# Convexity, Fourier transforms, and lattice point discrepancy 

Michael Greenblatt

February 27, 2024


#### Abstract

In a well-known paper by Bruna, Nagel and Wainger [BNW], Fourier transform decay estimates were proved for smooth hypersurfaces of finite line type bounding a convex domain. In this paper, we generalize their results in the following ways. First, for a surface that is locally the graph of a convex real analytic function, we show that a natural analogue holds even when the surface in question is not of finite line type. Secondly, we show a result for a general surface that is locally the graph of a convex $C^{2}$ function, or a piece of such a surface defined through real analytic equations, that implies an analogous Fourier transform decay theorem in situations where the oscillatory index is less than 1 . This result has implications for lattice point discrepancy problems, which we describe.


## 1 Background and surface measure Fourier transform theorem statements.

### 1.1 Introduction.

We consider Fourier transforms of hypersurface measures in $\mathbb{R}^{n+1}, n \geq 1$, where the surface is locally the graph of a convex function. Specifically, we let $S$ be a bounded $C^{2}$ hypersurface in $\mathbb{R}^{n+1}$, such that for each $x_{0} \in S$, there is a composition of a translation and a rotation, which we call $A_{x_{0}}$, such that $A_{x_{0}}\left(x_{0}\right)=0$, the normal to $S^{\prime}=A_{x_{0}}(S)$ at 0 is the vector $(0, \ldots, 0,1)$, and there is a ball $D=B\left(0, r_{0}\right)$ centered at 0 such that above $D$ the surface

[^0]$S^{\prime}$ is the graph of a $C^{2}$ convex function $f\left(x_{1}, \ldots, x_{n}\right)$ on a neighborhood of the closure $\bar{D}$. We localize the problem by letting $\phi\left(x_{1}, \ldots, x_{n}\right)$ be a $C^{1}$ real-valued function supported in $D$ and we look at the Fourier transform of the measure $\mu$ on $S^{\prime}$ where the surface measure is localized by the cutoff function $\phi\left(x_{1}, \ldots, x_{n}\right)$ at the point on $S^{\prime}$ above $\left(x_{1}, \ldots, x_{n}\right)$. To be precise, we are looking at
\[

$$
\begin{equation*}
\hat{\mu}\left(\lambda_{1}, \ldots, \lambda_{n+1}\right)=\int_{\mathbb{R}^{n}} e^{-i \lambda_{1} x_{1}-\ldots-i \lambda_{n} x_{n}-i \lambda_{n+1} f\left(x_{1}, \ldots, x_{n}\right)} \phi\left(x_{1}, \ldots, x_{n}\right) d x_{1} \ldots d x_{n} \tag{1.1}
\end{equation*}
$$

\]

Note that $A_{x_{0}}$ is such that $f(0)=0$ and $\nabla f(0)=0$.
The goal here is to understand the decay of $|\hat{\mu}(\lambda)|$ for large $|\lambda|$. For this, it is helpful to consider $\hat{\mu}(\lambda)$ for $\lambda$ on rays $r(v)=\{t v: t \in \mathbb{R}\}$ for various directions $v=\left(v_{1}, \ldots, v_{n+1}\right)$ with $|v|=1$. Since replacing $v$ by $-v$ just replaces the integrand of (1.1) by its complex conjugate, it suffices to assume that $v_{n+1} \geq 0$. If $v$ is such that the angle between $v$ and the normal to the surface at every point in $S^{\prime}$ above $D$ is at least some $\epsilon>0$, then by repeated integrations by parts one has that $|\hat{\mu}(\lambda)| \leq C_{\epsilon, N}|\lambda|^{N}$ for any $N$. Thus the focus is on directions $v$ that are either normal to the surface at some point above $D$ or close to some such direction.

A general heuristic when examining an oscillatory integral for large values of a parameter such as $\lambda$ is that the magnitude of the oscillatory integral should be of the same order of magnitude as the maximal measure of the points in the domain for which the phase is within a single period. When this principle is applied to the analysis of the decay of $|\hat{\mu}(\lambda)|$ in a direction $v$ perpendicular to the tangent plane $T_{y}\left(S^{\prime}\right)$ for a point $y$ on $S^{\prime}$ above the closure $\bar{D}$, this heuristic suggests the following. Let $\pi$ denote the projection onto the first $n$ variables. For any $v$ with $|v|=1$ and $v_{n+1} \geq 0$, we define $s(v)$ to be the "lowest" height in the $v$ direction achieved on the portion of $S^{\prime}$ above $\bar{D}$. That is, we make the definition

$$
\begin{equation*}
s(v)=\min \left\{x \cdot v: x \in S^{\prime}, \pi(x) \in \bar{D}\right\} \tag{1.2}
\end{equation*}
$$

Then in view of the convexity of $f$, if the disk $D$ is small enough this maximal measure is bounded by the following, where $m$ denotes the surface measure on $S$.

$$
\begin{equation*}
m\left(\left\{x \in S^{\prime}: \pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|\lambda|^{-1}\right\}\right) \tag{1.3}
\end{equation*}
$$

Hence heuristically we expect (1.1) to be bounded by a constant times (1.3). In addition, past experience leads one to expect that if $\phi$ is nonnegative and $s(v)$ is achieved at at least one point $x$ where $\phi(\pi(x))>0$, then (1.3) should give the correct order of magnitude for $\hat{\mu}(\lambda)$.

It follows from the classic paper [BNW] that for a smooth compact surface of finite line type bounding a convex domain (finite line type means that no line is tangent to the surface to infinite order), then for the $t$ and $v$ in question the function $|\hat{\mu}(t v)|$ is in fact bounded by a constant independent of $v$ times (1.3) for $\lambda=t v$. In [BNW] they make use of nonisotropic balls with a doubling property. Namely, if one defines $K(x, r)=\left\{x \in S^{\prime}\right.$ :
$\left.\pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|\lambda|^{-1}\right\}$, then the $K(x, r)$ are the balls for a metric space with a doubling property, which can be used to show various properties related to $\hat{\mu}$. (Actually they define the balls globally, where the same principle applies.) In this paper, we will see that although there are no families of balls here, one has estimates analogous to those of [BNW] in several settings.

### 1.2 Surface measure Fourier transform results.

The first class of surfaces we consider are surfaces which are locally the graph of a convex real analytic function, but which are not necessarily of finite line type. Canonical examples of such surfaces are cones given by equations $f\left(\frac{x_{1}}{x_{n+1}}, \frac{x_{2}}{x_{n+1}}, \ldots, \frac{x_{n}}{x_{n+1}}\right)=1$, where the $\left(x_{1}, \ldots, x_{n}\right)$ for which $f\left(x_{1}, \ldots, x_{n}\right)=1$ form a compact real analytic surface enclosing a convex domain (this surface will necessarily be of finite line type). Other examples include developable surfaces in three dimensions.

Our theorem in this situation is as follows.
Theorem 1.1. Suppose we are in the setting of (1.1), where $\phi(x)$ is $C^{n+1}$ and supported in $D$, and $f(x)$ is convex, real analytic, and not identically zero. Then for any $v$ with $|v|=1$ and $v_{n+1} \geq 0$, and any $t \neq 0$ we have an estimate

$$
\begin{equation*}
|\hat{\mu}(t v)| \leq C m\left(\left\{x \in S^{\prime}: \pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|t|^{-1}\right\}\right) \tag{1.4}
\end{equation*}
$$

Here $m$ denotes Euclidean surface measure and the constant $C$ depends only on $\phi$ and $f$.
While (1.4) holds for all $t$, it is only of interest for large $|t|$ since the result for $|t|<B$ for a fixed $B$ will follow immediately simply by taking absolute values inside the integrand and integrating. The same is true for the other theorems of this paper.

The natural geometric interpretation of the measure on the right-hand side of (1.4) is the volume of the intersection of the portion of $S^{\prime}$ over $D^{\prime}$ with the slab of width $|t|^{-1}$ perpendicular to $v$ whose lower side intersects this portion of $S^{\prime}$ at the lowest possible point in the $v$ direction. There are situations where for a given $v$ this does not give the best possible rate of decay, but when $v$ is perpendicular to the tangent plane of $S^{\prime}$ at a point above $D$ it typically does.

In a sense, Lemmas 4.1 and 4.3 will provide a substitute for real analytic functions for the finite type condition of [BNW], so that an analogous theorem can be proven. If one has a compact real analytic surface $S$ enclosing a convex domain (which will necessarily be of finite line type), then one can recover the Fourier transform bounds of [BNW] for $S$ by using a partition of unity to write the surface measure Fourier transform as a finite sum of integrals of the form (1.1), after appropriate translations and rotations. For a given piece, one applies Theorem 1.1 in directions that are normal or nearly normal to the tangent planes of the surface piece, and simply integrates by parts repeatedly in the oscillatory integral for
directions that are not nearly normal to the surface piece. Adding the results will then give the Fourier transform bounds of [BNW] for that surface.

Our second theorem holds for all surfaces that are graphs of convex functions that are at least $C^{2}$, and where $\phi$ is at least $C^{1}$, as well as graphs of portions of such surfaces carved out by real analytic functions. Specifically, we assume $f(x)$ is $C^{2}$ and convex, $\phi(x)$ is $C^{1}$, and $g_{1}(x), \ldots, g_{m}(x)$ are real analytic functions defined on a neighborhood of $\bar{D}$, none identically zero. We define $I(\lambda)$ to be the Fourier transform of a surface measure carved out by the the $g_{i}$ on the surface over $D$ whose graph is $f(x)$. Specifically, we define

$$
\begin{equation*}
I(\lambda)=\int_{\left\{x \in \mathbb{R}^{n}: g_{i}(x)<0 \text { for all } i\right\}} e^{-i \lambda_{1} x_{1}-\ldots-i \lambda_{n} x_{n}-i \lambda_{n+1} f\left(x_{1}, \ldots, x_{n}\right)} \phi\left(x_{1}, \ldots, x_{n}\right) d x_{1} \ldots d x_{n} \tag{1.5}
\end{equation*}
$$

Note that by choosing $g_{2}=-g_{1}+1$, the case where there are no $g_{i}(x)$ is included in the above. Our theorem for $I(\lambda)$ is as follows.

Theorem 1.2. For any $v=\left(v_{1}, \ldots, v_{n+1}\right)$ with $|v|=1$ and $v_{n+1} \geq 0$, and any $t \neq 0$ we have an estimate

$$
\begin{gather*}
|I(t v)| \leq C m\left(\left\{x \in S^{\prime}: \pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|t|^{-1}\right\}\right) \\
+C \sum_{j=1}^{\infty} 2^{-j} m\left(\left\{x \in S^{\prime}: \pi(x) \in \bar{D}, s(v)+2^{j-1}|t|^{-1} \leq x \cdot v \leq s(v)+2^{j}|t|^{-1}\right\}\right) \tag{1.6}
\end{gather*}
$$

Here $m$ denotes Euclidean surface measure and the constant $C$ depends only on $\phi, f$, and the $g_{i}$.

The terms in (1.6) can be interpreted as volumes of intersections of slabs perpendicular to $v$ with the portion of $S^{\prime}$ above $\bar{D}$, similarly to Theorem 1.1. To help further understand the meaning of Theorem 1.2, suppose we are in a situation where for some $v$ with $|v|=1$ and $v_{n+1} \geq 0$, there is a $0<\alpha<1$ such that for all $t \neq 0$ we have an estimate

$$
\begin{equation*}
m\left(\left\{x \in S^{\prime}: \pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|t|^{-1}\right\}\right) \leq C|t|^{-\alpha} \tag{1.7}
\end{equation*}
$$

Then by adding this estimate over all $j$, (1.6) implies that $|I(t v)| \leq C^{\prime}|t|^{-\alpha}$ for all $t$. If the estimate (1.7) holds for all $t \neq 0$ and all directions $v$ with $v_{n+1} \geq 0$, then we will similarly have a uniform estimate $|I(t v)| \leq C^{\prime}|t|^{-\alpha}$ for all such $t$ and $v$. In this way if the optimal $\alpha$ for which one has a uniform estimate (1.7) is in the range $0<\alpha<1$, one retains this rate of decay in the surface measure Fourier transform $I(\lambda)$. We record this fact in the following theorem.

Theorem 1.3. Suppose we are in the setting of Theorem 1.2 and for some $C>0$ and some $0<\alpha<1$ the estimate (1.7) holds for all $t \neq 0$ and all $v$ with $|v|=1$ and $v_{n+1} \geq 0$. Then there exists a constant $C^{\prime}$ such that one has the estimate $|I(\lambda)| \leq C^{\prime}|\lambda|^{-\alpha}$

In the setting of Theorem 1.2 , the surface may be flat to infinite order at some point $x$. In a direction $v$ perpendicular to $T_{x}(S)$, the rate of decay of the left hand side of (1.6) may be slower than any $|t|^{-\alpha}$ for $\alpha>0$. Nonetheless, in some such scenarios one may still sum the series in (1.6) and obtain a result bounded by a constant times the first term in (1.6). In such a way, one can still bound $|\hat{\mu}(t v)|$ in terms of this first term, much as in Theorem 1.1.

Theorem 1.3 can be readily globalized when there are no $g_{i}(x)$ as follows. Suppose $S$ is a compact $C^{2}$ surface bounding a convex domain. Let $\nu$ denote Euclidean surface measure on $S$ and let $\psi(x)$ be a $C^{1}$ function defined on a neighborhood of $S$ in $\mathbb{R}^{n+1}$. Then one can bound $|\widehat{\psi \nu}(\lambda)|$ by using a partition of unity and then applying Theorem 1.3 to each term. Because one can get different $s(v)$ corresponding to each term, in order to have a reasonable theorem statement we proceed as follows. For a given direction $v$ and $t \neq 0$ we define $a(v, t)$ by

$$
\begin{equation*}
a(v, t)=\sup _{s \in \mathbb{R}} m\left(\left\{x \in S: s \leq x \cdot v \leq s+|t|^{-1}\right\}\right) \tag{1.8}
\end{equation*}
$$

Thus $a(v, t)$ is the maximal value of the measure of intersection of $S$ with any "slab" perpendicular to $v$ of width $|t|^{-1}$. Note that with any partition of unity, any term $m\left(\left\{x \in S^{\prime}\right.\right.$ : $\left.\left.\pi(x) \in \bar{D}, s(v) \leq x \cdot v \leq s(v)+|t|^{-1}\right\}\right)$ showing up from an application of Theorem 1.3 will be bounded by $a(v, t)$. Thus if we have the bound $a(v, t) \leq C|t|^{-\alpha}$ holding uniformly in $v$ for some $0<\alpha<1$, then the sum of all the terms in (1.6) will similarly be bounded by some $C^{\prime}|t|^{-\alpha}$. In other words, by adding over terms of the partition of unity, Theorem 1.3 implies that $|\widehat{\psi \nu}(\lambda)| \leq C^{\prime}|\lambda|^{-\alpha}$ here. This leads to the following result.

Theorem 1.4. Let $S$ be a compact $C^{2}$ surface bounding a convex domain. Let $\nu$ denote Euclidean surface measure on $S$ and let $\psi(x)$ be a $C^{1}$ function defined on a neighborhood of $S$ in $\mathbb{R}^{n+1}$. Suppose for some $0<\alpha<1$ and some $C>0$ we have $a(v, t) \leq C|t|^{-\alpha}$ for all directions $v$ and all $t \neq 0$. Then there is a constant $C^{\prime}$ depending on $C, S, \psi$, and $\alpha$ such that $|\widehat{\psi \nu}(\lambda)| \leq C^{\prime}|\lambda|^{-\alpha}$ for all $\lambda$.

It is worth pointing out that quantities related to $a(v, t)$ have appeared in other results connecting oscillatory integrals to sublevel set measures, such as the recent paper [BaGuZhZo]. In addition, conditions somewhat resembling those of Theorems 1.3 and 1.4 have appeared in related contexts, such as [G2] and [ISa]. There also has been quite a bit of other work on Fourier transforms of surface measures of surfaces that are locally graphs of convex functions. We mention [BakMVW] [BrHoI] [CoDiMaMu] [Gr] [Gre] as examples especially pertinent to the topic of this paper. We also mention the book [IL] which contains many results in this area and further references.

## 2 Lattice point discrepancy.

Suppose $S$ is a compact $C^{2}$ surface bounding a convex domain $S_{0}$ containing the origin. For $k>0$, let $k S_{0}$ denote the dilated surface $\{k x: x \in S\}$, and let $N(k)$ denote the number
of lattice points on or inside $k S_{0}$. Then one has $N(k) \sim k^{n+1} m\left(S_{0}\right)$ for large $k$, and a straightforward geometric argument gives that for some constant $C$ one also has an estimate $\left|N(k)-k^{n+1} m\left(S_{0}\right)\right| \leq C k^{n}$. If $S_{0}$ were a polyhedron instead of having $C^{2}$ boundary, then it is not hard to show that the flatness of the sides of $S_{0}$ ensures that the exponent $n$ in $C k^{n}$ is best possible.

It turns out that when $S$ is curved enough for Theorem 1.4 to hold for some $\alpha>0$ then one gets a better exponent than $n$. Specifically, there are well known methods (see p.383-384 of [ShS]) that give the following. Suppose $A$ is a smooth compact surface in $\mathbb{R}^{n+1}$ bounding an open set $V$ such that there exist positive constants $c$ and $\epsilon_{0}$ such that if $\epsilon<\epsilon_{0}$, whenever $x \in V$ and $|y|<\epsilon$ one has $x+y \in(1+c \epsilon) V$. Then if the Euclidean surface measure on $A$, which we denote by $\rho$, satisfies $|\hat{\rho}(\xi)| \leq C|\xi|^{-\beta}$ for some $\beta, C>0$, then the lattice point discrepancy corresponding to $V$ satisfies $\left|N(k)-k^{n+1} m(V)\right| \leq C^{\prime} k^{n-\frac{\beta}{n+1-\beta}}$ for some constant $C^{\prime}$.

When the conditions of Theorem 1.4 hold, the surfaces $S$ at hand satisfy the above conditions with $\beta=\alpha$; the fact that $S_{0}$ is convex with $C_{2}$ boundary ensures that the requisite condition on $S_{0}$ holds. Thus Theorem 1.4 immediately leads to the following consequence.
Theorem 2.1. Suppose $S$ is a compact $C^{2}$ surface bounding a convex domain $S_{0}$ containing the origin. Suppose for some $0<\alpha<1$ and some $C>0$ we have $|a(v, t)| \leq C|t|^{-\alpha}$ for all directions $v$ and all $t \neq 0$. Then for some constant $C^{\prime}$ one has the lattice point discrepancy bound $\left|N(k)-k^{n+1} m\left(S_{0}\right)\right| \leq C^{\prime} k^{n-\frac{\alpha}{n+1-\alpha}}$

Finding the optimal exponent for the lattice discrepancy for a given domain can be very difficult in general. The famous unsolved Gauss circle problem is to show that for the disk in two dimensions one has the bound $N(k)-\pi k^{2}=O\left(k^{\frac{1}{2}+\epsilon}\right)$ for any $\epsilon>0$; it was shown by Hardy in [H1] [H2] that one does not have an $O\left(k^{\frac{1}{2}}\right)$ bound. The surface Fourier transform method used above gives an exponent of $\frac{2}{3}$. The current best known exponent is $\frac{517}{834}$, due to Bourgain and Watt [BoWa], where modern techniques such as decoupling are used.

For spheres, the problem becomes less difficult as the dimension increases. While the best possible exponent is unknown for spheres in three dimensions, if $n \geq 3$ one can show that $N(k)-k^{n+1} m(V)=O\left(s^{n-1}\right)$ and the exponent $n-1$ is best possible. We refer to [Kr] for more information about lattice point discrepancy for spheres.

There has been a lot of work on lattice point discrepancy problems for more general convex domains. If the boundary of the domain $V$ has nonzero Gaussian curvature and is sufficiently smooth, then the Fourier transform method described above gives the bound $N(k)-k^{n+1} m(V)=O\left(k^{n-1+\frac{2}{n+2}}\right)$, as was proven by Hlawka [Hl1] [H12]. Various improvements have been made to this bound over time. There have also been various papers where the Gaussian curvature condition or the smoothness condition has been relaxed, such as the references [BrIT] [ISaSe1] [ISaSe2] [ISaSe3] [R1] [R2]. These papers use harmonic analysis techniques in the proofs.

## 3 Some examples.

Examples where Theorem 1.1 applies but which are not covered by [BNW] include conic surfaces given by equations $f\left(\frac{x_{1}}{x_{n+1}}, \frac{x_{2}}{x_{n+1}}, \ldots, \frac{x_{n}}{x_{n+1}}\right)=1$, where the points $\left(x_{1}, \ldots, x_{n}\right)$ for which $f\left(x_{1}, \ldots, x_{n}\right)=1$ form a compact real analytic surface enclosing a convex domain (this surface will necessarily be of finite line type). While one can analyze this example directly, it is also a good illustration of the statement of Theorem 1.1. Let $\rho$ be the Euclidean surface measure corresponding to such a surface, and let $\psi(x)$ be a $C^{n+1}$ function on $\mathbb{R}^{n+1}$ supported in the set $1<x_{n+1}<2$. Then the measure $\psi(x) \rho$ falls under the hypotheses of Theorem 1.1; one can use a partition of unity to reduce the situation to pieces which, after a suitable translation and rotation, become of the form (1.1).

We examine what Theorem 1.1 says in this situation. If $v$ makes an angle less than $\frac{\pi}{8}$ with the $x_{n+1}$ axis, one can get arbitrarily fast decay rate for the integral (1.1) simply by repeated integrations by parts in directions parallel to $S$. On the other hand, if $v$ makes an angle $\frac{\pi}{8}$ or more with the $x_{n+1}$ axis, one may look at the slab given by Theorem 1.1 of width $|t|^{-1}$ perpendicular to the $v$ direction and look at the intersection of this slab with the cross section of $S$ at a given height $x_{n+1}=c$. The intersection of this $n+1$ dimensional slab with this cross section of $S$ is the intersection of a $n$-dimensional slab of $\left\{x: x_{n+1}=c\right\}$ with this cross section of $S$, again with width comparable to $|t|^{-1}$. Thus if one wants an overall decay rate of the form $|\widehat{\psi(x) \rho}(\lambda)| \leq C|\lambda|^{-\alpha}$, one can seek an $\alpha$ for which (1.7) holds for all $t$ and $v$ for the surface $f\left(x_{1}, \ldots, x_{n}\right)=1$, since the same exponent will work for each surface $f\left(\frac{x_{1}}{c}, \ldots, \frac{x_{n}}{c}\right)=1$. Then one will get $|\hat{\rho}(t v)| \leq C|t|^{-\alpha}$ for all $t$ and all $v$ making an angle $\frac{\pi}{8}$ or more with the $x_{n}$ axis. Combining with the case of the angle being less than $\frac{\pi}{8}$, we see that overall one has a Fourier transform decay rate of $|\hat{\rho}(\lambda)| \leq C|\lambda|^{-\alpha}$. So for example, if we are looking at the cone $x_{1}^{2}+\ldots+x_{n}^{2}=x_{n+1}^{2}$, then the exponent $\alpha=\frac{n-1}{2}$ works.

We move to Theorem 1.2 and 1.3. Let $S_{1}, \ldots, S_{k}$ be distinct compact real analytic surfaces enclosing convex regions $V_{1}, \ldots, V_{k}$ respectively. Let $V=\cup_{i=1}^{k} V_{i}$ and let $S$ be the boundary of $V$. Then $S$ the finite union of pieces of different $S_{i}$, each of which falls under Theorem 1.2. Let $\sigma$ be the Euclidean surface measure on $S$ and $\sigma_{i}$ the Euclidean surface measure on $S_{i}$. Suppose $\alpha>0$ is such that for each $i$, Theorem 1.3 gives a bound of $\hat{\sigma}_{i}(\lambda) \leq C^{\prime}|\lambda|^{-\alpha}$. Then Theorem 1.3 implies one has the same bound of $C^{\prime}|\lambda|^{-\alpha}$ for the Fourier transform of the surface measure of the piece of $S$ deriving from $S_{i}$. Adding this up over all $i$, we see that $|\hat{\sigma}(\lambda)| \leq C|\lambda|^{-\alpha}$ as well.

Note that the above argument will also work for an $S$ that is the boundary of a finite intersections of $V_{i}$, or more generally finite unions of finite intersections of such $V_{i}$. One can sometimes improve the estimates obtained by examining the piece of $S_{i}$ occurring in a given $S$; one only needs the best exponent for a neighborhood in $S_{i}$ of this piece.

Next, we consider some curves in the plane with infinitely flat points. First we look at the curve $e^{-\frac{1}{x^{2}}}+y^{2}=c$, where $c>0$ is small enough to ensure that the curve encloses
a convex domain. Let $\tau$ denote the Euclidean surface measure corresponding to this curve. We can localize and use Theorem 1.2 here to analyze $\hat{\tau}(\lambda)$. If $v$ is a direction not parallel to the $y$ axis, by adding (1.6) over all $j$ we have the estimate $|\hat{\tau}(t v)| \leq C_{v}|t|^{-\frac{1}{2}}$ for all $t$. If $v$ is parallel to the $y$ axis, then we can still add (1.6) over all $j$ to get for say $|t|>2$ that $|\hat{\tau}(t v)| \leq C m\left(\left\{x: e^{-\frac{1}{x^{2}}} \leq|t|^{-1}\right\}\right)$, which translates into $|\hat{\tau}(t v)| \leq C(\ln |t|)^{\frac{1}{2}}$.

If on the other hand we are looking at the surface $e^{-\frac{1}{x^{2}}}+e^{-\frac{1}{y^{2}}}=c$ in $\mathbb{R}^{n+1}$ for small $c$, in directions $v$ parallel to one of the coordinate axes, one would similarly get estimates $|\hat{\tau}(t v)| \leq C(\ln |t|)^{\frac{1}{2}}$ for large $|t|$, where again we denote the surface measure by $\tau$, while in other directions, applying Theorem 1.2 and adding (1.6) over all $j$ would give estimates $|\hat{\tau}(t v)| \leq C_{v}|t|^{-\frac{1}{2}}$.

## 4 Proofs of Theorems 1.1 and 1.2.

### 4.1 Some lemmas used in the proof of Theorem 1.1.

In this section, we will make use of a couple of results from [G1] about real analytic functions (which are closely related to some results in [Mi]). They are as follows.

Lemma 4.1. (Theorem 2.1 of [G1].) Suppose $g\left(x_{1}, \ldots, x_{n}\right)$ is a real analytic function defined on a neighborhood of the origin, not identically zero. Then there is an $n$-1-dimensional ball $B_{n-1}(0, \eta)$ and a $k \geq 0$ such that for each $\left(x_{1}, \ldots, x_{n-1}\right)$ in $B_{n-1}(0, \eta)$ either $g\left(x_{1}, \ldots, x_{n}\right)=0$ for all $\left|x_{n}\right|<\eta$ or there is $a \leq l \leq k$, which may depend on $\left(x_{1}, \ldots, x_{n-1}\right)$, such that for all $\left|x_{n}\right|<\eta$ one has

$$
\begin{equation*}
0<\frac{1}{2}\left|\partial_{x_{n}}^{l} g\left(x_{1}, \ldots, x_{n-1}, 0\right)\right|<\left|\partial_{x_{n}}^{l} g\left(x_{1}, \ldots, x_{n}\right)\right|<2\left|\partial_{x_{n}}^{l} g\left(x_{1}, \ldots, x_{n-1}, 0\right)\right| \tag{4.1}
\end{equation*}
$$

The set of $\left(x_{1}, \ldots, x_{n-1}\right)$ for which $g\left(x_{1}, \ldots, x_{n}\right)=0$ for all $\left|x_{n}\right|<\eta$ has measure zero.
Lemma 4.2. (Corollary 2.1.2 of [G1].) Suppose $f_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, f_{l}\left(x_{1}, \ldots, x_{n}\right)$ are real analytic functions on a neighborhood of the origin, none identically zero. Then there is an $n-1$ dimensional ball $B_{n-1}(0, \eta)$ and a positive integer $p$ such that for each $s_{1}, \ldots, s_{l}$ and each $\left(x_{1}, \ldots, x_{n-1}\right) \in B_{n-1}(0, \eta)$, the set $\left\{x_{n}:\left|x_{n}\right|<\eta\right.$ and $f_{i}\left(x_{1}, \ldots, x_{n}\right)<s_{i}$ for each $\left.i\right\}$ consists of at most $p$ intervals.

Lemma 4.1 will be used in the proof of Theorem 1.1 in conjunction with the following result from [PS].

Lemma 4.3. (Lemma 1 of [PS].) Suppose $F \in C^{N}[\alpha, \beta]$, with $N \geq 1$, such that for a constant $C \geq 0$ one has

$$
\begin{equation*}
\sup _{\alpha \leq x \leq \beta}\left|F^{(N)}(x)\right| \leq C \inf _{\alpha \leq x \leq \beta}\left|F^{(N)}(x)\right| \tag{4.2}
\end{equation*}
$$

Then if $I$ is any subinterval of $[\alpha, \beta]$ of length $\delta$ and $I^{*}$ denotes its concentric double (as a subset of $[\alpha, \beta]$ ), we have

$$
\begin{gather*}
\sup _{x \in I^{*}}|F(x)| \leq a \sup _{x \in I}|F(x)|  \tag{4.3a}\\
\sup _{x \in I^{*}}\left|F^{\prime}(x)\right| \leq a \delta^{-1} \sup _{x \in I}|F(x)| \tag{4.3b}
\end{gather*}
$$

Here the constant a depends only on $N$ and $C$.

We now proceed to the proofs of Theorem 1.1 and 1.2 , starting with Theorem 1.1.

### 4.2 The proof of Theorem 1.1.

In equation (1.1), we write $v=\frac{\lambda}{|\lambda|}$ and $P\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, f\left(x_{1}, \ldots, x_{n}\right)\right)$. Then (1.1) can be rewritten as

$$
\begin{equation*}
\hat{\mu}(\lambda)=\int_{\mathbb{R}^{n}} e^{-i|\lambda| P(x) \cdot v} \phi(x) d x \tag{4.4}
\end{equation*}
$$

Since $|I(\lambda)|=|I(-\lambda)|$, without loss of generality we may assume $v=\left(v_{1}, \ldots, v_{n+1}\right)$ satisfies $v_{n+1} \geq 0$ as in the statement of the theorem. For such a $v$, one has $s(v)=\min _{x \in \bar{D}}(P(x) \cdot v)$. Choose any $x_{0}$ such that $s(v)=P\left(x_{0}\right) \cdot v$. Then (4.4) gives

$$
\begin{equation*}
|\hat{\mu}(\lambda)|=\left|\int_{\mathbb{R}^{n}} e^{-i|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)} \phi(x) d x\right| \tag{4.5}
\end{equation*}
$$

We integrate the integral of (4.5) in a polar coordinate system centered at $x_{0}$. Namely we rewrite this integral as

$$
\begin{equation*}
\hat{\mu}(\lambda)=c_{n} \int_{S^{n-1}} \int_{0}^{\infty} e^{-i|\lambda|\left(P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v\right)} r^{n-1} \phi\left(x_{0}+r \omega\right) d r d \omega \tag{4.6}
\end{equation*}
$$

Let $\psi(x)$ be a nonnegative smooth bump function on $\mathbb{R}$ supported on $[-1,1]$ with $\psi(x)=1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $\psi_{1}(x)=1-\psi(x)$. We write (4.6) as $I_{1}+I_{2}$, where
$I_{1}=c_{n} \int_{S^{n-1}} \int_{0}^{\infty} e^{-i|\lambda|\left(P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v\right)} r^{n-1} \phi\left(x_{0}+r \omega\right) \psi\left(|\lambda|\left(P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v\right)\right) d r d \omega$
$I_{2}=c_{n} \int_{S^{n-1}} \int_{0}^{\infty} e^{-i|\lambda|\left(P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v\right)} r^{n-1} \phi\left(x_{0}+r \omega\right) \psi_{1}\left(|\lambda|\left(P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v\right)\right) d r d \omega$
Note that by simply taking absolute values in the integral (4.7a), integrating, and then going back into rectangular coordinates, one sees that (4.7a) is bounded by the right-hand side of (1.4). Thus our concern here is bounding (4.7b).

To simplify notation we let $P_{x_{0}, \omega, v}(r)=P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v$, so that the inside integral of (4.7b) can be just written as

$$
\begin{equation*}
\int_{0}^{\infty} e^{-i|\lambda| P_{x_{0}, \omega, v}(r)} r^{n-1} \phi\left(x_{0}+r \omega\right) \psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right) d r \tag{4.8}
\end{equation*}
$$

We examine (4.8). Since $x_{0}$ was chosen so that $P\left(x_{0}\right) \cdot v$ achieves the minimal possible value of $P(x) \cdot v$ for $x \in \bar{D}$, one has that $P_{x_{0}, \omega, v}(0)=0$ and $\partial_{r} P_{x_{0}, \omega, v}(0) \geq 0$ (at least for $\omega$ for which there is a nonzero $r$ integral.) Since $f$ is convex, we also have that $\partial_{r r} P_{x_{0}, \omega, v}(r) \geq 0$ for all $r$ in (4.8), so that $\partial_{r} P_{x_{0}, \omega, v}(r) \geq \partial_{r} P_{x_{0}, \omega, v}(0) \geq 0$ for all $r$. Furthermore, since $f$ is real analytic here, either $\partial_{r} P_{x_{0}, \omega, v}(r)$ is identically zero in $r$ (in which case $P_{x_{0}, \omega, v}(r)$ is also identically zero in $r$ ), or $\partial_{r} P_{x_{0}, \omega, v}(r)>0$ for all $r>0$. Because of the $\psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right)$ factor in (4.8), in the former situations (4.8) is just zero, so for the purposes of our analysis we may exclude such cases and work under the assumption that $\partial_{r} P_{x_{0}, \omega, v}(r)>0$ for all $r>0$, which in turn implies $P_{x_{0}, \omega, v}(r)>0$ for all $r>0$.

Also, since $\partial_{r r} P_{x_{0}, \omega, v}(r) \geq 0$, and therefore $\partial_{r} P_{x_{0}, \omega, v}(s)$ is increasing, we have

$$
\begin{gathered}
P_{x_{0}, \omega, v}(r)=\int_{0}^{r} \partial_{r} P_{x_{0}, \omega, v}(s) d s \\
\leq r \partial_{r} P_{x_{0}, \omega, v}(r)
\end{gathered}
$$

We will normally use this in the form

$$
\begin{equation*}
\frac{1}{\left|\lambda r \partial_{r} P_{x_{0}, \omega, v}(r)\right|} \leq \frac{1}{\left|\lambda P_{x_{0}, \omega, v}(r)\right|} \tag{4.9}
\end{equation*}
$$

Next, we apply Lemma 4.1 to $\partial_{r}^{n+2} P_{x_{0, \omega}, v}(r)$, where one treats $r$ as the $x_{n}$ variable in the statement of the lemma, and the $x_{0}, \omega$, and $v$ variables as the remaining $n-1$ variables. While Lemma 4.1 is a local statement, since we are working on a compact set in all variables, it immediately implies a corresponding statement over all $\left(x_{0}, \omega, v, r\right)$. Namely there is some $k \geq n+2$ and $\delta>0$ such given any ( $\omega, x_{0}, v, r$ ), there is a $C_{x_{0}, \omega, v}$ (which may be zero) and an $n+2 \leq l_{x_{0}, \omega, v} \leq k$ such that for $\left|r^{\prime}-r\right|<\delta$ one has

$$
\begin{equation*}
\frac{1}{2} C_{x_{0}, \omega, v} \leq\left|\partial_{r}^{l_{0}, \omega, v} P_{x_{0}, \omega, v}(r)\right| \leq C_{x_{0}, \omega, v} \tag{4.10}
\end{equation*}
$$

We now apply Lemma 4.3 in conjunction with (4.10). By applying (4.3b) repeatedly, we get that there is a constant $b$ such that for all $1 \leq j \leq n+2$ and all $r>0$ one has

$$
\begin{equation*}
\sup _{t \in[0, r]}\left|\partial_{r}^{j} P_{x_{0}, \omega, v}(r)\right| \leq b r^{-j+1} \sup _{t \in[0, r]}\left|\partial_{r} P_{x_{0}, \omega, v}(r)\right| \tag{4.11}
\end{equation*}
$$

Since $\partial_{r} P_{x_{0}, \omega, v}(r)$ is nonnegative and increasing, this implies that

$$
\begin{equation*}
\left|\partial_{r}^{j} P_{x_{0}, \omega, v}(r)\right| \leq b r^{-j+1} \partial_{r} P_{x_{0}, \omega, v}(r) \tag{4.12}
\end{equation*}
$$

We now proceed to the analysis of (4.8). Since $\partial_{r} P_{x_{0}, \omega, v}(r)$ is positive and nondecreasing for $r>0, \partial_{r} P_{x_{0}, \omega, v}(r)$ is bounded below away from zero on the support of the $\psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right)$ factor appearing in (4.8). As a result, we may integrate by parts as follows, with no endpoint terms appearing. We write

$$
e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}=-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r) e^{-i|\lambda| P_{x_{0}, \omega, v}(r)} \times\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}
$$

In (4.8) we integrate $-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r) e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}$ back to $e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}$, and differentiate $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ times the remaining factors. We perform this integration by parts a total of $n+1$ times.

The idea is that after $k$ integrations by parts, the integrand incurs a factor bounded by a constant times $\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-k}$. To see why this is the case, we examine all the possibile factors the $r$ derivative may land in an integration by parts, and we will see that each time the integrand is multiplied a factor bounded by a constant times $\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$. We start with the situation where the derivative lands on a $\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{l}$ appearing in the denominator of a ratio of derivatives of $P_{x_{0}, \omega, v}(r)$, such as the original $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ in the first integration by parts. Then the $\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-l}$ becomes $\partial_{r r} P_{x_{0}, \omega, v}(r)\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-l-1}$, so that we have incurred a factor bounded by $C\left|\partial_{r r} P_{x_{0}, \omega, v}(r)\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}\right|$. By (4.12) this is at most $C^{\prime} r^{-1}$. When combined with the preexisting $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ factor in the integration by parts, we see we have a factor bounded by $C^{\prime}\left|\lambda r \partial_{r} P_{x_{0}, \omega, v}(r)\right|^{-1}$, which by (4.9) is bounded by the desired constant times $\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$, the factor we seek.

Next, we consider the case where the $r$ derivative lands on some $\partial_{r}^{k} P_{x_{0}, \omega, v}(r)$ appearing in the numerator of a ratio of derivatives of $P_{x_{0}, \omega, v}(r)$. Then the $\partial_{r}^{k} P_{x_{0}, \omega, v}(r)$ becomes a $\partial_{r}^{k+1} P_{x_{0}, \omega, v}(r)$. Whereas before $\left|\partial_{r}^{k} P_{x_{0}, \omega, v}(r)\right|$ was being estimated using (4.12) with $j=k$, we now estimate $\left|\partial_{r}^{k+1} P_{x_{0}, \omega, v}(r)\right|$ with (4.12) with $j=k+1$. Thus we incur an additional factor of $C r^{-1}$. Hence again when combined with the preexisting $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ factor, we obtain the desired factor of a constant times $\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$.

Next, we look at when the $r$ derivative lands on the $\psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right)$ factor. We obtain a factor bounded by a constant times $|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)$, which is exactly cancelled out by the preexisting $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ factor. Thus it would seem we just incur a factor bounded by a constant here. However, the derivative turns the $\psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right)$ into a $\psi_{1}^{\prime}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right)$ and $\psi_{1}^{\prime}$ is compactly supported. Hence we may simply insert $1 \leq C\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$ here to obtain the desired factor of a constant times $\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$.

We move on to the case where the derivative lands on $r^{n-1}$. This gives a factor of $C r^{-1}$, which like in the earlier cases gives us the factor we seek. Lastly, we consider the case where the derivative lands on the $\phi\left(x_{0}+r \omega\right)$ factor. Here we simply incur a bounded factor, which will be better than the $C r^{-1}$ we need.

We have now considered all possibilities and we see that with each integration by parts incurs a factor bounded by $C\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1}$. Thus if we integrate by parts in this fashion $n+1$ times, we see that the inside integral (4.8) is bounded by the following, where
$K$ denotes the interval of integration in this integral.

$$
\begin{equation*}
C \int_{K} r^{n-1} \psi_{1}\left(\left|\lambda P_{x_{0}, \omega, v}(r)\right|\right) \frac{1}{\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{n+1}} d r \tag{4.13}
\end{equation*}
$$

Here the constant $C$ will depend on $f$ and $\phi$, but not on $x_{0}, \omega$, $v$, or $\lambda$. Next, since $\psi_{1}(x)$ is supported on $|x| \geq \frac{1}{2}$, we have that (4.13) is bounded by

$$
\begin{equation*}
C \int_{\left\{r \in K:\left|\lambda P_{x_{0}, \omega, v}(r)\right| \geq \frac{1}{2}\right\}} \frac{r^{n-1}}{\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{n+1}} d r \tag{4.14}
\end{equation*}
$$

We split (4.14) dyadically as

$$
\begin{equation*}
C \sum_{j=0}^{\infty} \int_{\left\{r \in K: 2^{j-1}|\lambda|^{-1} \leq P_{x_{0}, \omega, v}(r)<2^{j}|\lambda|^{-1}\right\}} \frac{r^{n-1}}{\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{n+1}} d r \tag{4.15}
\end{equation*}
$$

Let $\left[a_{j}, b_{j}\right]$ denote the $r$ interval of integration in (4.15) corresponding to $j$ and denote its length $b_{j}-a_{j}$ by $l_{j}$. Since $\partial_{r r} P_{x_{0}, \omega, v}(r) \geq 0$ and therefore $\partial_{r} P_{x_{0}, \omega, v}(r)$ is increasing, the lengths of the intervals of integration in (4.15) increase no faster than if $P_{x_{0}, \omega, v}(r)$ were linear, so that $l_{j} \leq 2^{j} l_{0}$ and $r^{n-1} \leq 2^{j(n-1)} b_{0}^{n-1}$ on $\left[a_{j}, b_{j}\right]$. (This can be seen rigorously by applying the mean-value theorem to $\left(P_{x_{0}, \omega, v}\right)^{-1}(t)$ on the intervals $\left[2^{j-1}|\lambda|^{-1}, 2^{j}|\lambda|^{-1}\right]$ for $j \geq 0$ and comparing.)

Thus the $j$ th term of (4.15) is bounded by $C^{\prime} 2^{j(n-1)} b_{0}^{n-1} \times 2^{j} l_{0} \times 2^{-j(n+1)}=C^{\prime} 2^{-j} b_{0}^{n-1} l_{0}$. Adding this over all $j$ gives that (4.15) is bounded by $C^{\prime \prime} b_{0}^{n-1} l_{0}$, which is bounded by a constant times the $j=0$ term of (4.15), where one has $\left|\lambda P_{x_{0}, \omega, v}(r)\right| \sim 1$. As a result, we have that (4.15) is bounded by

$$
\begin{equation*}
C^{\prime \prime \prime} \int_{\left\{r \in K:\left|\lambda P_{x_{0}, \omega, v}(r)\right| \leq 1\right\}} r^{n-1} d r \tag{4.16}
\end{equation*}
$$

Going back from polar to retangular coordinates and integrating (4.16) in the $\omega$ variables leads to a bound of $C^{\prime \prime \prime} m\left(\left\{x \in \bar{D}:|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)<1\right\}\right)$. Since the above measure for $P(x) \cdot v-P\left(x_{0}\right) \cdot v=P(x) \cdot v-s(v)$ on $D$ corresponds to, up to a constant factor, the corresponding measure for $x \cdot v-s(v)$ on the surface $S^{\prime}$, and $|\lambda|$ here corresponds to $|t|$ in the statement of Theorem 1.1, this is exactly the desired right hand side of (1.4) and we are done.

### 4.3 The proof of Theorem 1.2.

We define $E=\left\{x \in \bar{D}: g_{i}(x)<0\right.$ for all $\left.i\right\}$ and our goal is to bound $I(\lambda)$ given in (1.5) by

$$
\begin{equation*}
I(\lambda)=\int_{E} e^{-i \lambda_{1} x_{1}-\ldots-i \lambda_{n} x_{n}-i \lambda_{n+1} f\left(x_{1}, \ldots, x_{n}\right)} \phi\left(x_{1}, \ldots, x_{n}\right) d x_{1} \ldots d x_{n} \tag{4.17}
\end{equation*}
$$

Once again we let $v=\frac{\lambda}{\mid \lambda}$, and since $|I(\lambda)|=|I(-\lambda)|$, once again we may assume $v_{n+1} \geq 0$ as in the statement of the theorem. Once again we subtract $s(v)$ from the phase, where $s(v)=\min _{x \in \bar{D}} P(x) \cdot v$, and $x_{0}$ is any point such that $s(v)=P\left(x_{0}\right) \cdot v$. So like (4.5) we have

$$
\begin{equation*}
|I(\lambda)|=\left|\int_{\mathbb{R}^{n}} e^{-i|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)} \phi(x) d x\right| \tag{4.18}
\end{equation*}
$$

Next, we let $A=\left\{x \in D:\left|\lambda\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)\right|<1\right\}$ and $2 A$ its $x_{0}$-centered double $\left\{x_{0}+2\left(x-x_{0}\right): x \in A\right\}$. We let $\chi_{1}(x)$ be the characteristic function of $2 A$ and $\chi_{2}(x)=$ $1-\chi_{1}(x)$, the characteristic function of $(2 A)^{c}$. Denote the integral in (4.18) by $I_{0}(\lambda)$. We write $I_{0}(\lambda)=I_{1}(\lambda)+I_{2}(\lambda)$, where for $j=1,2$ we have

$$
\begin{equation*}
I_{j}(\lambda)=\int_{\mathbb{R}^{n}} \chi_{j}(x) e^{-i|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)} \phi(x) d x \tag{4.19}
\end{equation*}
$$

We bound $\left|I_{1}(\lambda)\right|$ simply by taking absolute values inside the integrand of (4.19) and integrating. We see that $\left|I_{1}(\lambda)\right|$ is bounded by the first term on the right-hand side of (1.6), recalling that $P\left(x_{0}\right) \cdot v=s(v)$ and that the measure of a subset of $S^{\prime}$ is comparable to the measure of its projection onto the first $n-1$ coordinates. Hence we devote our attention to bounding $\left|I_{2}(\lambda)\right|$.

In (4.19) for $j=2$, we switch to polar coordinates, and as in the proof of Theorem 1.1 we define

$$
P_{x_{0}, \omega, v}(r)=P\left(x_{0}+r \omega\right) \cdot v-P\left(x_{0}\right) \cdot v
$$

Analogous to (4.6) we have

$$
\begin{equation*}
I(\lambda)=\int_{S^{n-1}} \int_{E_{\omega, x_{0}}} e^{-i|\lambda| P_{x_{0}, \omega, v}(r)} r^{n-1} \phi\left(x_{0}+r \omega\right) d r d \omega \tag{4.20}
\end{equation*}
$$

Here $E_{\omega, x_{0}}$ is the cross section of $E$ being integrated over in $r$. We claim that there is a fixed $N$ such that $E_{\omega, x_{0}}$ is comprised of at most $N$ intervals for every $\omega$ and $x_{0}$. To see this, we locally apply Lemma 4.2 to the functions $g_{i}\left(x_{0}+r \omega\right)$, where $r$ corresponds to the $x_{n}$ variable in Lemma 4.2 and the $x_{0}$ and $\omega$ variables correspond to the $x_{i}$ variables for $i<n$; a compactness argument then gives the statement over all $x_{0}, r$, and $\omega$. Note that the fact that we are restricting to $\bar{D}$ does not matter since $\bar{D}$ is convex and won't increase the number of intervals. Similarly the fact that we are restricting to the points where $|\lambda| P_{x_{0}, \omega, v}(r)>1$ doesn't increase the number of intervals since this function is increasing.

The domain of integration of the inner integral of (4.20), if nonempty, is a collection of at most $N$ intervals. On a given interval, which we call $J$, we integrate by parts as we did in the proof of Theorem 1.1, but only once this time. The function $\partial_{r} P_{x_{0}, \omega, v}(r)$ will never be zero on $J$ since for $I_{2}(\lambda)$ we are on the set where $|\lambda| P_{x_{0}, \omega, v}(r)>1$; since $P_{x_{0}, \omega, v}(0)=0$ and $\partial_{r} P_{x_{0}, \omega, v}(0)$ is nonnegative and increasing, the latter due to the convexity of $f(x)$, we have that if $|\lambda| P_{x_{0}, \omega, v}(r)>1$ then $\partial_{r} P_{x_{0}, \omega, v}(r)>0$. Thus we may proceed as in the proof of Theorem 1.1 and write

$$
e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}=-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r) e^{-i|\lambda| P_{x_{0}, \omega, v}(r)} \times\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}
$$

In the inner integral of (4.20) we integrate $-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r) e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}$ back to $e^{-i|\lambda| P_{x_{0}, \omega, v}(r)}$, and differentiate $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ times the remaining factors. We also get two endpoint terms this time. If we denote the endpoints of $J$ by $\alpha_{J}$ and $\beta_{J}$, then the terms obtained at these endpoints are of magnitude bounded by $C\left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\alpha_{J}\right)\right)^{-1}\left(\alpha_{J}\right)^{n-1}$ and $C\left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\beta_{J}\right)\right)^{-1}\left(\beta_{J}\right)^{n-1}$ respectively.

We now look at the several places the $r$ derivative may land in this integration by parts and the effect it has. In every case we will simply take absolute values of the resulting integrand and bound the resulting integral. First, suppose the derivative lands on the $r^{n-1}$ factor. Then we incur a factor of $C r^{-1}$, and the absolute value of the resulting term is at most

$$
\begin{equation*}
C \int_{J}\left(|\lambda| r \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1} r^{n-1} d r \tag{4.21}
\end{equation*}
$$

Next, we consider the case where the derivative lands on the $\left(-i|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$ factor. The resulting term is at most

$$
\begin{equation*}
C \int_{J}\left(|\lambda|\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}\right)^{2}\left(|\lambda| \partial_{r r} P_{x_{0}, \omega, v}(r)\right) r^{n-1} d r \tag{4.22}
\end{equation*}
$$

Here we are using that $\partial_{r r} P_{x_{0}, \omega, v}(r) \geq 0$ due to the convexity of $f(x)$. In (4.22) we integrate by parts again, this time integrating the $\left(|\lambda|\left(\partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}\right)^{2}\left(|\lambda| \partial_{r r} P_{x_{0}, \omega, v}(r)\right)$ factor back to $\left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1}$. We get one integral term bounded by (4.21), and two endpoint terms that like before are bounded by $C\left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\alpha_{J}\right)\right)^{-1}\left(\alpha_{J}\right)^{n-1}$ and $C\left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\beta_{J}\right)\right)^{-1}\left(\beta_{J}\right)^{n-1}$.

Lastly, the $r$ derivative may land on the $\phi\left(x_{0}+r \omega\right)$ factor. In this case one incurs at most a constant factor, which is better than the $C r^{-1}$ factor incurred when the derivative lands on the $r^{n-1}$ factor. Hence once again (4.21) serves as a bound for the term in question.

Next, we bound the above endpoint terms in terms of an integral resembling (4.21). Namely, since $\partial_{r} P_{x_{0, \omega}, v}(r)$ is increasing, we have

$$
\begin{align*}
& \left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\alpha_{J}\right)\right)^{-1}\left(\alpha_{J}\right)^{n-1} \leq C \int_{\frac{\alpha_{J}}{2}}^{\alpha_{J}}\left(|\lambda| r \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1} r^{n-1} d r \\
& \left(|\lambda| \partial_{r} P_{x_{0}, \omega, v}\left(\beta_{J}\right)\right)^{-1}\left(\beta_{J}\right)^{n-1} \leq C \int_{\frac{\beta_{J}}{2}}^{\beta_{J}}\left(|\lambda| r \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1} r^{n-1} d r \tag{4.23}
\end{align*}
$$

We see that we have boundedly many terms, all bounded by integrals of the form (4.21) or (4.23). The intervals $J$ are all derived from the characteristic function of $(2 A)^{c}$ above (4.19), so any of the intervals in either (4.21) or (4.23) come from the characteristic function of $A^{c}$. Since $A=\left\{x \in D:\left|\lambda\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)\right|<1\right\}$, this means the intervals all are a subset of the points where $\left|\lambda P_{x_{0}, \omega, v}(r)\right| \geq 1$. As a result, given that there are boundedly many intervals $J$, the sum of all of the terms (4.21) and (4.23) bounding our inner integral of (4.20) is bounded by

$$
\begin{equation*}
C^{\prime} \int_{\left|\lambda P_{x_{0}, \omega, v}(r)\right|>1}\left(|\lambda| r \partial_{r} P_{x_{0}, \omega, v}(r)\right)^{-1} r^{n-1} d r \tag{4.24}
\end{equation*}
$$

Equation (4.9) holds exactly as in the proof of Theorem 1.1, so (4.24) in turn is bounded by

$$
\begin{equation*}
C^{\prime \prime} \int_{\left|\lambda P_{x_{0}, \omega, v}(r)\right|>1}\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1} r^{n-1} d r \tag{4.25}
\end{equation*}
$$

Thus we have an overall bound for $I_{2}(\lambda)$ of

$$
\begin{equation*}
\left|I_{2}(\lambda)\right| \leq C^{\prime \prime} \int_{S^{n-1}} \int_{\left|\lambda P_{x_{0}, \omega, v}(r)\right|>1}\left|\lambda P_{x_{0}, \omega, v}(r)\right|^{-1} r^{n-1} d r d \omega \tag{4.26}
\end{equation*}
$$

Converting back into rectangular coordinates, we obtain

$$
\begin{equation*}
\left|I_{2}(\lambda)\right| \leq C^{\prime \prime \prime} \int_{\left\{x \in \bar{D}:|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)>1\right\}}\left(|\lambda|\left(P(x) \cdot v-P\left(x_{0}\right) \cdot v\right)\right)^{-1} d x \tag{4.27}
\end{equation*}
$$

We recall that $P\left(x_{0}\right) \cdot v=s(v)$ and decompose (4.27) dyadically to obtain a bound

$$
\begin{equation*}
\left|I_{2}(\lambda)\right| \leq C^{\prime \prime \prime \prime} \sum_{j=1}^{\infty} 2^{-j} m\left(\left\{x \in \bar{D}: 2^{j-1}|\lambda|^{-1} \leq(P(x) \cdot v-s(v))<2^{j}|\lambda|^{-1}\right\}\right) \tag{4.28}
\end{equation*}
$$

Since the above measure for $P(x) \cdot v-s(v)$ on $D$ corresponds to, up to a constant factor, the corresponding measure for $x \cdot v-s(v)$ on the surface $S^{\prime}$, and $|\lambda|$ here corresponds to $|t|$ in the statement of Theorem 1.2, we see that $\left|I_{2}(\lambda)\right|$ is bounded by the infinite series in (1.6). This concludes the proof of Theorem 1.2.

## 5 References.

[BakMVW] J. Bak, D. McMichael, J. Vance, S. Wainger, Fourier transforms of surface area measure on convex surfaces in $\mathbb{R}^{3}$, Amer. J. Math. 111 (1989), no.4, 633-668.
[BaGuZhZo] S. Basu, S. Guo, R. Zhang, P. Zorin-Kranich, A stationary set method for estimating oscillatory integrals, to appear, J. Eur. Math. Soc.
[BoWa] J. Bourgain, N. Watt, Mean square of zeta function, circle problem and divisor problem revisited, preprint. arxiv 1709.04340.
[BrHoI] L. Brandolini, S. Hoffmann, A. Iosevich, Sharp rate of average decay of the Fourier transform of a bounded set, Geom. Funct. Anal. 13 (2003), no. 4, 671-680.
[BrIT] L. Brandolini, A. Iosevich, G. Travaglini, Planar convex bodies, Fourier transform, lattice points, and irregularities of distribution, Trans. Amer. Math. Soc. 355 (2003), no.9, 3513-3535.
[BNW] J. Bruna, A. Nagel, and S. Wainger, Convex hypersurfaces and Fourier transforms, Ann. of Math. (2) 127 no. 2, (1988), 333-365.
[CoDiMaMu] M. Cowling, S. Disney, G. Mauceri, D. Muller, Damping oscillatory integrals, Invent. Math. 101 (1990), no.2, 237-260.
[G1] M. Greenblatt, A method for bounding oscillatory integrals in terms of non-oscillatory integrals, submitted.
[G2] M. Greenblatt, Hyperplane integrability conditions and smoothing for Radon transforms, J. Geom. Anal. 31 (2021), no.4, 3683-3697.
[Gr] J. Green, Uniform oscillatory integral estimates for convex phases via subleve set estimates, preprint.
[Gre] P. Gressman, Scalar oscillatory integrals in smooth spaces of homogeneous type, Rev. Mat. Iberoam. 31 (2015), no. 1, 215-244.
[Ha1] G. H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Math. 46 (1915), 263-283.
[Ha2] G. H. Hardy, On Dirichlet's divisor problem, Proc. London Math. Soc. (2) 15 (1916) 1-25.
[H11] E. Hlawka, Über Integrale auf konvexen Körpern. I. Monatsh. Math. 54 (1950), 1-36.
[H12] E. Hlawka, Über Integrale auf konvexen Körpern. II. Monatsh. Math. 54 (1950), 81-99.
[IL] A. Iosevich, E Liflyand, Decay of the Fourier transform, analytic and geometric aspects, Birkhäuser/Springer, Basel, 2014. xii+222 pp.
[ISa] A. Iosevich, E. Sawyer, Maximal averages over surfaces, Adv. Math. 132 (1997), no. 1, 46-119.
[ISaSe1] A. Iosevich, E. Sawyer, A. Seeger, Two problems associated with convex finite type domains, Publ. Mat. 46, no. 1 (2002), 153-177.
[ISaSe2] A. Iosevich, E. Sawyer, A. Seeger, Mean square discrepancy bounds for the number of lattice points in large convex bodies, J. Anal. Math. 87 (2002), 209-230.
[ISaSe3] A. Iosevich, E. Sawyer, A. Seeger, Mean lattice point discrepancy bounds. II. Convex domains in the plane, J. Anal. Math. 101 (2007), 25-63.
[Kr2] E. Krätzel, Analytische Funktionen in der Zahlentheorie, [Analytic functions in number theory] Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 139. B. G. Teubner, Stuttgart, 2000. 288 pp. ISBN: 3-519-00289-2.
[Mi] D. J. Miller, A preparation theorem for Weierstrass systems, Trans. Amer. Math. Soc. 358 (2006), no. 10, 4395-4439.
[PS] D. H. Phong, E. M. Stein, The Newton polyhedron and oscillatory integral operators, Acta Mathematica 179 (1997), 107-152.
[R1] B. Randol, A lattice point problem I, Trans. Amer. Math. Soc. 121 (1966), 257-268.
[R2] B. Randol, A lattice point problem II, Trans. Amer. Math. Soc. 125 (1966), 101-113.
[ShS] R. Shakarchi, E. Stein, Functional analysis. Introduction to further topics in analysis. Princeton Lectures in Analysis 4. Princeton University Press, Princeton, NJ, 2011. xviii+423 pp. ISBN: 978-0-691-11387-6.

Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
322 Science and Engineering Offices

851 S. Morgan Street
Chicago, IL 60607-7045
greenbla@uic.edu


[^0]:    This work was supported by a grant from the Simons Foundation.

