rate of part b), but which does not give the best estimates in all directions.

Theorem 1.4. Suppose (x_0, y_0) is in the support of $m(x, y)$ and let ϵ and d be as in Lemma 1.1 as applied to the $g(x, y)$ associated with $m(x_0 + x, y_0 + y)$. If $\epsilon \geq \frac{1}{2}$, but there is at least one direction for which $\delta_v < \frac{1}{2}$, then the following hold.

a) There are at most finitely many directions for which the corresponding δ_v is less than $\frac{1}{2}$, and each such direction must be perpendicular to one of the lines $l_1, ..., l_p$. For each such direction, we have the same estimate as in Theorem 1.3: if the support of $\phi(x, y)$ is sufficiently small there is a constant C depending $g(x, y)$, $\phi(x, y)$, H, l, and A such that for (t, u) in the strip l_H one has

$$|K_{\phi,x_0,y_0}(t, u)| \leq C(2 + |(t, u)|)^{-\delta_v}(\ln(2 + |(t, u)|))^{e_v} \tag{1.9}$$

This estimate is sharp in the same sense as in Theorem 1.3 c).

For the remaining directions, we still have the (usually nonsharp) estimate that in place of (1.9) one has

$$|K_{\phi,x_0,y_0}(t, u)| \leq C(2 + |(t, u)|)^{-\frac{1}{2}}(\ln(2 + |(t, u)|))^{2} \tag{1.10}$$

b) The statement of part b) of Theorem 1.3 holds and is sharp in the same sense as in Theorem 1.3.

Our next theorem says that in the case that all δ_v are at least $\frac{1}{2}$, one still gets an exponent of at least $\frac{1}{2}$ in any direction, and also for the overall decay rate. As a result, Theorems 1.3 and 1.4 give the best overall decay rate whenever it is less than $\frac{1}{2}$.

Theorem 1.5. Let ϵ and d be as in Theorems 1.3 and 1.4. If $\delta_v \geq \frac{1}{2}$ for all directions v, then there is a constant C depending on $g(x,y)$, ϕ, and A such that one has the estimate

$$|K_{\phi,x_0,y_0}(t,u)| \leq C(2+|(t,u)|)^{-\frac{1}{2}}(\ln(2+|(t,u)|))^2 \tag{1.11}$$

The above theorems give local estimates for the convolution kernel associated to a given $m(x,y)$ of the type treated in the paper. One can then use a partition of unity to write $m(x,y) = \sum_{i=1}^{K} m_i(x,y)$, where one of the above theorems provides estimates for each $m_i(x,y)$, thereby giving global estimates for this kernel. When one obtains a sharp estimate for any $m_i(x,y)$, one typically obtains a sharp estimate for $m(x,y)$ as well; cancellation does not typically occur. We describe this phenomenon in the next theorem.

Theorem 1.6. Suppose $m(x,y) = \sum_{i=1}^{K} m_i(x,y)$ such that each $m_i(x,y)$ is localized enough so that Theorem 1.3 or Theorem 1.4 applies to $m_i(x,y)$. Suppose there is a $c > 0$ such that each function $\alpha(x,y)$ of (1.1) corresponding to any $m_i(x,y)$ satisfies $\alpha(x,y) > c$ on the support of $m_i(x,y)$. Suppose further that when adding the estimates given by Theorems 1.3 or 1.4 the resulting estimate is one that is stated by Theorem 1.3 or 1.4 to be sharp for at least one of the $m_i(x,y)$ that it came from. Then this estimate is also sharp for $m(x,y)$ in the same sense that it was stated to be sharp for any such $m_i(x,y)$.

To help understand heuristically why in general one will not get a better exponent than $\frac{1}{2}$ than in Theorems 1.3-1.6, we focus on Theorems 1.3a) and 1.4a) and consider the case where $E = \{(x,y) \in D : x > 0,\ x^2 < y < 2x^2\}$, where D is a small disk centered at the origin, and assume there are two $f_i(x,y)$, given by $f_1(x,y) = x$ and $f_2(x,y) = y - x^2$. We make no restrictions on γ_1, and let $\gamma_2 = -1 + \eta$ for some small η. Assume $\alpha(x,y)$ is identically equal to 1. Then the convolution kernel associated to the multiplier in this case is given by

$$K(t,u) = \int_D x^{\gamma_1}(y - x^2)^{-1+\eta} e^{itx+iuy}\, dx\, dy \tag{1.12}$$

Changing variables from y to $y + x^2$ and setting $t = 0$, we get

$$K(0,u) = \int_{\{(x,y)\in D:\, x>0,\ 0<y<x^2\}} x^{\gamma_1} y^{-1+\eta} e^{iux^2+iuy} \tag{1.13}$$

When η is very small, the $y^{-1+\eta}$ factor ensures that one gets very little decay in $K(t,u)$ due to the iuy term in the exponential; the behavior is driven by the x integral in (1.13) for fixed values of y. Stationary phase can be readily used on each dyadic piece of this x integral and the result is

$$|K(0,u)| \leq C \int_{\{(x,y)\in D:\, x>0,\ 0<y<x^2\}} x^{\gamma_1} y^{-1+\eta} \min\left(1, \frac{1}{|ux^2|^{\frac{1}{2}}}\right) \tag{1.14}$$

Converting back to the original variables and using that $y \sim x^2$ on the domain of integration yields

$$|K(0,u)| \leq C \int_{\{(x,y)\in D:\, x>0,\ x^2<y<2x^2\}} x^{\gamma_1} (y-x^2)^{-1+\eta} \min\left(1, \frac{1}{|uy|^{\frac{1}{2}}}\right) \tag{1.15}$$

Because of the exponent $\frac{1}{2}$ in the $\frac{1}{|uy|^{\frac{1}{2}}}$ factor in (1.15), in the u direction one can never get a better decay rate than $|u|^{-\frac{1}{2}}$ in (1.15). The u direction here corresponds to a direction perpendicular to a l_i in Theorems 1.3-1.4. At the same time, one may select γ_1 such that the exponent δ_v in Theorem 1.3-1.4 is a given value greater than $\frac{1}{2}$. While there is a slight improvement over the above heuristics due to the iuy term in (1.13), as η goes to zero, this improvement vanishes. Hence the statements of Theorems 1.3a) and 1.4a) will not hold in generality if we replace $\frac{1}{2}$ by any larger exponent. Similar considerations apply concerning the optimality of this exponent in the other parts of Theorems 1.3-1.6.

Examples like the above show that the sharp estimates of Theorems 1.3 and 1.4 do not hold in general if the exponents are greater than $\frac{1}{2}$. However, the sharpness proofs we will give in section 4 do extend to any $\delta_v > 0$ and $\epsilon > 0$ situations, meaning that in such situations one cannot prove better estimates than the above sharp estimates either. It is unclear if there is a general statement that can be stated simply that covers index ranges beyond $\frac{1}{2}$. We will however prove a theorem which does give at least some estimates in these ranges:

Theorem 1.7. Suppose $1 < p \leq \infty$ is such that for each (x_0, y_0) in the support of $m(x,y)$, the function $g(x,y) = \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$ is in $L^p(N)$ for some neighborhood N of the origin. Then if p' denotes the complementary exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, for some constant C depending on $m(x,y)$ and p one has that $|K(t,u)| \leq C(2 + |(t,u)|)^{-\frac{1}{p'}}$ when $p < \infty$, and $|K(t,u)| \leq C(2 + |(t,u)|)^{-1} \ln(2 + |(t,u)|)$ if $p = \infty$.

2. Resolution of singularities in two dimensions and some consequences.

We will make use of the real analytic case of the resolution of singularities theorem of [G1]- [G2], which goes as follows. Let $S(x,y) = \sum_{\alpha,\beta} S_{\alpha\beta} x^\alpha y^\beta$ be a real analytic function on a neighborhood of the origin, not identically zero, satisfying $S(0,0) = 0$.

Divide the xy plane into eight triangles by slicing the plane using the x and y axes and two lines through the origin, one of the form $y = mx$ for some $m > 0$ and one of the form $y = mx$ for some $m < 0$. One must ensure that these two lines are not ones on which the function $\sum_{\alpha+\beta=o} S_{\alpha\beta}x^\alpha y^\beta$ vanishes other than at the origin. After reflecting about the x and/or y axes and/or the line $y = x$ if necessary, each of the triangles becomes of the form $T_b = \{(x, y) \in \mathbf{R}^2 : x > 0,\, 0 < y < bx\}$ (modulo an inconsequential boundary set of measure zero). The version of the real analytic case of Theorem 2.1 of [G1] that we need (which is a restatement of Theorem 3.1 of [G2]) is as follows.

Theorem 2.1. Let $T_b = \{(x, y) \in \mathbf{R}^2 : x > 0,\, 0 < y < bx\}$ be as above. Abusing notation slightly, use the notation $S(x, y)$ to denote the reflected function $S(\pm x, \pm y)$ or $S(\pm y, \pm x)$ corresponding to T_b. Then there is a $a > 0$ and a positive integer N such that if F_a denotes $\{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq a,\, 0 \leq y \leq bx\}$, then one can write $F_a = \cup_{i=1}^n cl(D_i)$, such that for to each i there is a $k_i(x) = l_ix^{s_i} + ...$ with $k_i(x^N)$ real analytic and $s_i \geq 1$ such that after a coordinate change of the form $\eta_i(x, y) = (x, \pm y + k_i(x))$, the set D_i becomes a set D_i' on which the function $S \circ \eta_i(x, y)$ approximately becomes a monomial $d_ix^{\alpha_i}y^{\beta_i}$, α_i a nonnegative rational number and β_i a nonnegative integer in the following sense.

a) $D_i' = \{(x, y) : 0 < x < a,\, g_i(x) < y < G_i(x)\}$, where $g_i(x^N)$ and $G_i(x^N)$ are real analytic. If we expand $G_i(x) = H_ix^{M_i} + ...$, then $M_i \geq 1$ and $H_i > 0$, and consists of a single term $H_ix^{M_i}$ when $\beta_i = 0$.

b) Suppose $\beta_i = 0$. Then $g_i(x) = 0$. The set D_i' can be constructed such that for any predetermined $\eta > 0$ there is a $d_i \neq 0$ such that on D_i', for all $0 \leq l \leq \alpha_i$ one has

$$|\partial_x^l(S \circ \eta_i)(x, y) - d_i\alpha_i(\alpha_i - 1)...(\alpha_i - l + 1)x^{\alpha_i - l}| < \eta|d_i|x^{\alpha_i - l} \tag{2.1}$$

c) If $\beta_i > 0$, then $g_i(x)$ is either identically zero or $g_i(x)$ can be expanded as $h_ix^{m_i} + ...$ where $h_i > 0$ and $m_i > M_i$. The D_i' can be constructed such that such that for any predetermined $\eta > 0$ there is a $d_i \neq 0$ such that on D_i', for all $0 \leq l \leq \alpha_i$ and all $0 \leq m \leq \beta_i$ one has

$$|\partial_x^l\partial_y^m(S \circ \eta_i)(x, y) - \alpha_i(\alpha_i - 1)...(\alpha_i - l + 1)\beta_i(\beta_i - 1)...(\beta_i - m + 1)d_ix^{\alpha_i - l}y^{\beta_i - m}|$$
$$\leq \eta|d_i|x^{\alpha_i - l}y^{\beta_i - m} \tag{2.2}$$

It should be pointed out that the development of Theorem 2.1 of [G2] was influenced by the philosophy of [PS] where one divides a neighborhood of the origin into wedges on which $S(x, y)$ and its derivatives are well-behaved, as well as the general philosophy of resolution of singularities where one does changes of variables to monomialize a given real analytic function of interest. It should also be pointed out that in [G1] (but not [G2]), before proving Theorem 2.1 one assumes that one has rotated coordinates so that $\partial_y^oS(0, 0)$ and $\partial_x^oS(0, 0)$ are nonzero, where o is the order of the zero of $S(x, y)$ at $(0, 0)$. This was done to make the exposition of the smooth situation easier and is not necessary for the arguments to work.

For the purposes of proving our theorems, we will need to simultaneously resolve the singularities of several functions. As is well-known in the subject of resolution of singularities, one can often simultaneously resolve the singularities of several functions by resolving the singularities of their product. This is the case here as well, as can be seen from the following theorem.

Theorem 2.2. Suppose $S_1(x,y), ..., S_k(x,y)$ are real analytic functions on a neighborhood of the origin, none identically zero, with $S_j(0,0) = 0$ for each j. Let D_i', α_i, and β_i be as in Theorem 2.1 applied to $\prod_{j=1}^k S_j(x,y)$. Then one can further divide each D_i' into finitely many pieces D_{il}', such that on each D_{il}' an additional coordinate change of the form $(x,y) \rightarrow (x, y - c_{il}x^{M_i})$ or $(x, y - c_{il}x^{m_i})$, $c_{il} \geq 0$, will result in each $S_j(x,y)$ satisfying the conclusions of Theorem 2.1, with one difference: the domains D_{il}' with $\beta_i = 0$ now are only assumed to have the same form as the domains where $\beta_i > 0$. That is, D_{il}' is the form $\{(x,y) : 0 < x < a,\, g_{il}(x) < y < G_{il}(x)\}$, where $g_{il}(x^N)$ and $G_{il}(x^N)$ are real analytic, $G_{il}(x) = H_{il}x^{M_{il}} + ...$, and $g_{il}(x) = h_{il}x^{m_{il}} + ...$ where $1 \leq M_{il} < m_{il}$ and $h_{il} \geq 0, H_{il} > 0$.

Proof. Let $S_{ij}(x,y)$ denote $S_j \circ \eta_i(x,y)$; that is, the function S_j in the coordinates of D_i'. The idea of the proof is as follows. We will do a change of variables that converts D_i' into a domain which is in a certain sense comparable to a rectangle $\{(X,Y) : 0 < X < a,\, 0 < Y < b\}$ on which in the new coordinates the function $\prod_{j=1}^k S_{ij}(X,Y)$ is of the form $c(X,Y)m(X,Y)$, where $m(X,Y)$ is a monomial and $|c(X,Y)| > \epsilon$ for some $\epsilon > 0$. This will imply that for each j the function $S_{ij}(X,Y)$ is of the same form, and translating this back into the coordinates of D_i' will give us what we need.

The main point of the proof is to show that (2.1) and (2.2) hold. We break the domains D_i' into three cases. The first are the domains D_i' for which $\beta_i > 0$ and the lower boundary of D_i' is not the x-axis. The second are the domains for which $\beta_i > 0$ and he lower boundary of D_i' is the x-axis. The third are the domains for which $\beta_i = 0$.

Case 1. Analysis for D_i' for which $\beta_i > 0$ and the lower boundary of D_i' is not the x-axis.

We change variables by letting $Y = \frac{y}{x^{M_i}}$ and $X = \frac{x^{m_i}}{y}$. So $x = (XY)^{\frac{1}{m_i - M_i}}$ and $y = X^{\frac{M_i}{m_i - M_i}} Y^{\frac{m_i}{m_i - M_i}}$ here. So the upper boundary curve of D_i' gets sent to a curve of the form $Y = H_i + O(X^\epsilon)$ for some positive ϵ and the lower boundary curve of D_i' gets sent to a curve of the form $X = \frac{1}{h_i} + O(Y^\epsilon)$ for some positive ϵ. Thus D_i' transforms into a region D_i'' bounded by the X and Y axes and the curves $Y = H_i + O(X^\epsilon)$, $X = \frac{1}{h_i} + O(Y^\epsilon)$, and $(XY)^{\frac{1}{m_i - M_i}} = a$, where a as in the statement of Theorem 2.1.

Note that x and y are obtained from X and Y through monomials with rational nonnegative exponents. As a result, by the form of the coordinate changes in Theorem 2.1, in the X, Y variables the function $\prod_{j=1}^k S_{ij}$ will be real analytic in $X^{\frac{1}{K}}$ and $Y^{\frac{1}{K}}$ for some large positive integer K. Furthermore, by (2.2) the function $\prod_{j=1}^k S_{ij}$ will be equal to a monomial in $X^{\frac{1}{K}}$ and $Y^{\frac{1}{K}}$ plus a smaller error term. Because of this, along with the fact that $\prod_{j=1}^k S_{ij}$ is real analytic in $X^{\frac{1}{K}}$ and $Y^{\frac{1}{K}}$ and the fact that D'_{ij} contains a square of the form $(0,b)\times(0,b)$, on D''_i, the function $\prod_{j=1}^k S_{ij}$ must in fact be of the form $c_i(X,Y)X^{a_i}Y^{b_i}$ where $|c_i(X,Y)| > \epsilon$ for some $\epsilon > 0$. Since the product of the functions S_{ij} in the X-Y coordinates are of this form and each S_{ij} in these coordinates is a real analytic function of $X^{\frac{1}{K}}$ and $Y^{\frac{1}{K}}$, each S_{ij} must also be of the form $c_{ij}(X,Y)X^{a_{ij}}Y^{b_{ij}}$ with $|c_{ij}(X,Y)| > \epsilon$ for some $\epsilon > 0$.

We now go back into the (x,y) coordinates of D'_i. Here $S_{ij}(x,y)$ is of the form $c_{ij}(\frac{x^{m_i}}{y}, \frac{y}{x_i^M})x^{\alpha_{ij}}y^{\beta_{ij}}$ for some α_{ij} and β_{ij}. We must have that $\alpha_{ij}, \beta_{ij} \geq 0$; the terms of the Taylor series of $S_{ij}(x,y)$ in the coordinates of D'_i transform into a power series in X and Y under the above coordinate change; because of the form $c_{ij}(X,Y)X^{a_{ij}}Y^{b_{ij}}$ for S_{ij} in the X-Y variables one of the terms of the Taylor series of $S_{ij}(x,y)$ becomes a multiple of $X^{a_{ij}}Y^{b_{ij}}$ and the others become multiples of $X^{a'_{ij}}Y^{b'_{ij}}$ for $a'_{ij} \geq a_{ij}$ and $b'_{ij} \geq b_{ij}$. In particular $\alpha_{ij}, \beta_{ij} \geq 0$ since there must be a term of $S_{ij}(x,y)$'s Taylor series of the form $cx^{\alpha_{ij}}y^{\beta_{ij}}$.

We next write $D'_i = E_i^1 \cup E_i^2 \cup E_i^3$, where for a large N to be determined by our arguments we define $E_i^1 = \{(x,y) \in D'_i : Nx^{m_i} < y < \frac{1}{N}x^{M_i}\}$, $E_i^2 = \{(x,y) \in D'_i : y > \frac{1}{N}x^{M_i}\}$, and $E_i^3 = \{(x,y) \in D'_i : y < Nx^{m_i}\}$ We will see that E_i^2, and E_i^3 can be subdivided into domains after which a coordinate change of the form $(x,y) \rightarrow (x, y - a_{il}x^{M_i})$ or $(x,y) \rightarrow (x, y - a_{il}x^{m_i})$ respectively gives domains satisfying the conditions we need. We will see that E_i^1 already satisfies the conclusions of this theorem; if N is large enough then $S_{ij}(x,y)$ is already approximately $c_{ij}(0,0)x^{\alpha_{ij}}y^{\beta_{ij}}$ in the sense that (2.1) or (2.2) holds.

We start with E_i^1. Writing $S_{ij}(x,y) = c_{ij}(\frac{x^{m_i}}{y}, \frac{y}{x_i^M})x^{\alpha_{ij}}y^{\beta_{ij}}$ again, note that $Nx^{m_i} < y < \frac{1}{N}x^{M_i}$ is equivalent to $X, Y < \frac{1}{N}$, so if N is chosen sufficiently large then (2.1) or (2.2) holds for zeroth derivatives with $(\alpha_{ij}, \beta_{ij})$ being what is called (α_i, β_i) in (2.1) or (2.2) and $c_{ij}(0,0)$ being what is called d_i. Next, observe that $\frac{\partial}{\partial x}[c_{ij}(\frac{x^{m_i}}{y}, \frac{y}{x^{M_i}})x^{\alpha_{ij}}y^{\beta_{ij}}]$ is equal to

$$m_i \frac{1}{x}\left(\frac{x^{m_i}}{y}\right)\frac{\partial c_{ij}}{\partial x}\left(\frac{x^{m_i}}{y}, \frac{y}{x_i^M}\right)x^{\alpha_{ij}}y^{\beta_{ij}} - M_i \frac{1}{x}\left(\frac{y}{x^{M_i}}\right)\frac{\partial c_{ij}}{\partial y}\left(\frac{x^{m_i}}{y}, \frac{y}{x_i^M}\right)x^{\alpha_{ij}}y^{\beta_{ij}}$$

$$+\alpha_{ij}\frac{1}{x}\left(c_{ij}\left(\frac{x^{m_i}}{y}, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}}y^{\beta_{ij}}\right) \tag{2.3}$$

Since $Nx^{m_i} < y < \frac{1}{N}x^{M_i}$ on E_i^1, given any $\delta > 0$, if N is large enough the factors $\frac{x^{m_i}}{y}$ and $\frac{y}{x^{M_i}}$ in the first line of (2.3) will be of absolute value less than $\delta|c_{ij}(0,0)|$. Since $c_{ij}(x,y)$ is continuous and nonzero at $(0,0)$, this implies that given $\delta > 0$, if N is large enough the whole expression (2.3) will be within $\delta|c_{ij}(0,0)|x^{\alpha_{ij}-1}y^{\beta_{ij}}$ of $\alpha_{ij}\frac{1}{x}(c_{ij}(0,0)x^{\alpha_{ij}}y^{\beta_{ij}}) = \alpha_{ij}c_{ij}(0,0)x^{\alpha_{ij}-1}y^{\beta_{ij}}$. This is exactly what is required for (2.1) or (2.2) to hold.

A similar analysis holds for other derivatives. We now do the first y derivative explicitly. Note that $\partial_y[c_{ij}(\frac{x^{m_i}}{y}, \frac{y}{x^{M_i}})x^{\alpha_{ij}}y^{\beta_{ij}}]$ is equal to

$$-\frac{1}{y}\left(\frac{x^{m_i}}{y}\right)\frac{\partial c_{ij}}{\partial x}\left(\frac{x^{m_i}}{y}, \frac{y}{x_i^M}\right)x^{\alpha_{ij}}y^{\beta_{ij}} + \frac{1}{y}\left(\frac{y}{x^{M_i}}\right)\frac{\partial c_{ij}}{\partial y}\left(\frac{x^{m_i}}{y}, \frac{y}{x_i^M}\right)x^{\alpha_{ij}}y^{\beta_{ij}}$$

$$+\beta_{ij}\frac{1}{y}\left(c_{ij}\left(\frac{x^{m_i}}{y}, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}}y^{\beta_{ij}}\right) \tag{2.4}$$

As with the x derivative, if N is large enough the two terms of the first line of (2.4) can be made smaller than any $\delta|c_{ij}(0,0)|x^{\alpha_{ij}}y^{\beta_{ij}-1}$, while the second line of (2.4) can be made within $\delta|c_{ij}(0,0)|x^{\alpha_{ij}}y^{\beta_{ij}-1}$ of $\beta_{ij}c_{ij}(0,0)x^{\alpha_{ij}}y^{\beta_{ij}-1}$. Hence if N is large enough then (2.1) or (2.2) will hold as needed.

Higher order derivatives are done similarly; we get a main term given by the corresponding derivative of $c_{ij}(0,0)x^{\alpha_{ij}}y^{\beta_{ij}}$, and various other smaller terms which either have additional $\frac{y}{x^{M_i}}$ factors or additional $\frac{x^{m_i}}{y}$ factors. Then the continuity of c_{ij} at $(0,0)$ gives that (2.1) or (2.2) holds.

This concludes our analysis on the domain E_i^1.

We now examine E_i^2. On E_i^2 we change variables to $Y = \frac{y}{x_i^M}$ and $X = x$. Note that $y = X^{M_i}Y$ in this situation. The upper boundary curve of E_i^2 gets sent to a curve of the form $Y = H_i + O(X^\epsilon)$ for some positive ϵ, while the lower boundary curve of E_i^2 gets sent to the line $y = \frac{1}{N}$. Thus the domain E_i^2 transforms into the rectangle-like region F_i bounded by the curves $Y = \frac{1}{N}$ and $Y = H_i + O(X^\epsilon)$, and the lines $X = 0$ and $X = a$, where a is as in the statement of Theorem 2.1.

By our use of Theorem 2.1, $\prod_{j=1}^k S_{ij}(x,y)$ is of the form $d_i x^{\alpha_i}y^{\beta_i}$ plus a smaller error term on E_i^2, so in the X-Y coordinates $\prod_{j=1}^k S_{ij}$ is of the form $d_i X^{\alpha_i + M_i\beta_i}Y^{\beta_i}$ plus a smaller error term. Due to the fact that F_i contains some square $(0,b)\times(0,b)$ and the fact that $\prod_{j=1}^k S_{ij}$ is a real analytic function of some $X^{\frac{1}{K}}$ and $Y^{\frac{1}{K}}$, analogous to the E_i^1 case $\prod_{j=1}^k S_{ij}$ can be written in the form $e_i(X,Y)X^{\alpha_i+M_i\beta_i}Y^{\beta_i}$ where $|e_i(X,Y)| > \epsilon$ for some positive ϵ. So we can argue as in the E_i^1 case and say that each $S_{ij}(x,y)$ can be similarly written in the form $e_{ij}(X,Y)X^{a'_{ij}}Y^{b'_{ij}}$, and we can transform back into the x-y coordinates and write

$$S_{ij}(x,y) = e_{ij}(x, \frac{y}{x^{M_i}})x^{\alpha'_{ij}}y^{\beta'_{ij}} \tag{2.5}$$

Here $\alpha'_{ij}, \beta'_{ij} \geq 0$ and $|e_{ij}(x, \frac{y}{x^{M_i}})| > \epsilon$ for some positive ϵ. We rewrite (2.5) as

$$S_{ij}(x,y) = f_{ij}(x, \frac{y}{x^{M_i}})x^{\alpha'_{ij}+M_i\beta'_{ij}} \tag{2.6}$$

Here $f_{ij}(x,y) = y^{\beta'_{ij}}e_{ij}(x,y)$. Note that $|f_{ij}(x, \frac{y}{x^{M_i}})|$ is bounded below by some positive number since $y > \frac{1}{N}x^{M_i}$ on E_i^2. If one does a variable change $(x,y) \to (x, y - cx^{M_i})$ for some positive c, then (2.6) will hold with $f_{ij}(x,y)$ replaced by $f_{ij}(x,y-c)$. Thus if we can show that $S_{ij}(x,y)$ satisfies (2.1) for x and $|\frac{y}{x^{M_i}}|$ sufficently small under the assumption that $f(0,0) \neq 0$ then we are done; shrinking a if necessary, by a compactness argument the domain E_i^2 can be written as the union of finitely many slivers on each of which a coordinate change $(x,y) \rightarrow (x, y - cx^{M_i})$ turns the sliver into a domain on which (2.1) holds. Since these are amongst the variable changes are allowed by this theorem, we will then be done.

We proceed to show that $S_{ij}(x,y)$ satisfies (2.1) for x and $|\frac{y}{x^{M_i}}|$ sufficently small, under the assumption that $f_{ij}(0,0) \neq 0$. Since $X = x$ and $Y = \frac{y}{x^{M_i}}$, the continuity of f_{ij} is enough to ensure (2.1) holds for zeroth derivatives when x and $|\frac{y}{x^{M_i}}|$ are sufficently small. Next, we take an x derivative of (2.6) and get

$$\frac{\partial S_{ij}}{\partial x}(x,y) = \frac{\partial f_{ij}}{\partial x}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha'_{ij}+M_i\beta'_{ij}} - M_i\frac{y}{x^{M_i}}\frac{\partial f_{ij}}{\partial y}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha'_{ij}+M_i\beta'_{ij}-1}$$
$$+(\alpha'_{ij} + M_i\beta'_{ij})f_{ij}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha'_{ij}+M_i\beta'_{ij}-1} \tag{2.7}$$

The first term will be much smaller than the last if x is small enough since the exponent on x is larger. Given any $\delta > 0$, if $|\frac{y}{x^{M_i}}|$ is small enough the second term will be smaller than $\delta x^{\alpha'_{ij}+M_i\beta'_{ij}-1}$. By continuity of f_{ij}, if x and $|\frac{y}{x^{M_i}}|$ are small enough then $f_{ij}(x, \frac{y}{x^{M_i}})$ will be similarly within δ of $f_{ij}(0,0)$. Hence given any $\delta > 0$, if x and $|\frac{y}{x^{M_i}}|$ are small enough then $\frac{\partial S_{ij}}{\partial x}(x,y)$ is within $\delta x^{\alpha'_{ij}+M_i\beta'_{ij}-1}$ of $(\alpha'_{ij} + M_i\beta'_{ij})f_{ij}(0,0)x^{\alpha'_{ij}+M_i\beta'_{ij}-1}$. This is exactly (2.1) for first derivatives. (Since the possible $|f_{ij}(0,0)|$ are bounded below we can replace δ by $\delta|f_{ij}(0,0)|$ as in (2.1).)

Higher derivatives are treated similarly; we get a main term given by the corresponding derivative of $f_{ij}(0,0)x^{\alpha'_{ij}}y^{\beta'_{ij}}$, and various other smaller terms which either have additional $\frac{y}{x^{M_i}}$ factors or higher powers of x in them. Then the continuity of f_{ij} at $(0,0)$ gives that (2.1) holds. This concludes our analysis of the domain E_i^2.

The domains E_i^3 are done exactly the same way as the domains E_i^2, replacing M_i by m_i and the condition $\frac{y}{x^{M_i}} > \frac{1}{N}$ by the condition $\frac{y}{x^{m_i}} < N$. This concludes our discussion of Case 1.

Case 2. Analysis for D'_i for which $\beta_i > 0$ and the lower boundary of D'_i is the x-axis.

The argument here is very similar to the analysis of E_i^2 above. We write $D'_i = G_i^1 \cup G_i^2$, where for a large N to be determined by our arguments $G_i^1 = \{(x,y) \in D'_i : y < \frac{1}{N}x^{M_i}\}$ and $G_i^2 = \{(x,y) \in D'_i : y > \frac{1}{N}x^{M_i}\}$. The argument for G_i^2 is exactly the same as the argument for E_i^2 in case 1 and we do not repeat it here.

As for G_i^1, we once again change variables to $Y = \frac{y}{x_i^M}$ and $X = x$. The upper boundary curve of G_i^1 gets sent to the line $y = \frac{1}{N}$ and the lower boundary curve of G_i^1, the

x-axis, gets sent to the X-axis. So the domain G_i^1 transforms into the rectangular region G_i' bounded by the X and Y axes, the line $Y = \frac{1}{N}$, and the line $X = a$.

Exactly as in E_i^2 analysis, we have that in the X-Y coordinates each S_{ij} is of the form $c_{ij}(X,Y)X^{a_{ij}}Y^{b_{ij}}$ and that $S_{ij}(x,y) = c_{ij}(x, \frac{y}{x^{M_i}})x^{\alpha_{ij}}y^{\beta_{ij}}$. Here $\alpha_{ij}, \beta_{ij} \geq 0$ and $|c_{ij}(X,Y)| > \epsilon$ for some positive ϵ.

Our goal is to show (2.1) or (2.2) holds for x and $\frac{y}{x^{M_i}}$ sufficiently small, since shrinking x is equivalent to shrinking a in Theorem 2.1, which we may do, and shrinking $\frac{y}{x^{M_i}}$ is equivalent to increasing N, which we may also do. For zeroth derivatives, like in the analysis of E_i^2, that (2.2) holds for x and $\frac{y}{x^{M_i}}$ sufficiently small is an immediate consequence of the continuity of c_{ij}. For first x derivatives, we have an expression for $\frac{\partial S_{ij}}{\partial x}(x,y)$ analogous to (2.7), namely

$$\frac{\partial S_{ij}}{\partial x}(x,y) = \frac{\partial c_{ij}}{\partial x}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}}y^{\beta_{ij}} - M_i\frac{y}{x^{M_i}}\frac{\partial c_{ij}}{\partial y}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}-1}y^{\beta_{ij}}$$

$$+\alpha_{ij}c_{ij}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}-1}y^{\beta_{ij}} \tag{2.8}$$

Like in the E_i^2 situation, the third term dominates the first for x small enough since the power of x in the first term is greater. In addition, if N is large enough, which we may assume, the $\frac{y}{x^{M_i}}$ factor in the second term ensures that the second term can be made less than any $\delta|c_{ij}(0,0)|x^{\alpha_{ij}-1}y^{\beta_{ij}}$ in absolute value. Hence by continuity of $c_{ij}(x,y)$, the expression (2.8) can be assumed to be within $\delta|c_{ij}(0,0)|x^{\alpha_{ij}-1}y^{\beta_{ij}}$ of $\alpha_{ij}c_{ij}(0,0)x^{\alpha_{ij}-1}y^{\beta_{ij}}$. This gives (2.1) for first x derivatives.

We next look at $\frac{\partial S_{ij}}{\partial y}(x,y)$, given by

$$\frac{\partial S_{ij}}{\partial y}(x,y) = \frac{y}{x^{M_i}}\frac{\partial c_{ij}}{\partial y}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}}y^{\beta_{ij}-1} + \beta_{ij}c_{ij}\left(x, \frac{y}{x^{M_i}}\right)x^{\alpha_{ij}}y^{\beta_{ij}-1} \tag{2.9}$$

Since $\frac{y}{x^{M_i}} < \frac{1}{N}$ and c_{ij} is continuous at $(0,0)$, again if N is sufficiently large and a is sufficiently small then (2.2) will hold for first y derivatives.

Higher order derivatives are done similarly; we get a main term given by the corresponding derivative of $c_{ij}(0,0)x^{\alpha_{ij}}y^{\beta_{ij}}$, and various other smaller terms which either have additional $\frac{y}{x^{M_i}}$ factors or higher powers of x in them. Then the continuity of c_{ij} at $(0,0)$ gives that (2.1) or (2.2) holds. This completes the argument for case 2.

Case 3. Analysis for D_i' for which $\beta_i = 0$.

For this case, the argument for E_i^2 in case 1 works if we replace β_i by 0. For in this situation the β_{ij} and β_{ij}' all become zero. We omit the details for brevity.

This concludes the proof of Theorem 2.2.

The proof above gives the following strengthening of Theorem 2.2:

Corollary 2.3. For any given K, however large, for any predetermined $\eta > 0$ the D_i' can be constructed so that (2.1) and (2.2) hold for all $\alpha_i, \beta_i < K$.

Note that Corollary 2.3 explains why the β_i in Theorem 2.2 must all be non-negative integers; if some β_i were not a positive integer, then by taking sufficiently many y derivatives one would get an unbounded function. However, since S is real analytic its y derivatives are bounded and therefore the same is true for $S \circ \eta_i(x, y) = S(x, y \pm k_i(x))$.

Proof of Lemma 1.1.

If each $f_i(0,0) \neq 0$, the result easily follows by finding the area of the portion of E within distance r of the origin, so we assume at least one $f_i(0,0) \neq 0$. We can also replace each f_i for which $f_i(0,0) \neq 0$ by the constant function 1, so without loss of generality we can remove these functions and assume that each $f_i(0,0) = 0$.

It suffices to prove (1.5) replacing integrals over discs centered at the origin with integrals over rectangles of fixed edge length ratio, and this is what we will do. We apply Theorem 2.2 to $f_1, ..., f_n$, and the result is a rectangle centered at the origin on which Theorem 2.2 holds. We will show (1.5) for dilations of this rectangle. Theorem 2.2 provides slivers of the form $S = \{(x, y) : 0 < x < a, g_{il}(x) < y < G_{il}(x)\}$ with $g_{il}(x^N)$ and $G_{il}(x^N)$ real analytic for some positive integer N. On this set, in the new coordinates each $|f_i(x, y)|$ is within a constant factor of some $x^{\alpha_i} y^{\beta_i}$. Thus the product $\prod_{i=1}^n |f_i(x, y)|^{\gamma_i}$ is also within a constant factor of some $M(x, y) = x^\alpha y^\beta$.

If one integrates $M(x, y)$ over the set $\{(x, y) : 0 < x < r, g_{il}(x) < y < G_{il}(x)\}$, one obtains an expression of the form $cr^a(\ln r)^b + o(r^a(\ln r)^b)$, where the $r^a(\ln r)^b$ term is derived from the leading terms of the Taylor expansions of $g_{il}(x)$ and $G_{il}(x)$. Here $b = 0$ or 1. Since the coordinate changes of Theorem 2.2 all have Jacobian 1, the integral of $M(x, y)$ over this sliver in its original coordinates will be of the same form.

If one now inserts a $\chi_E(x, y)$ factor and looks at the integral of $\chi_E(x, y)M(x, y)$ over the sliver S in the original coordinates, and transfers to the new coordinates, instead of integrating over S in the new coordinates, one integrates over a portion cut out by at most finitely many functions of the form $y = g(x)$ where some $g(x^N)$ is real analytic. Again direct integration reveals that the result is of the same form $cr^a(\ln r)^b + o(r^a(\ln r)^b)$. Hence the integral of $M(x, y)$ over $S \cap E$, in the original coordinates or final coordinates, is of this form. Since $\chi_E(x, y) \prod_{i=1}^n |f_i(x, y)|^{\gamma_i}$ is within a bounded factor of $M(x, y)$, we conclude that the integral of $\chi_E(x, y) \prod_{i=1}^n |f_i(x, y)|^{\gamma_i}$ is also within a constant factor of some $cr^a(\ln r)^b$. Adding this over all slivers gives (1.5), completing the proof of Lemma 1.1.

Proof of Lemma 1.2.

Let l be a line segment centered at the origin with direction v such that each $f_i(x,y)$ and each of the functions defining E is defined on a neighborhood of l. Using a partition of unity, we let $p_1, ..., p_k$ be points on l such that to each p_j there is a rectangle R_j centered at p_j such that either the product $\prod_i^n f_i(x,y)$ is nonzero on a neighborhood of $cl(R_j)$ or such that Theorem 2.2 holds for the product of the nonzero $f_i(x,y)$ on the rectangle R_j when we center at p_j and have rotated so that the v direction has become the x direction. It suffices to prove (1.6) for the portion of the integral contained in a given R_j since the overall result will follow simply by adding these statements over all j.

As in part a), the estimates for the rectangles where $\prod_i^n f_i(x,y)$ is nonzero follow from a straightforward integration, so we assume at least one $f_i(x,y)$ is zero at p_j. Analogous to part a) we can assume the partition of unity is such that we may replace all of the $f_i(x,y)$ which are nonzero at p_j by the constant function 1. Thus without loss of generality we can assume each $f_i(p_j) = 0$. Since we have rotated so that v is the x direction, our goal is to understand as a function of r the integral of $\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$ over the portion of R_j for which $|y| < r$. As in part a), it suffices to show (1.6) for the portion of the integral over $|y| < r$ coming from each of the slivers arising from Theorem 2.2, as the overall result will then follow via adding over all slivers.

If the sliver is one of the ones adjacent to the upper or lower boundaries of the rectangle R_j, then the coordinate changes of Theorem 2.2 turn the lines $y = \pm r$ into the line $x = r$, and the situation reduces to the one considered in part a), so we have the desired estimates in this situation. Assume therefore that the sliver is one of the ones adjacent to the right or left boundaries of R_j. The overall coordinate change in Theorem 2.2 is of the form $(x,y) \to (\pm x, \pm y + k(x))$, where some $k(x^N)$ is real analytic. If $k(x)$ happens to be the zero function, then $\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$ is already comparable in magnitude to some $M(x,y)$ of the form $x^c y^d$, so one may perform a direct integration of $M(x,y)$ to get an expression of the form $cr^a(\ln r)^b + o(r^a(\ln r)^b)$. The presence of a $\chi_E(x,y)$ factor will not change the resulting form, for the same reasons as in part a).

If $k(x)$ is not identically zero, we denote by p the degree of the initial term of the Taylor expansion of $k(x)$ at the origin. Cutting off the sliver at height $y = r$ or $y = -r$ in the original coordinates has a similar effect as cutting off the sliver with a vertical line $x = r^{\frac{1}{p}}$ or $-r^{\frac{1}{p}}$; when $k(x)$ is not identically zero, by construction the sliver in the original coordinates is always contained within a wedge $c_1|x|^p < y < c_2|x|^p$ that is in one of the four quadrants. In view of the monomial form of the functions in the final coordinates, the integral of $\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$ over the portion of the sliver in the original coordinates where $y < r$ will therefore be within a constant factor of the integral over the portion of the sliver in the original coordinates where $x < r^{\frac{1}{p}}$. This can then be computed directly in the same way as one computed the integral over the portion where $x < r$ for the first kind of sliver, and in part a) of this lemma. The result will once again be comparable to $r^a(\ln r)^b$ for some a and b. Thus we see that we have such an expression for all slivers, and the proof of Lemma 1.2 is complete.

3. Proofs of the estimates of Theorem 1.3, 1.4, 1.5 and Theorem 1.7.

We start with the well-known Van der Corput lemma (see p. 334 of [S]).

Lemma 3.1. Suppose $k \geq 2$ and $h(x)$ is a C^k function on the interval $[a,b]$ with $|h^{(k)}(x)| > A$ on $[a,b]$ for some $A > 0$. Let $\phi(x)$ be C^1 on $[a,b]$. If $k \geq 2$ there is a constant c_k depending only on k such that

$$\left|\int_a^b e^{ih(x)}\phi(x)\,dx\right| \leq c_k A^{-\frac{1}{k}}\left(|\phi(b)| + \int_a^b |\phi'(x)|\,dx\right)$$

If $k = 1$, the same is true if we also assume that $h'(x)$ is monotonic on $[a,b]$.

Throughout most of this section, we will be focusing on local behavior near a given (x_0, y_0). Namely, using the notation of (1.1), for $\phi(x,y)$ supported on a small neighborhood of (x_0, y_0) and various sets S we will be looking at quantities of the form

$$\left|\int_S \phi(x_0 + x, y_0 + y)\alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

To simplify notation, we will just write $\alpha(x,y)$ in place of $\phi(x_0 + x, y_0 + y)\alpha(x,y)$ with the understanding that $\alpha(x,y)$ is to be supported on a sufficiently small neighborhood of $(0,0)$ for our arguments to work.

Our next lemma provides the key Fourier transform estimate for a given sliver arising from Theorem 2.2. Theorems 1.3 and 1.4 will be proven by adding these estimates over all slivers and interpreting the result in an appropriate way.

Lemma 3.2. Let S be a sliver in the original coordinates arising from an application of Theorem 2.2 to $f_1(x,y)$,...,$f_n(x,y)$, and real analytic functions whose zero sets contain all the boundary curves of E on a neighborhood of the origin (recall such functions always exist). Then if the function $\alpha(x,y)$ in (1.1) is supported on the neighborhood of the origin on which we are applying Theorem 2.2 and S is one of the slivers coming from the $|y| < b|x|$ region, we have the estimate

$$\left|\int_S \alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

$$\leq C\int_S (1 + |(t,u)\cdot v||(x,y)| + |u||(x,y)\cdot v^{\perp}|)^{-\frac{1}{2}}\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \qquad (3.1)$$

Here v denotes a unit vector tangent to the sliver S at the origin, and $v^\perp$ a normal vector; in the case where the two boundary curves of S at the origin have different tangents (i.e. S is a "wedge") then v denotes the tangent to the boundary curve of S nearest to the x-axis. The constant C here depends on the function $\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}$, E, the application of Theorem 2.2 being used and the constant A of $(1.2)-(1.3)$. If S is a sliver from the $|y| > b|x|$ region the corresponding estimate holds with the $|u|$ factor replaced by $|t|$ and one replaces the x-axis with the y-axis in the above.

Proof. We examine the integral $\int_S \alpha(x,y)\,\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy$ in the new coordinates after applying Theorem 2.2. The coordinate change transferring old coordinates to new is either of the form $(x,y)\to(\pm x, \pm y + k(x))$, or consists of a reflection $(x,y)\to(y,x)$ followed by a mapping of such form. Here $k(x^N)$ is real analytic for some positive integer N. We will consider only the case where it is of the form $(x,y)\to(x,y+k(x))$ as Lemma 3.2 for the other situations follow from this case as applied to reflected versions of $f_1(x,y),...,f_n(x,y)$ and the real analytic functions defining the boundary curves of E.

In the new coordinates, $\int_S \alpha(x,y)\,\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy$ becomes

$$\int_D \alpha(x,y+k(x))\,\chi_E(x,y+k(x))\,\prod_{i=1}^n |f_i(x,y+k(x))|^{\gamma_i} e^{itx+iuy+iuk(x)}\,dx\,dy$$

Here D denotes the sliver in the new coordinates (what is called D_{il}' in the notation of Theorem 2.2). Because the real analytic functions defining the boundary curves have had their singularities resolved, those functions are comparable to monomials in the new coordinates. In particular, they cannot have zeroes in D. Hence $\chi_E(x,y+k(x))$ is either identically zero or identically 1 on D. Clearly we need only consider the case where it is identically 1. In addition, since the order of the zero of $k(x)$ at the origin is at least one, $\alpha(x,y+k(x))$ satisfies the estimates $(1.2)-(1.3)$ since $\alpha(x,y)$ does. So we denote $\alpha(x,y+k(x))$ by $\beta(x,y)$ and we are considering the following expression, where $\beta(x,y)$ satisfies $(1.2)-(1.3)$.

$$\int_D \beta(x,y)\,\prod_{i=1}^n |f_i(x,y+k(x))|^{\gamma_i} e^{itx+iuy+iuk(x)}\,dx\,dy \tag{3.2}$$

Note that each $f_i(x,y+k(x))$ here is comparable to a monomial in the sense of Theorem 2.2. Next, since order of the zero of $k(x)$ at the origin is at least 1, we may write $k(x)=cx+l(x)$, where $l(x)$ has a zero of order greater than one at the origin. Here c and/or $l(x)$ may be zero. Accordingly, (3.2) can be rewritten as

$$\int_D \beta(x,y)\,\prod_{i=1}^n |f_i(x,y+k(x))|^{\gamma_i} e^{i(t+cu)x+iuy+iul(x)}\,dx\,dy \tag{3.3}$$

We denote the expression (3.3) by I, and we divide the integral I dyadically in the x and y variables. Namely, for a nonnegative smooth compactly-supported function $s(x)$ on $\mathbf{R}$

that vanishes on a neighborhood of 0, we write $I = \sum_{jk} I_{jk}$, where

$$I_{jk} = \int_D s(2^j x)s(2^k y)\beta(x,y) \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i} e^{i(t+cu)x+iuy+iul(x)}\, dx\, dy \tag{3.4}$$

We will apply the Van der Corput lemma (Lemma 3.1) in (3.4) in the x and/or y direction. Adding the result over all j and k will give the needed bounds for I. We start with the y-direction, which it will turn out will only be needed when $l(x)$ is identically zero. We apply the Van der Corput Lemma for first derivatives in the y-direction. By applying Corollary 2.3 for first y derivatives on each monomial-like $f_i(x, y + k(x))$, we see that taking a y derivative of $\prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}$ introduces a factor of magnitude at most $C\frac{1}{y}$. By (1.3) we have $|\partial_y \beta(x,y)| \leq C\frac{1}{y}$, and the support condition on $s(y)$ ensures that the y derivative of the $s(2^j y)$ factor introduces a factor satisfying the same upper bounds. Thus if Q_{jk} denotes the rectangle $[2^{-j-1}, 2^{-j}] \times [2^{-k-1}, 2^{-k}]$, the Van der Corput lemma for first derivatives leads to a bound of

$$|I_{jk}| \leq C \int_{Q_{jk}} \frac{1}{|uy|} \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.5}$$

Just taking absolute values and integrating in (3.4) leads to the bound

$$|I_{jk}| \leq C \int_{Q_{jk}} \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.6}$$

Thus combining (3.5) and (3.6) we obtain

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|uy|}\right) \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.7}$$

For our purposes however, we only need the weaker statement

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|uy|^{\frac{1}{2}}}\right) \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.8}$$

Next, in the event that $l(x)$ is not identically zero, we apply the Van der Corput lemma for second derivatives in the x direction. Note $l''(x) \sim \frac{l(x)}{x^2}$ on a small enough neighborhood of the origin (which we may assume we are in). By Corollary 2.3, applying an x derivative to $\prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}$ yields a factor of at most $C\frac{1}{x}$. This time, by (1.3) we have $|\partial_x \beta(x,y)| \leq C\frac{1}{x}$, and the support condition on $s(x)$ ensures that taking the x derivative of the $s(2^j x)$ factor incurs a factor satisfying the same upper bounds. Thus applying the Van der Corput lemma we get

$$|I_{jk}| \leq C \int_{Q_{jk}} \frac{1}{|ul(x)|^{\frac{1}{2}}} \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.9}$$

(The $\frac{1}{x^2}$ factor one gets from taking the second derivative of $l(x)$ is exactly enough to compensate for the $\frac{1}{x}$ that one normally gets in such applications of the Van der Corput lemma.) As in the steps from (3.5) – (3.8), this leads to

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|ul(x)|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.9$'$}$$

Lastly, suppose that on the domain of integration in (3.4) one has $\inf |(t + cu)x| > B \sup |ul(x)|$ (such as when $l(x)$ is identically zero), where the constant B is large enough to ensure that if we are on a sufficiently small neighborhood of the origin, which we may assume, the absolute value of the first x-derivative of the phase in (3.4) is bounded below by $\frac{1}{2}|t + cu|$. In this situation, we may apply the Van der Corput lemma for first derivatives in the x direction. This time we obtain a bound of

$$|I_{jk}| \leq C \int_{Q_{jk}} \frac{1}{|(t + cu)x|} \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.10}$$

Like in the steps from (3.5) – (3.8) this implies that

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|(t + cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.10$'$}$$

Combining (3.9′) and (3.10′), we have for all (j, k) that

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|ul(x)|^{\frac{1}{2}}}, \frac{1}{|(t + cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.11}$$

Finally, combining with (3.8), we see that for each (j, k) we have

$$|I_{jk}| \leq C \int_{Q_{jk}} \min\left(1, \frac{1}{|uy|^{\frac{1}{2}}}, \frac{1}{|ul(x)|^{\frac{1}{2}}}, \frac{1}{|(t + cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.12}$$

In view of the shape of D as given by Theorem 2.2 (where it is called D'_{il}), summing (3.12) over all (j, k) leads to

$$|I| \leq C \int_{D} \min\left(1, \frac{1}{|uy|^{\frac{1}{2}}}, \frac{1}{|ul(x)|^{\frac{1}{2}}}, \frac{1}{|(t + cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i}\, dx\, dy \tag{3.13}$$

We are now in a position to prove (3.1). First suppose $k(x)$ is identically zero. Then $l(x)$ is identically zero and $c = 0$, and (3.13) becomes

$$|I| \leq C \int_{D} \min\left(1, \frac{1}{|uy|^{\frac{1}{2}}}, \frac{1}{|tx|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y)|^{\gamma_i}\, dx\, dy \tag{3.14}$$

By the form of D given by Theorem 2.2, one has $v = (1, 0)$ in (3.1) when $k(x)$ is identically zero (see the discussion at the end of the proof for the case when D is a wedge.) Therefore equation (3.14) is equivalent to (3.1) and we are done. So we move to the case where $k(x)$ is not identically zero. Then (3.13) implies

$$|I| \leq C \int_D \min\left(1, \frac{1}{|ul(x)|^{\frac{1}{2}}}, \frac{1}{|(t+cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y + k(x))|^{\gamma_i} \, dx \, dy \tag{3.15}$$

Doing the variable change $(x, y) \to (x, y - k(x))$ to turn the sliver back into its original coordinates, (3.15) becomes

$$|I| \leq C \int_S \min\left(1, \frac{1}{|ul(x)|^{\frac{1}{2}}}, \frac{1}{|(t+cu)x|^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y)|^{\gamma_i} \, dx \, dy \tag{3.16}$$

The quantity $|(t+cu)x| = |(t, u) \cdot (1, c)||x|$ is within a bounded factor of $|(t, u) \cdot (1, c)||(x, y)|$ since we are assuming the sliver S is from the $|y| < b|x|$ region in Theorem 2.2. Also, note that $(1, c)$ is tangent to the sliver. Hence $|(t, u) \cdot (1, c)||(x, y)|$ is within a bounded factor of $|(t, u) \cdot v||(x, y)|$, where v is a unit tangent vector as in the statement of Lemma 3.2. On the other hand, the quantity $l(x)$ is the vertical drop between (x, y) and the line with direction v through the origin, and since the sliver is in the $|y| < b|x|$ region this vertical drop is within a bounded factor of the distance from (x, y) to this line, which is given by $|(x, y) \cdot v^{\perp}|$. Hence $|ul(x)|$ is within a bounded factor of $|u||(x, y) \cdot v^{\perp}|$. Thus (3.16) implies

$$|I| \leq C \int_S \min\left(1, \frac{1}{(|u||(x, y) \cdot v^{\perp}|)^{\frac{1}{2}}}, \frac{1}{(|(t, u) \cdot v||(x, y)|)^{\frac{1}{2}}}\right) \prod_{i=1}^{n} |f_i(x, y)|^{\gamma_i} \, dx \, dy \tag{3.17}$$

This is equivalent to (3.1) as desired.

As for the statement in Lemma 3.2 concerning which v to choose when S is a wedge-shaped region with two tangent lines at the origin, such an S can arise in two ways. We focus on the wedges where $x > 0$ and $|y| < bx$ as the other cases are very similar. One way for such a wedge to arise occurs at the beginning of the resolution process of Theorem 2.1 when S is of the form $\{(x, y) : 0 < x < a,\ hx^m < y < Hx^M\}$, for $h, H \geq 0$, $m > M$ or $\{(x, y) : 0 < x < a,\ hx^m > y > Hx^M\}$ for $h, H \leq 0$, $m > M$. In these cases $k(x)$ is always identically zero, so the correct tangent line to choose for S is the one closest to the x-axis. The other way such an S can arise is again early in the resolution process when S is of the form $\{(x, y) : 0 < x < a, hx < y < Hx\}$ for some $h \neq H$ and the resolution process is such that $k(x)$ takes the x-axis to the nearer boundary curve of S via a map of the form $(x, y) \to (x, \pm y + cx)$ for an appropriate c. Once again the correct boundary curve of S to choose is the one nearest the x-axis. This completes the proof of Lemma 3.2.

Lemma 3.3. Suppose we are not in the trivial situation where E contains a neighborhood of the origin and each $f_i(0, 0) \neq 0$. Then a sufficiently small disk B centered at the origin can be written in the form $B = \cup_{i=1}^{M} B_i$, where each B_i is a wedge bounded by lines through the origin and the boundary of B, such that each sliver S of Lemma 3.2 is contained in some B_i, and such that each B_i is of one of the following two forms.

1) Let n_i denote the order of the zero of f_i at the origin. Then on the first type of wedge, for some positive constants c_i and c_i', $f_i(x,y)$ satisfies

$$c_i(x^2+y^2)^{\frac{n_i}{2}} < |f_i(x,y)| < C_i(x^2+y^2)^{\frac{n_i}{2}} \tag{3.18}$$

Furthermore, the boundary curves of E do not intersect the closure of B_i and one has

$$\left|\int_{B_i} \alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

$$\leq C\int_{B_i}(1+|tx|+|uy|)^{-1}\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \tag{3.19}$$

2) Let $F_i(x,y)$ denote the sum of the terms of $f_i(x,y)$'s Taylor expansion of lowest degree. If B_i is the second type of wedge, there is a line l_i through the origin intersecting B_i that is either part of the zero set of one of the $F_i(x,y)$ or tangent to one of the boundary curves of E at the origin. Furthermore, if v denotes a unit vector in the direction of l_i then we have

$$\left|\int_{B_i} \alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

$$\leq C\int_{B_i}(1+|(t,u)\cdot v||(x,y)|+|(t,u)\cdot v^{\perp}||(x,y)\cdot v^{\perp}|)^{-\frac{1}{2}}\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \tag{3.20}$$

Proof. We apply Theorem 2.2 to all of the $f_i(x,y)$ as well as real analytic functions whose zero set contains the boundary of E. We let the first type of B_i be certain wedges which can be described in terms of the resolution of singularities process of Theorem 2.1 as follows.

Let $F(x,y)$ denote the sum of terms of lowest degree of the Taylor expansion at the origin of the product of functions, call it $f(x,y)$, whose zero set is being resolved. At the beginning of the resolution of singularities process of Theorem 2.1, one isolates small wedges surrounding the zeroes of the $F(x,y)$. These wedges are bounded by two lines through the origin and a vertical or horizontal line. Outside these small wedges, $f(x,y)$ will satisfy $f(x,y) \sim (x^2+y^2)^{\frac{n}{2}}$, where n denotes the order of the zero at the origin of $f(x,y)$.

Since the zeroes of $F(x,y)$ are the union of the zeroes of the $G_i(x,y)$, where $G_i(x,y)$ denotes the sum of terms of lowest degree of one of the functions in the product, the complement of the union of these wedges is away from the zeroes of any $G_i(x,y)$ as well. Denote this complement by A. Then A itself as a union of wedges through the origin, and we declare that any intersection of one of these wedges with the disk B is a B_i of the first type of in Lemma 3.2. Because they are away from the zeroes of any $G_i(x,y)$, equation (3.18) holds. Furthermore one could have taken $k(x)$ to be zero for these wedges, since no resolution of singularities is needed. Equation (3.19) is therefore a consequence of (3.7) and (3.10), summed over j and k.

The complement of the union of the B_i above is, modulo boundaries, a finite union of disjoint wedges. By the constructions of Theorem 2.1 and Theorem 2.2, each sliver S that is not in one of the wedges B_i above is contained in one of these new wedges. Furthermore, each such wedge contains exactly one line through the origin which is in the zero set of $F(x,y)$. Since this zero set is the union of the zero sets of the $G_i(x,y)$, the line in question is either a zero set of an $F_i(x,y)$ coming from an $f_i(x,y)$, a tangent line at the origin to a boundary curve of E, or a tangent line at the origin to one of the other curves which are in the zero set of the real analytic functions whose zero sets contains the boundary curves to E, but which is not also one of the earlier tangent lines. If the line is of the last variety, we let this wedge be a B_i of the first kind, and (3.18) – (3.19) holds exactly as before. All other wedges are declared to be wedges of the second kind.

Thus in order to prove part 2 of this lemma it suffices to show (3.20) for the second kind of wedge, where l_i is the line through the origin contained in the closure of B_i which is in the zero set of $F(x,y)$.

For each sliver S contained in a B_i of the second type, the coordinate change $(x,y) \rightarrow (x, y+k(x))$ satisfies $k(x) = cx+$ higher order terms where by the resolution of singularities process of Theorem 2.2 the line $y = cx$ is contained in the zero set of $F(x,y)$. Thus the v provided by Lemma 3.2 is exactly of the type needed in part b) of this lemma. We now add (3.1) over all slivers S contained in a given B_i of the second type and obtain

$$\left| \int_{B_i} \alpha(x,y)\, \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\, dx\, dy \right|$$

$$\leq C \int_{B_i} (1 + |(t,u)\cdot v||(x,y)| + |u||(x,y)\cdot v^{\perp}|)^{-\frac{1}{2}} \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\, dx\, dy \qquad (3.21)$$

This is almost the same as (3.20). The one difference is that instead of having a $|(t,u)\cdot v^{\perp}|$ factor as in (3.20) we have a $|u|$ factor. Suppose we could show that for some constant e the following inequality holds on B_i.

$$|(t,u)\cdot v||(x,y)| \geq e|(t,u)\cdot v^{\perp}||(x,y)\cdot v^{\perp}| \qquad (3.22)$$

Then the $|(t,u)\cdot v||(x,y)|$ term alone is enough for (3.21) to imply (3.20). This would only not hold if (t,u) is nearly in the $v^{\perp}$ direction. In this case $|(t,u)\cdot v^{\perp}|$ is of comparable magnitude to $|(t,u)|$. Because v is in the direction of $(1,c)$ for fixed c, there's a M such that if $|t| > M|u|$ then (3.22) holds. Otherwise, $|(t,u)|$ is of comparable magnitude to $|u|$, so $|(t,u)\cdot v^{\perp}|$ is also of comparable magnitude to $|u|$. In this case (3.21) once again implies (3.20) as needed. This concludes the proof of Lemma 3.3.

Note that since on the wedge B_i we have $|(x,y)| \sim |(x,y)\cdot v|$, one can write (3.20) in the symmetric form

$$\left|\int_{B_i} \alpha(x,y)\,\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

$$\leq C\int_{B_i}(1+|(t,u)\cdot v||(x,y)\cdot v| + |(t,u)\cdot v^{\perp}||(x,y)\cdot v^{\perp}|)^{-\frac{1}{2}}\chi_E(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \tag{3.23}$$

In this form it is readily apparent how the estimate (3.20) is independent of the resolution of singularities process being used.

Next, we give the following corollary to Lemma 3.3 which we will need for the paper [G3].

Corollary 3.4. Suppose that E is a disk centered at the origin, but we are not in the trivial situation where each $f_i(0,0) \neq 0$. Let $p_1 \neq p_2$, and let V be any of the four wedges with vertex $(0,0)$ formed by the lines $y = p_1 x$ and $y = p_2 x$. Suppose that each $F_i(x,y)$ has no zeroes on set $cl(V) \cap (\mathbf{R} - \{0\})^2$. Then if the function $\alpha(x,y)$ in (1.1) is supported on the disk B where Lemma 3.3 applies, then we have the following simplified version of (3.1).

$$\left|\int_{B\cap V} \alpha(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}dx\,dy\right| \leq C\int_{B\cap V}(1+|tx|+|uy|)^{-\frac{1}{2}}\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}dx\,dy \tag{3.24a}$$

If each $F_i(x,y)$ has no zeroes on all of $(\mathbf{R} - \{0\})^2$, one has

$$\left|\int_{B} \alpha(x,y)\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right| \leq C\int_{B}(1+|tx|+|uy|)^{-\frac{1}{2}}\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \tag{3.24b}$$

Proof. The second part follows immediately from the first, as in the setting of the second part one can write $\mathbf{R}^2$ as the union of four V on which $(3.24a)$ applies. Then $(3.24b)$ follows by addition. As for part a), one applies Lemma 3.3 to $f_1(x,y), ..., f_n(x,y)$. As long as the B_i of the second type were chosen to be narrow enough, each $B_i \cap V$ for a B_i of the second type will be empty unless the line l_i is the x or y axis; the other l_i are zeroes of some $F_i(x,y)$ which lie outside of $cl(V)$. In this situation we can define the B_i so that $B \cap V$ is a union of some $B_i \cap V$ where each B_i of either of the first type or of the second type with $v = (1,0)$ or $(0,1)$. Then adding (3.19) or (3.20) over all B_i gives the corollary.

The the next lemma will help go from Lemma 3.3 to the estimates of Theorems 1.3-1.5.

Lemma 3.5. Let B_i be one of the domains of part 2 of Lemma 3.3, and v a unit vector in the direction of the associated line l_i.

a) Let (ϵ, d) be as in (1.5). For any $a > 0$, let $F_a = \{(t,u) : |(t,u) \cdot v| > a|(t,u) \cdot v^{\perp}|\}$. Then if $\epsilon < \frac{1}{2}$ there is a constant C_a such that for $(t,u) \in F_a$ one has an estimate

$$\left| \int_{B_i} \alpha(x,y)\, \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\, dx\, dy \right| \leq C_a (2 + |(t,u)|)^{-\epsilon} (\ln(2 + |(t,u)|))^d \tag{3.25}$$

If $\epsilon = \frac{1}{2}$ one gets the estimate obtained by replacing d by $d+1$ in (3.25), and if $\epsilon > \frac{1}{2}$ one has $(2 + |(t,u)|)^{-\frac{1}{2}}$ in place of $(2 + |(t,u)|)^{-\epsilon} (\ln(2 + |(t,u)|))^d$.

b) Let (δ_v, e_v) be as in (1.6), and let $G = \{(t,u) : |(t,u) \cdot v| < |(t,u) \cdot v^{\perp}|\}$. Then if $\delta_v < \frac{1}{2}$ there is a constant D such that for $(t,u) \in G$ one has an estimate

$$\left| \int_{B_i} \alpha(x,y)\, \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\, dx\, dy \right| \leq D (2 + |(t,u)|)^{-\delta_v} (\ln(2 + |(t,u)|))^{e_v} \tag{3.26}$$

If $\delta_v = \frac{1}{2}$ one gets the estimate obtained by replacing e_v by $e_v + 1$, and if $\delta_v > \frac{1}{2}$ one has $(2 + |(t,u)|)^{-\frac{1}{2}}$ in place of $(2 + |(t,u)|)^{-\delta_v} (\ln(2 + |(t,u)|))^{e_v}$.

Proof. We start with part a). We can assume that $|(t,u)| > 4$ say, since the case where $|(t,u)| \leq 4$ is immediate. On the domain F_a, there is a constant a' such that $|(t,u) \cdot v| > a'|(t,u)|$. Thus ignoring the $|(t,u) \cdot v^{\perp}||(x,y) \cdot v^{\perp}|$ term in (3.20), we see that on F_a, (3.20) implies that

$$\left| \int_{B_i} \alpha(x,y)\, \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\, dx\, dy \right|$$

$$\leq C \int_{B_i} (1 + |(t,u)||(x,y)|)^{-\frac{1}{2}} \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i}\, dx\, dy \tag{3.27}$$

We divide the integral (3.27) into $|(x,y)| < |(t,u)|^{-1}$ and $|(x,y)| > |(t,u)|^{-1}$ parts. The integral over the first part is

$$C \int_{\{(x,y) \in B_i :\, |(x,y)| < |(t,u)|^{-1}\}} \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i}\, dx\, dy \tag{3.28}$$

By Lemma 1.1 this is bounded by the desired bound of $C_a (2 + |(t,u)|)^{-\epsilon} (\ln(2 + |(t,u)|))^d$. For the $|(x,y)| > |(t,u)|^{-1}$ part, we divide the integral dyadically in $|(x,y)|$, obtaining a bound of

$$C \sum_{j=1}^{\lfloor \log_2 |(t,u)| - 1 \rfloor} 2^{-\frac{j}{2}} \int_{\{(x,y) \in B_i :\, 2^j |(t,u)|^{-1} \leq |(x,y)| < 2^{j+1} |(t,u)|^{-1}\}} \chi_E(x,y) \prod_{i=1}^{n} |f_i(x,y)|^{\gamma_i}\, dx\, dy \tag{3.29}$$

We can use $\lfloor \log_2 |(t,u)| - 1 \rfloor$ here because our domain is contained in a small neighborhood of the origin. Inserting (1.5) into the above provides a bound of

$$C \sum_{j=1}^{\lfloor \log_2 |(t,u)|-1 \rfloor} 2^{-\frac{j}{2}} (2^j |(t,u)|^{-1})^{\epsilon} |\ln(2^j |(t,u)|^{-1})|^d \tag{3.30}$$

If $\epsilon < \frac{1}{2}$ the summands decrease exponentially and we obtain the bound given by the first term, namely $C|(t,u)|^{-\epsilon}|\ln|(t,u)||^d$. If $\epsilon > \frac{1}{2}$ the summands increase exponentially and we obtain the bound given by the last term, namely $|(t,u)|^{-\frac{1}{2}}$. When $\epsilon = \frac{1}{2}$, we get a summation of powers of $|\ln(2^j|(t,u)|^{-1})|^d$ which results in a bound of $C|(t,u)|^{-\frac{1}{2}}|\ln|(t,u)||^{d+1}$. This is equivalent to the statement of Lemma 3.5a) since we can assume that $|(t,u)| > 4$. This completes the proof of part a).

The proof of part b) is rather similar. As before, it suffices to assume $|(t,u)| > 4$. In the domain G, since $|(t,u) \cdot v| < |(t,u) \cdot v^{\perp}|$, one has that $(t,u) \cdot v^{\perp} \sim |(t,u)|$ and the right-hand side of (3.20) can be bounded by

$$C \int_{B_i} (1 + |(t,u)||(x,y) \cdot v^{\perp}|)^{-\frac{1}{2}} \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i} \, dx \, dy \tag{3.31}$$

We break the integral in (3.31) into $|(x,y) \cdot v^{\perp}| < |(t,u)|^{-1}$ and $|(x,y) \cdot v^{\perp}| \geq |(t,u)|^{-1}$ parts. By Lemma 1.2, the first part is bounded by $C|(t,u)|^{-\delta_v}(\ln|(t,u)|)^{e_v}$. We again break up the second integral dyadically, obtaining a bound of

$$\sum_{j=1}^{\lfloor \log_2 |(t,u)|-1 \rfloor} 2^{-\frac{j}{2}} \int_{\{(x,y) \in B_i :\, 2^j |(t,u)|^{-1} \leq |(x,y) \cdot v^{\perp}| \leq 2^{j+1} |(t,u)|^{-1}\}} \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i} \, dx \, dy \tag{3.32}$$

Using Lemma 1.2, this leads to a bound of

$$C \sum_{j=1}^{\lfloor \log_2 |(t,u)|-1 \rfloor} 2^{-\frac{j}{2}} (2^j |(t,u)|^{-1})^{\delta_v} |\ln(2^j |(t,u)|^{-1})|^{e_v} \tag{3.33}$$

One then argues as after (3.30), and we see that we get a bound of $C|(t,u)|^{-\delta_v}|\ln|(t,u)||^{e_v}$ when $\delta_v < \frac{1}{2}$, a bound of $C|(t,u)|^{-\frac{1}{2}}$ when $\delta_v > \frac{1}{2}$, and $C|(t,u)|^{-\delta_v}|\ln|(t,u)||^{e_v+1}$ when $\delta_v = \frac{1}{2}$. This completes the proof of Lemma 3.5.

For the B_i of part a) of Lemma 3.3, we have estimates at least as strong as those that hold for the B_i of part b) of Lemma 3.3:

Lemma 3.6. If B_i is a wedge from part a) of Lemma 3.3 and $\epsilon < 1$, then for all (t,u) one has an estimate

$$\left| \int_{B_i} \alpha(x,y) \, \chi_E(x,y) \prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy} \, dx \, dy \right| \leq C_a (2 + |(t,u)|)^{-\epsilon} (\ln(2 + |(t,u)|))^d \tag{3.34}$$

If $\epsilon = 1$ one gets the estimate obtained by replacing d by $d+1$ in (3.34), and if $\epsilon > 1$ one has $(2+|(t,u)|)^{-1}$ in place of $(2+|(t,u)|)^{-\epsilon}(\ln(2+|(t,u)|))^d$.

Proof. Equation (3.19) implies

$$\left|\int_{B_i} \alpha(x,y)\,\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy\right|$$

$$\leq C\int_{B_i} (1+|(t,u)||(x,y)|)^{-1}\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i}\,dx\,dy \tag{3.35}$$

This is exactly (3.27) with the exponent $-\frac{1}{2}$ replaced by -1. The steps from (3.27) through the paragraph after (3.30) lead to the statement of this lemma, with this modification due to the new exponent.

Proofs of Theorem 1.3-1.5.

We now are in a position to prove Theorems 1.3-1.5. If E contains a neighborhood of $(0,0)$ and each $f_i(0,0) \neq 0$ the results are easy, so we assume that this is not the case. We start with Theorems 1.3a) and 1.4a). First of all, note that the directions of the lines l_i in Theorems 1.3 and 1.4 are exactly the directions v in Lemma 3.3 corresponding to the domains B_i of the second type. If l is not perpendicular to one of these directions, then for each B_i of the second type, the set l_H of Theorems 1.3 and 1.4 is contained in one of the sets F_a of Lemma 3.5, except for an inconsequential part near the origin. Thus for $(t,u) \in l_H$, the estimates of part a) of Lemma 3.5 hold, possibly with different constants. By Lemma 3.6 they also hold for $(t,u) \in l_H$ when B_i is of the type of the first part of Lemma 3.3. Hence they hold for all B_i. Adding this over all i therefore results in the bounds of Lemma 3.5a) whenever $(t,u) \in l_H$, giving an estimate $|K_{\phi,x_0,y_0}(t,u)| \leq C(2+|(t,u)|)^{-\epsilon}(\ln(2+|(t,u)|))^d$ on l_H when l is not in one of the $v^\perp$ directions. In section 4, we will show the best possible power of $(2+|(t,u)|)$ that can appear in such an estimate is $-\delta_v$. Since $\delta_v \leq \epsilon$ for any direction, we have $\epsilon = \delta_v$ here as needed in Theorems 1.3a) and 1.4a). This provides the estimates of Theorems 1.3a) and 1.4a) when l is not perpendicular to the direction of a v corresponding to the second type of B_i.

If l is in the $v^\perp$ direction for some B_{i_0} of the second type, when $(t,u) \in l_H$ one has the same estimates given by Lemma 3.5a) for $i \neq i_0$, for the same reasons as before. For B_{i_0} however, the arguments above will not work since l_H is a subset of no F_a. However, besides an inconsequential region with small $|(t,u)|$, l_H is a subset of the set called G in Lemma 3.5b). So for $(t,u) \in l_H$ one has the possibly weaker bounds of Lemma 3.5b) in place of those of Lemma 3.5a). Adding this to the estimates over the other B_i therefore gives the overall bound given by part b) of Lemma 3.5. This gives the statements of Theorems 1.3a) and 1.4a) when l is one of the $v^\perp$ directions.

Moving on to the overall decay rates of Theorems 1.3b, 1.4b, and 1.5, for a given B_i of the second type, each (t,u) is either in a F_a of Lemma 3.5a) or a set G of

Lemma 3.5b). (The exact value of a is not important for our purposes.) Thus the quantity $|\int_{B_i} \alpha(x,y)\,\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy|$ is always bounded by the worse of the two estimates, namely the estimate given by Lemma 3.5b). The corresponding estimate for the B_i of the first type, provided by Lemma 3.6, is at least as good as this, so adding over all i we have that $|\alpha(x,y)\,\chi_E(x,y)\,\prod_{i=1}^n |f_i(x,y)|^{\gamma_i} e^{itx+iuy}\,dx\,dy|$ is bounded by the worst over all i of the estimates given by Lemma 3.5b). These are exactly the overall decay rates of Theorems 1.3b, 1.4b, and 1.5. This completes the proofs of Theorems 1.3, 1.4, and 1.5.

Proof of Theorem 1.7.

It suffices to prove that given any (x_0, y_0) in the support of $m(x,y)$ there is a neighborhood U of (x_0, y_0) such that if the cutoff function $\phi(x,y)$ is supported in U, then each term of (1.4) satisfies the bounds stipulated in Theorem 1.7 as the estimates for $K(t,u)$ then follow by addition. Let V denote the set of all (t,u) within angle $\frac{\pi}{8}$ of the lines $y = x$ or $y = -x$. We will prove the estimates for (t,u) in the closure of V. The result for the remaining (t,u) will follow by applying the resolution of singularities theorem in the coordinates obtained after rotating by 45 degrees.

Let S be any of the slivers arising from the application of Theorem 2.2 as in the previous lemmas. We will examine the contribution to $F(t,u)$ coming from S and see that it satisfies the needed bounds, so that adding over all slivers gives the desired estimates. We focus as before on slivers coming from the region $|y| < b|x|$ as the other slivers are treated in an entirely analogous fashion.

For a given sliver S, we move into the new coordinates and use (3.7). We add (3.7) over all j and k and call the result $F_S(t,u)$. In view of the shape of the domains provided by Theorem 2.2, we get that

$$|F_S(t,u)| \leq C \int_D \min\left(1, \frac{1}{|uy|}\right) \prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}\,dx\,dy \tag{3.36}$$

Here D denotes the sliver S in the new coordinates. Since the coordinate changes have Jacobian 1, the function $\prod_{i=1}^n |f_i(x, y + k(x))|^{\gamma_i}$ is in $L^p(D)$, where p is as in the statement of Theorem 1.7. We apply Hölder's inequality in (3.36), obtaining

$$|F_S(t,u)| \leq C\left(\int_D \min\left(1, \frac{1}{|uy|^{p'}}\right)\right)^{\frac{1}{p'}} \tag{3.37}$$

$$\leq C\left(\int_{[0,1]\times[0,1]} \min\left(1, \frac{1}{|uy|^{p'}}\right)\right)^{\frac{1}{p'}} \tag{3.38}$$

Doing the integral in (3.38) gives $C\min(1, |u|^{-\frac{1}{p'}})$ when $p < \infty$, and $C\min(1, |u|^{-1}\ln|u|)$ if $p = \infty$. Since we are considering (t,u) within angle at most $\frac{\pi}{8}$ of the lines $y =$

x or $y = -x$, this gives the desired bound of $C\min(1, |(t,u)|^{-\frac{1}{p'}})$ when $p < \infty$ and $C\min(1, |(t,u)|^{-1}\ln|(t,u)|)$ if $p = \infty$. We add this over all slivers S and we are done.

4. Proofs of sharpness statements.

The directions of part a) of Theorems 1.3 and 1.4 are given by the same direction, and we will prove sharpness of both simultaneously. So suppose (1.7a) or (1.9) holds with the exponent δ_v replaced by some $\delta > \delta_v$; we will arrive at a contradiction. Since the estimate is to hold on the whole strip l_H, it must hold on the ray with direction $v^\perp$ emanating from the origin. In other words, we have the following estimate in the s variable.

$$|K_{\phi,x_0,y_0}(sv^\perp)| \leq C(2+|s|)^{-\delta} \tag{4.1}$$

Let $\psi(x)$ be a smooth function on $\mathbf{R}$ whose Fourier transform is a compactly supported nonnegative function equal to 1 on a neighborhood of the origin. Let $0 < \eta < \delta$ such that $\eta + \delta_v < \delta$. For a large L we look at

$$I_L = \int K_{\phi,x_0,y_0}(sv^\perp)\psi\left(\frac{s}{L}\right)|s|^{\delta-1-\eta}\,ds \tag{4.2}$$

Inserting (4.1) into (4.2) gives that for all L we have

$$I_L \leq C\int (2+|s|)^{-\delta}|s|^{\delta-1-\eta}\psi\left(\frac{s}{L}\right)ds \tag{4.3}$$

Because $\eta > 0$, the integrand in (4.3) is integrable for large $|s|$, and because $\eta < \delta$ the integrand in (4.3) is integrable for small $|s|$. Hence the I_L are uniformly bounded in L. On the other hand, I_L is given by

$$I_L = \int \phi(x_0+x, y_0+y)m(x_0+x, y_0+y)e^{is((x,y)\cdot v^\perp)}|s|^{\delta-1-\eta}\psi\left(\frac{s}{L}\right)ds\,dx\,dy \tag{4.4}$$

Performing the s integral in (4.4) leads to

$$I_L = \int \phi(x_0+x, y_0+y)m(x_0+x, y_0+y)L^{\delta-\eta}\xi(L[(x,y)\cdot v^\perp])\,dx\,dy \tag{4.5}$$

Here $\xi(x)$ is the Fourier transform of $\psi(s)|s|^{\delta-1-\eta}$. Since the Fourier transform of $\psi(s)$ is nonnegative and compactly supported, and the Fourier transform of $|s|^{\delta-1-\eta}$ is of the form $c|x|^{\eta-\delta}$, we have that $\xi(x)$ is of the form $c\tilde{\xi}(s)$ where $\tilde{\xi}(s)$ is nonnegative and decays as $|s|^{\eta-\delta}$ as $|s| \rightarrow \infty$. Since we are assuming $\alpha(x,y)$ in (1.1) is bounded below by a positive value on some neighborhood of the origin, as long as the support of $\phi(x_0+x, y_0+y)$ is contained in this neighborhood, the $\phi(x_0+x, y_0+y)m(x_0+x, y_0+y)$ factor in (4.5) is nonnegative and we can we can rewrite (4.5) as

$$|I_L| = |c|\int \phi(x_0+x, y_0+y)m(x_0+x, y_0+y)L^{\delta-\eta}\tilde{\xi}(L[(x,y)\cdot v^\perp])\,dx\,dy \tag{4.5$'$}$$

Letting N be a neighborhood of the origin on which $\phi(x_0+x, y_0+y)$ and $\alpha(x,y)$ are both bounded below by a positive number, there is a constant C such that

$$|I_L| \geq C \int_N g(x,y) L^{\delta-\eta} \tilde{\xi}(L[(x,y)\cdot v^\perp])\, dx\, dy \tag{4.6}$$

As a result, for any $r > 0$ we have

$$\sup_L |I_L| \geq C \int_{\{(x,y)\in N:\, r<|(x,y)\cdot v^\perp|<2r\}} g(x,y) L^{\delta-\eta} \tilde{\xi}(L[(x,y)\cdot v^\perp])\, dx\, dy \tag{4.7}$$

Note that the left-hand side of (4.7) is finite. Since $\tilde{\xi}(s)$ is nonnegative and decays as $|s|^{\eta-\delta}$ as $|s| \to \infty$, if we take the limit as $L \to \infty$ in the right-hand side of (4.7) we obtain

$$\sup_L |I_L| \geq C \int_{\{(x,y)\in N:\, r<|(x,y)\cdot v^\perp|<2r\}} g(x,y) |(x,y)\cdot v^\perp|^{\eta-\delta}\, dx\, dy \tag{4.8}$$

As result we have

$$\sup_r r^{\eta-\delta} \int_{\{(x,y)\in N:\, r<|(x,y)\cdot v^\perp|<2r\}} g(x,y)\, dx\, dy < \infty \tag{4.9}$$

Since we are assuming η was chosen so that $\eta-\delta < -\delta_v$, equation (4.9) contradicts Lemma 1.2. Hence we conclude the estimates of parts a) Theorems 1.3 and 1.4 are sharp as desired.

The proof of sharpness of parts b) of the two theorems is very similar, so we omit the full details. One assumes the result holds for some $\epsilon' > \epsilon$, chooses some $\eta > 0$ with $\eta + \epsilon < \epsilon'$ and instead of using (4.2), one uses

$$I_L = \int K_{\phi,x_0,y_0}(t,u) \psi\bigg(\frac{|(t,u)|}{L}\bigg) |(t,u)|^{\epsilon'-2-\eta}\, ds \tag{4.10}$$

Then the steps analogous to (4.2) – (4.9) lead to

$$\sup_r r^{\eta-\epsilon'} \int_{\{(x,y)\in N:\, r<|(x,y)|<2r\}} g(x,y)\, dx\, dy < \infty \tag{4.11}$$

Since the exponent $\eta - \epsilon'$ is less than $-\epsilon$, equation (1.5) is contradicted and we must have sharpness. This completes the proofs of the sharpness statements of Theorems 1.3 and 1.4.

As for Theorem 1.6, one can readily reduce it to the above sharpness statements. Suppose $m(x,y) = \sum_i m_i(x,y)$ satisfies the conditions of Theorem 1.6, and i_0 is an index such the estimate for $m_{i_0}(x_0+x, y_0+y)$ given by Theorem 1.3 or 1.4 is stated to be sharp and such that the estimate for $K(t,u)$ of (1.4) given by adding the estimates for all $K_{\phi,x_0,y_0}(t,u)$ over all i is the estimate for the $K_{\phi,x_0,y_0}(t,u)$ corresponding to $i = i_0$. We suppose for argument's sake the estimate for $m_{i_0}(x_0 + x, y_0 + y)$ derives from part a) of Theorem 1.3; the other cases are dealt with similarly.

If $K(t,u)$ satisfied a better estimate $|K(t,u)| \leq C(2+|(t,u)|)^{-\delta}$ on the strip l_H of Theorem 1.3a), where $\delta > \delta_v$, then in place of (4.1) we would have $|K(sv^\perp)| \leq C(2+|s|)^{-\delta}$. Instead of defining I_L as in (4.2), for a constant a to be determined, one uses

$$I_L = \int K(sv^\perp)\psi\left(\frac{s}{L}\right)e^{-isa}|s|^{\delta-1-\eta}\,ds \tag{4.12}$$

One gets that $\sup_L |I_L| < \infty$ exactly as before. Performing the steps from (4.2) – (4.9) this time leads to

$$\sup_r r^{\eta-\delta}\int_{\{(x,y)\in N:\, r<|((x,y)-av^\perp)\cdot v^\perp|<2r\}} m(x,y)\,dx\,dy < \infty \tag{4.13}$$

By the assumptions of Theorem 1.6, each $m_i(x,y)$ is nonnegative, so we must also have

$$\sup_r r^{\eta-\delta}\int_{\{(x,y)\in N:\, r<|((x,y)-av^\perp)\cdot v^\perp|<2r\}} m_{i_0}(x,y)\,dx\,dy < \infty \tag{4.14}$$

We choose a so that

$$\{(x,y)\in N:\, r<|((x+x_0,y+y_0)-av^\perp)\cdot v^\perp|<2r\}$$

$$= \{(x,y)\in N:\, r<|(x,y)\cdot v^\perp|<2r\} \tag{4.15}$$

Then changing variables from (x,y) to $(x+x_0, y+y_0)$ in (4.14) leads to

$$\sup_r r^{\eta-\delta}\int_{\{(x,y)\in N:\, r<|(x,y)\cdot v^\perp|<2r\}} m_{i_0}(x_0+x,y_0+y)\,dx\,dy < \infty \tag{4.16}$$

Since $\alpha(x,y)$ is bounded below by a positive constant, (4.16) implies (4.9), and we get a contradiction like before. Thus the estimate for $K(t,u)$ here is in fact sharp.

Although the above dealt with the situation when the estimate for $m(x,y)$ derives from a sharp estimate for $m_{i_0}(x_0 + x, y_0 + y)$ derived from part a) of Theorem 1.3 or 1.4, the other situations are dealt with in the analogous manner. This concludes the proof of Theorem 1.6.

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Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
322 Science and Engineering Offices
851 S. Morgan Street
Chicago, IL 60607-7045
greenbla@uic.edu