Fourier transforms of powers of well-behaved 2D real analytic functions

Michael Greenblatt

May 27, 2016

Abstract

This paper is a companion paper to [G4], where sharp estimates are proven for Fourier transforms of compactly supported functions built out of two-dimensional real-analytic functions. The theorems of [G4] are stated in a rather general form. In this paper, we expand on the results of [G4] and show that there is a class of "well-behaved" functions that contains a number of relevant examples for which such estimates can be explicitly described in terms of the Newton polygon of the function. We will further see that for a subclass of these functions, one can prove noticeably more precise estimates, again in an explicitly describable way.

1. Introduction and Definitions.

In this paper we consider Fourier transforms of powers of local two-dimensional real analytic functions. Namely we consider integrals

$$F(\lambda_1, \lambda_2) = \int_{\mathbf{R}^2} \phi(x_1, x_2)|f(x_1, x_2)|^{-\rho} e^{-i\lambda_1 x_1 - i\lambda_2 x_2}\, dx_1\, dx_2 \tag{1.1}$$

Here $f(x_1, x_2)$ is real analytic near the origin with $f(0,0) = 0$, $\rho > 0$ such that $|f(x_1, x_2)|^{-\rho}$ is integrable on a neighborhood of the origin, and $\phi(x_1, x_2)$ is supported on a neighborhood of the origin and C^1 on $(\mathbf{R} - \{0\})^2$ such that for some constant $A > 0$, on $(\mathbf{R} - \{0\})^2$ we have

$$|\phi(x_1, x_2)| < A \qquad\qquad |\nabla \phi(x_1, x_2)| < \frac{A}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \tag{1.2}$$

The prototype for $\phi(x_1, x_2)$ would be a smooth cutoff function on a neighborhood of the origin, but since the more general form is no more difficult we stipulate this condition. Note that we can multiply $\phi(x_1, x_2)$ by the characteristic function of any quadrant and (1.2) still holds. This allows us for example to estimate the Fourier transform of $|f(|x_1|, |x_2|)|^{-\rho}$ by adding the estimates for a given quadrant.

In the paper [G4], we provide various sharp estimates that can be proven for the functions $F(\lambda_1, \lambda_2)$. The theorems of [G4] are stated in a rather general form, and as a result sometimes the estimates of that paper are not amenable to being written out directly in terms of explicit properties of $f(x_1, x_2)$.

In this paper, we will expand on the results of [G4] and define a class of "well-behaved" functions that contains a number of relevant examples for which such estimates can be explicitly described. Specifically, we will see that for a range of ρ, for these well-behaved $f(x_1, x_2)$ we will be able to find optimal estimates of the form $|F(\lambda)| < C|\lambda_i|^{-\epsilon_i}$ for $i = 1, 2$, which immediately lead to optimal estimates of the form $|F(\lambda)| < C|\lambda|^{-\epsilon}$. Here λ denotes (λ_1, λ_2). The ϵ_i will be explicitly describable in terms of the Newton polygons of $f(x_1, x_2)$. We will further see that for a subclass of these functions, these estimates hold for all ρ (even when $\rho < 0$) and furthermore we even have estimates $|F(\lambda)| < C\alpha(\lambda)$, where again the estimates can be explicitly expressed in terms of the Newton polygon of $f(x_1, x_2)$.

In order to state our theorems, we now give some terminology that is frequently used in the subject of two-dimensional oscillatory integrals.

Definition 1.1. Let $f(x_1, x_2) = \sum_{a,b} f_{ab} x^a y^b$ denote the Taylor expansion of $f(x_1, x_2)$ at the origin. For any (a, b) for which $f_{ab} \neq 0$, let Q_{ab} be the quadrant $\{(x_1, x_2) \in \mathbf{R}^2 : x \geq a, y \geq b\}$. Then the *Newton polygon* $N(f)$ of $f(x_1, x_2)$ is defined to be the convex hull of the union of all Q_{ab}.

In general, the boundary of $N(f)$ consists of finitely many (possibly none) bounded edges of negative slope as well as an unbounded vertical ray and an unbounded horizontal ray.

A key role in our paper is played by the following polynomials.

Definition 1.2. Suppose e is a compact edge of $N(f)$. Define $f_e(x_1, x_2)$ by $f_e(x_1, x_2) = \sum_{(a,b)\in e} f_{ab} x^a y^b$. In other words $f_e(x_1, x_2)$ is the sum of the terms of the Taylor expansion of f corresponding to (a, b) on the edge e.

It is an important point that one only considers compact edges of $N(f)$ in the above definition. Next, our theorem statements will make use the following notion.

Definition 1.3. The *Newton distance* $d(f)$ of $f(x_1, x_2)$ is defined to be $\inf\{t : (t, t) \in N(f)\}$.

Our well-behavedness condition is then given by the following.

Definition 1.4. $f(x_1, x_2)$ is said to be *well-behaved* if the order of any zero of any $f_e(x_1, x_2)$ in $(\mathbf{R} - \{0\})^2$ is less than $d(f)$, and if there is an edge e of slope -1 then that $f_e(x_1, x_2)$ has no zeroes at all in $(\mathbf{R} - \{0\})^2$.

This condition is related to the concept of adapted coordinates in the subject of two-dimensional oscillatory integrals as initiated in [V]. Namely, $f(x_1, x_2)$ is in adapted coordinates if the zeroes of each $f_e(x_1, x_2)$ in $(\mathbf{R} - \{0\})^2$ have order less than or equal to $d(f)$. It turns out that that the scalar oscillatory index of $f(x_1,x_2)$ at the origin (see [AGV] for the relevant definitions) is equal to $\frac{1}{d(f)}$ if and only if $f(x_1,x_2)$ is in adapted coordinates, and thus in this situation one can readily compute this index in terms of $N(f)$. The reference [AGV] has a wealth of information on related matters. For the purpose of this paper, we are most concerned with the following (closely related) fact.

Lemma 1.1. **([G1])** If $f(x_1,x_2)$ is well-behaved, then $|f(x_1,x_2)|^{-\rho}$ is integrable on a neighborhood of the origin whenever $\rho < \frac{1}{d(f)}$, and is not integrable on any neighborhood of the origin whenever $\rho > \frac{1}{d(f)}$.

Next, we define the function $f^*(x_1,x_2)$, which will be a regularized version of $f(x_1,x_2)$ whose general behavior will be the same as $f(x_1,x_2)$ when $f(x_1,x_2)$ is well-behaved but for which many relevant quantities such as integrals are quite a bit easier to compute.

Definition 1.5. $f^*(x_1,x_2)$ denotes the function $\sum_{(v_1,v_2)\ a\ vertex\ of N(f)} |x_1|^{v_1}|x_2|^{v_2}$.

A useful fact concerning $f^*(x_1,x_2)$ is the following.

Lemma 1.2. Suppose $0 < \rho < \frac{1}{d(f)}$ and $f(x_1,x_2)$ is well-behaved. Then there are positive constants C_1 and C_2 depending on ρ and f such that if R is a dyadic rectangle one has

$$C_1 \int_R |f|^{-\rho} \leq \int_R (f^*)^{-\rho} \leq C_2 \int_R |f|^{-\rho} \tag{1.3}$$

Proof. The n-dimensional version of this was proven in [G2]. Specifically, by Lemma 2.1 of [G2], one has the existence of a constant C for which $|f(x)| \leq Cf^*(x)$ for all x, which gives the right-hand side of (1.3). The left-hand side follows from (4.15) of [G3], taking $\epsilon = 1$, since the left hand inequality of (1.3) for the portion of R where $|f(x)| > f^*(x)$ is immediate.

As a consequence of Lemmas 1.1-1.2, if $f(x_1,x_2)$ is well-behaved $(f^*(x_1,x_2))^{-\rho}$ is integrable on a neighborhood of the origin if $\rho < \frac{1}{d(f)}$ and integrable on no neighborhood of the origin if $\rho > \frac{1}{d(f)}$ since the same is true for $|f(x_1,x_2)|$. It can be shown that the same is also true for $f(x_1,x_2)$ that is not well-behaved.

2. Main Results.

Our first lemma defines some quantities used in the statement of our first theorem. In the following, we denote the edges of $N(f)$ by $e_0, ..., e_n$, where e_0 is the horizontal edge, e_n is the vertical edge, and the e_i are listed in order of decreasing slope. We write the slope of e_i as $-\frac{1}{m_i}$ where for $i = 0$ we take $m_i = \infty$ and for $i = n$ we take $m_i = 0$. Thus $m_{i+1} < m_i$ for all i. We denote by v_i the vertex of $N(f)$ between edges e_{i-1} and e_i, and write $v_i = (v_1^i, v_2^i)$.

Lemma 2.1. Suppose that $(f^*(x_1,x_2))^{-\rho}$ is integrable on a neighborhood of the origin, where $\rho > 0$. Then there exists an $\epsilon \geq 0$ and a $d = 0$ or 1, both depending on ρ and f, such that if $0 < r_0 < \frac{1}{2}$ there are positive constants c_1, c_2 depending on ρ, f, and r_0, such that if $0 < r < r_0$ one has one has $c_1 r^{-\epsilon}|\ln r|^d \leq \int_0^{r_0}(f^*(r,x_2))^{-\rho}\,dx_2 \leq c_2 r^{-\epsilon}|\ln r|^d$.

In addition, ϵ and d can be explicitly computed by finding the dominant term in the sum

$$r^{-\rho v_1^1}\int_0^{r^{m_1}} x_2^{-\rho v_2^1}\,dx_2 + \sum_{i=2}^{n-1} r^{-\rho v_1^i}\int_{r^{m_{i-1}}}^{r^{m_i}} x_2^{-\rho v_2^i}\,dx_2 + r^{-\rho v_1^n}\int_{r^{m_{n-1}}}^{1} x_2^{-\rho v_2^n}\,dx_2 \tag{2.1}$$

In the event that $n = 2$, one excludes the middle term of (2.1), and in the event $n = 1$ we replace (2.1) by $r^{-\rho v_1^1}\int_0^1 x_2^{-\rho v_2^1}\,dx_2$.

Proof. We first consider the portion of the integral $\int_0^r (f^*(r,x_2))^{-\rho}\,dx_2$ between any $x_2 = r^{m_{i-1}}$ and $x_2 = r^{m_i}$ for $i \geq 2$ that occurs. We claim that in this range, the quantity $r^{{v_1}^i} x_2^{{v_2}^i}$ is at least as large as $r^{{v_1}^j} x_2^{{v_2}^j}$ for any $j \neq i$. To see why this is the case, we look at the ratio $(r^{{v_1}^i} x_2^{{v_2}^i})/(r^{{v_1}^j} x_2^{{v_2}^j}) = r^{v_1^i - v_1^j} x_2^{v_2^i - v_2^j}$. If $v_2^i > v_2^j$, since $x_2 \geq r^{m_{i-1}}$ we have

$$r^{v_1^i - v_1^j} x_2^{v_2^i - v_2^j} \geq r^{v_1^i - v_1^j + m_{i-1}(v_2^i - v_2^j)} \tag{2.2}$$

Because (v_1^j, v_2^j) is on or above the edge e_{i-1}, whose slope is $-\frac{1}{m_{i-1}}$, we have that $v_2^j + m_{i-1}v_1^j \geq v_2^i + m_{i-1}v_1^i$. Thus the exponent in (2.2) is negative. Hence $r^{{v_1}^i} x_2^{{v_2}^i} \geq r^{{v_1}^j} x_2^{{v_2}^j}$ as needed. If on the other hand $v_2^i \leq v_2^j$, since $x_2 \leq r^{m_i}$, in place of (2.2) we can use

$$r^{v_1^i - v_1^j} x_2^{v_2^i - v_2^j} \geq r^{v_1^i - v_1^j + m_i(v_2^i - v_2^j)} \tag{2.3}$$

This time, we use that since (v_1^j, v_2^j) is on or above the edge e_i we have $v_2^j + m_i v_1^j \geq v_2^i + m_i v_1^i$. Thus the exponent in (2.3) is again negative and $r^{{v_1}^i} x_2^{{v_2}^i} \geq r^{{v_1}^j} x_2^{{v_2}^j}$ as desired.

Hence we have seen that on the portion of the integral where $x_1^{m_{i-1}} \leq x_2 \leq x_1^{m_i}, i \geq 2$, we have that $r^{{v_1}^i} x_2^{{v_2}^i} \geq r^{{v_1}^j} x_2^{{v_2}^j}$ for $j \neq i$. Thus $f^*(r,x_2)$ is the sum of several positive terms, the largest of which is $r^{{v_1}^i} x_2^{{v_2}^i}$. Hence there are constants C_1 and C_2 depending on $N(f)$ and ρ such that whenever $x_1^{m_i} \leq x_2 \leq x_1^{m_{i-1}}$ we have

$$C_1 r^{{v_1}^i} x_2^{{v_2}^i} < f^*(r,x_2) < C_2 r^{{v_1}^i} x_2^{{v_2}^i} \tag{2.4}$$

Next, we will prove an analogue of (2.4) that holds on $x_2 < x_1^{m_1}$. This time $i = 1$ is the dominant term. To see why, note that since $v_2^1 \leq v_2^j$ for all $j \neq 1$ and $x_2 < x_1^{m_1}$, we have that (2.3) holds for $i = 1$ and all $j \neq 1$. So since each (v_1^j, v_2^j) is on or above the edge e_1, we have $v_2^j + m_1 v_1^j \geq v_2^1 + m_1 v_1^1$ and like before we have $r^{{v_1}^1} x_2^{{v_2}^1} \geq r^{{v_1}^j} x_2^{{v_2}^j}$. The analogue to (2.4) that we get for the points were $y < x^{m_1}$ is therefore

$$C_1' r^{{v_1}^1} x_2^{{v_2}^1} < f^*(r,x_2) < C_2' r^{{v_1}^1} x_2^{{v_2}^1} \tag{2.5}$$

Similarly, if $x_2 > r^{m_{n-1}}, n \geq 2$, one can argue as in the above cases and show that we have

$$C_1'' r^{{v_1}^n} x_2^{{v_2}^n} < f^*(r, x_2) < C_2'' r^{{v_1}^n} x_2^{{v_2}^n} \tag{2.6}$$

Equations (2.4) – (2.6) cover the entire y range of integration in $\int_0^{r_0} (f^*(r,x_2))^{-\rho}\, dx_2$, except when $N(f)$ has exactly one vertex. But in this case $f^*(r,x_2) = r^{{v_1}^1} x_2^{{v_2}^1}$ which serves as a substitute for (2.4) – (2.6).

Equation (2.1) follows from (2.4) – (2.6) in short order; one simply takes the monomial from (2.4) – (2.6), raises it to the $-\rho$ power, and integrates in x_2 over its domain. Adding over all domains gives (2.1). This completes the proof of Lemma 2.1.

Note that by the proof of Lemma 2.1, one has that $f^*(x_1,x_2)$ is always within a constant factor of some dominant $x_1^{v_1^i} x_2^{v_2^i}$ which can be readily determined at a given (x_1,x_2). This will prove useful later.

Note also that the ϵ given by the expression (2.1) is a continuous function of ρ at any value of ρ where the expression is finite. As a result, when (2.1) is finite for $\rho = \frac{1}{d(f)}$ this ϵ must be 1. This true for the following reason. Since $|f^*(x_1,x_2)|^{-\rho}$ is integrable on a neighborhood of the origin when $\rho < \frac{1}{d(f)}$ by Lemma 1.1, ϵ must be less than 1 for such ρ. By continuity ϵ is therefore at most 1 when $\rho = \frac{1}{d(f)}$. If it were strictly less, the continuity of ϵ in the expression (2.1) implies that we could integrate $|f^*(x_1,x_2)|^{-\rho}$ to a finite value on a neighborhood of the origin for some $\rho > \frac{1}{d(f)}$, which is not possible by Lemma 1.1. Hence $\epsilon = 1$ when $\rho = \frac{1}{d(f)}$ whenever (2.1) is finite.

As a result, the continuity of ϵ in ρ says there will be an interval on which $\epsilon > \frac{1}{2}$ as long as (2.1) is finite at $\rho = \frac{1}{d(f)}$, which is the typical situation (but not always; see Example 1 below.) By symmetry the same will be true when $f^*(x_1,x_2)$ is replaced by $f^*(x_2,x_1)$. This justifies the $\epsilon_i > \frac{1}{2}$ conditions in the statement of Theorem 2.2, our first main theorem, which we now come to.

Theorem 2.2. Suppose $\rho > 0$ and $f(x_1,x_2)$ is well-behaved and the ϵ of Lemma 2.1 is greater than $\frac{1}{2}$ for both $f^*(x_1,x_2)$ and $f^*(x_2,x_1)$.

a) Let (ϵ_1, d_1) be as in Lemma 2.1 applied to $f(x_1,x_2)$, and let (ϵ_2, d_2) as in Lemma 2.1 applied to $f(x_2,x_1)$. Then there is a neighborhood N of the origin such that if the function $\phi(x_1,x_2)$ in (1.1) is supported in N then we have the following estimates, where C is a constant depending on f, ρ, N, and the constant A of (1.2).

$$|F(\lambda_1,\lambda_2)| < C(2+|\lambda_1|)^{\epsilon_1 - 1}(\ln(2+|\lambda_1|))^{d_1} \tag{2.7a}$$

$$|F(\lambda_1,\lambda_2)| < C(2+|\lambda_2|)^{\epsilon_2 - 1}(\ln(2+|\lambda_2|))^{d_2} \tag{2.7b}$$

Thus $|F(\lambda_1,\lambda_2)| < C(2+|\lambda|)^{\epsilon_j - 1}(\ln(2+|\lambda|))^{d_j}$ where (ϵ_j, d_j) denotes the slower of the two decay rates.

b) When $\phi(x_1, x_2)$ is bounded below by a positive constant on a neighborhood of the origin, then the exponents ϵ_i of $(2.7a)$ – $(2.7b)$ are best possible whenever $\epsilon_i < 1$; one does not have an estimate $|F(\lambda_1, \lambda_2)| < C(2 + |\lambda_i|)^{\epsilon' - 1}$ for $\epsilon' < \epsilon_i$.

Although we won't prove it here, there is a variation of Theorem 2.2 for the case when $f(x_1, x_2)$ is not well-behaved, but where instead the sum of terms of lowest degree of $f(x_1, x_2)$ has no zeroes on $V = \{(x_1, x_2) : |x_2| < c|x_1|\} \cap (\mathbf{R} - \{0\})^2$ for some $c > 0$. If instead of $F(\lambda_1, \lambda_2)$ one looks at the Fourier transform of $|f(x_1, x_2)|^{-\rho}\chi_V(x_1, x_2)$, one can show that Theorem 2.2 holds where the new (ϵ_1, d_1) and (ϵ_2, d_2) are defined by the variation on Lemma 2.1 where the integral $\int_0^{r_0} (f^*(r, x_2))^{-\rho}\, dx_2$ in the statement is replaced by $\int_0^{cx_1} (f^*(r, x_2))^{-\rho}\, dx_2$ and where the integral $\int_0^1 (f^*(x_1, r))^{-\rho}\, dx_2$ in the statement is replaced by $\int_{\frac{x_2}{c}}^1 (f^*(x_1, r))^{-\rho}\, dx_1$. This variant of Theorem 2.2 allows us to divide a neighborhood of the origin via lines through the origin, resulting in several wedges W_i. One can rotate each W_i to turn it into a set of the form V. If this variation of Theorem 2.2 applies on each such V, then can estimate $F(\lambda_1, \lambda_2)$ by adding the Fourier transform estimates for each $|f(x_1, x_2)|^{-\rho}\chi_V(x_1, x_2)$.

Example 1. Suppose $f(x_1, x_2) = x_1^a x_2^b$ for some a and b not both zero. Then $|f(x_1, x_2)|^{-\rho}$ is integrable on a neighborhood of the origin if $\rho < \frac{1}{max(a,b)}$. Here $f^*(x_1, x_2) = |x_1|^a |x_2|^b$, and one can compute (ϵ_1, d_1) using the integral $\int_0^1 (r^a x_2^b)^{-\rho}\, dx_2 = Cr^{-a\rho}$. So $\epsilon_1 = a\rho$ and $d_1 = 0$ here. By symmetry, $\epsilon_2 = b\rho$ and $d_2 = 0$. Hence Theorem 2.2 says that if $\frac{1}{2\min(a,b)} < \rho < \frac{1}{\max(a,b)}$ one has estimates $|F(\lambda_1, \lambda_2)| \leq C|\lambda_1|^{a\rho - 1}$ and $|F(\lambda_1, \lambda_2)| \leq C|\lambda_2|^{b\rho - 1}$, leading to an overall decay rate of $|F(\lambda_1, \lambda_2)| \leq C|\lambda|^{\max(a\rho, b\rho) - 1}$. Note that if $\min(a, b) \leq \frac{1}{2}\max(a, b)$, the conditions of Theorem 2.2 will never hold for this example.

Example 2. Suppose $f(x_1, x_2) = |x_1|^a + |x_2|^b$ for some a and b neither of which is zero. Then $f^*(x_1, x_2) = |x_1|^a + |x_2|^b$. The Newton polygon $N(f)$ has two vertices, $(a, 0)$ and $(0, b)$, and three edges: the vertical and horizontal edges, and a compact edge of slope $-\frac{b}{a}$. So $m_0 = \infty$, $m_1 = \frac{a}{b}$, and $m_2 = 0$. Then (ϵ_1, d_1) is computed using

$$r^{-\rho a} \int_0^{r^{\frac{a}{b}}} 1\, dx_2 + \int_{r^{\frac{a}{b}}}^1 x_2^{-\rho b}\, dx_2$$

If $\rho \neq \frac{1}{b}$, this is equal to $r^{-\rho a + \frac{a}{b}} + \frac{1}{1-\rho b}(1 - r^{(1-\rho b)\frac{a}{b}}) = \frac{1}{1-\rho b} - \frac{\rho b}{1-\rho b} r^{-\rho a + \frac{a}{b}}$. The second term dominates if $\rho > \frac{1}{b}$, and the first term dominates if $\rho < \frac{1}{b}$. Hence $(\epsilon_1, d_1) = (\rho a - \frac{a}{b}, 0)$ if $\rho > \frac{1}{b}$ and $(\epsilon_1, d_1) = (0, 0)$ if $\rho < \frac{1}{b}$. If $\rho = \frac{1}{b}$, the sum of the two integrals is $r^{-\rho a + \frac{a}{b}} - \frac{a}{b} \ln r$. Thus the second term dominates, and $(\epsilon_1, d_1) = (0, 1)$. By symmetry, $(\epsilon_2, d_2) = (\rho b - \frac{b}{a}, 0)$ if $\rho > \frac{1}{a}$, $(\epsilon_2, d_2) = (0, 0)$ if $\rho < \frac{1}{a}$, and $(\epsilon_2, d_2) = (0, 1)$ if $\rho = \frac{1}{a}$.

$f(x_1, x_2)$ is integrable over a neighborhood of the origin if $\epsilon_1 < 1$, which in the current situation is equivalent to the statement that $\epsilon_2 < 1$. The condition works out to $\rho < \frac{1}{a} + \frac{1}{b}$. One has $\epsilon_1 > \frac{1}{2}$ if $\rho a - \frac{a}{b} > \frac{1}{2}$ or equivalently $\rho > \frac{1}{2a} + \frac{1}{b}$. Similarly, one has $\epsilon_2 > \frac{1}{2}$ if $\rho > \frac{1}{a} + \frac{1}{2b}$. Hence Theorem 2.2 applies for ρ on the smaller of the two intervals $(\frac{1}{2a} + \frac{1}{b}, \frac{1}{a} + \frac{1}{b})$ or $(\frac{1}{a} + \frac{1}{2b}, \frac{1}{a} + \frac{1}{b})$. For such ρ one has estimates $|F(\lambda_1, \lambda_2)| \leq C|\lambda_1|^{\rho a - \frac{a}{b} - 1}$ and $|F(\lambda_1, \lambda_2)| \leq C|\lambda_2|^{\rho b - \frac{b}{a} - 1}$. The overall estimate obtained is then $|F(\lambda_1, \lambda_2)| \leq C|\lambda|^{\max(\rho a - \frac{a}{b} - 1, \rho b - \frac{b}{a} - 1)}$. If one works it out, one sees that one uses the exponent $\rho a - \frac{a}{b} - 1$ if $a \leq b$ and the exponent $\rho b - \frac{b}{a} - 1$ if $a \geq b$.

This example also satisfies the conditions of Theorem 2.3. As a result the estimate $|F(\lambda_1, \lambda_2)| \leq C|\lambda|^{\max(\rho a - \frac{a}{b} - 1, \rho b - \frac{b}{a} - 1)}$ (as well as more precise estimates) will hold for any (a, b) and any ρ.

Our second main theorem will give more precise information than Theorem 2.2 when each $f_e(x_1, x_2)$ has no zeroes on $(\mathbf{R} - \{0\})^2$. Instead of (1.2) we will assume that $\phi(x_1, x_2)$ is C^∞ on $(\mathbf{R} - \{0\})^2$ and there are constants A and $A_{a,b}$ such that

$$|\phi(x_1, x_2)| \leq A \qquad |\partial_{x_1}^a \partial_{x_2}^b \phi(x_1, x_2)| \leq A_{a,b}|x_1|^{-a}|x_2|^{-b} \quad (\forall a\, \forall b) \tag{2.8}$$

Theorem 2.3. Suppose (2.8) holds and that each $f_e(x_1, x_2)$ has no zeroes on $(\mathbf{R} - \{0\})^2$. Then there is a neighborhood U of the origin such that if $\phi(x_1, x_2)$ is supported in U, then equations $(2.7a) - (2.7b)$ hold for all ρ, even if $\rho < 0$. In fact, one has the following stronger estimate, where C is a constant depending on f, ρ, U, and the constant A of (1.2).

$$|F(\lambda_1, \lambda_2)| \leq C \int_{\{(x_1,x_2) \in U:\, |x_1| < |\lambda_1|^{-1},\, |x_2| < |\lambda_2|^{-1}\}} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{2.9}$$

Up to a constant factor, one can explicitly determine the integral (2.9) similarly to in Theorem 2.2, proceeding as in Lemma 2.1 where one divides a neighborhood of the origin into domains on each of which $f^*(x_1, x_2)$ is within a constant factor of some explicitly determinable $x_1^{v_1^i} x_2^{v_2^i}$. On each such domain, the right-hand side of (2.9) will be within a bounded factor of $|\lambda_1|^a |\ln \lambda_1|^{d_1} |\lambda_2|^b |\ln \lambda_2|^{d_2}$ for some a and b and $d_i = 0$ or 1.

It can also be shown similarly to the proof of Theorem 2.2b) that if $a < 0$ and $b < 0$ then on a domain $\{(\lambda_1, \lambda_2) : |\lambda_1|^{-m_i} < |\lambda_2|^{-1} < |\lambda_1|^{-m_{i+1}}\}$ for compact edges e_i and e_{i+1}, the exponents a and b are best possible.

Example 1. Let $f(x_1, x_2) = x_1^a x_2^b$ where a and b are not both zero. Then the right-hand side of (2.9) is given by $C \int_0^{|\lambda_1|^{-1}} \int_0^{|\lambda_2|^{-1}} x_1^{-a\rho} x_2^{-b\rho}\, dx_2\, dx_1$, which equals $C|\lambda_1|^{a\rho - 1}|\lambda_2|^{b\rho - 1}$ when it is finite. Note the improvement over the estimate for the same example after Theorem 2.2.

Example 2. Suppose $f(x_1, x_2) = |x_1|^a + |x_2|^b$ for some a and b neither equal to zero. Theorem 2.3 then gives

$$|F(\lambda_1, \lambda_2)| \leq C \int_0^{|\lambda_1|^{-1}} \int_0^{|\lambda_2|^{-1}} (x_1^a + x_2^b)^{-\rho}\, dx_2\, dx_1 \tag{2.10}$$

One divides the integral along the curve $x_2 = x_1^{\frac{a}{b}}$, and (2.10) becomes

$$|F(\lambda_1, \lambda_2)| \leq C \int_0^{|\lambda_1|^{-1}} \int_0^{\min(x_1^{\frac{a}{b}}, |\lambda_2|^{-1})} x_1^{-a\rho}\, dx_2\, dx_1$$

$$+C \int_0^{|\lambda_1|^{-1}} \int_{\min(x_1^{\frac{a}{b}}, |\lambda_2|^{-1})}^{|\lambda_2|^{-1}} x_2^{-b\rho}\, dx_2\, dx_1 \tag{2.11}$$

One can readily perform the integrations in (2.11) to get explicit formulas. One gets two different formulas depending on whether or not $|\lambda_1|^{-a} \leq |\lambda_2|^{-b}$.

3. Theorem proofs.

Proof of Theorem 2.2.

Suppose we are in the setting of Theorem 2.2. The key fact that we use here is Corollary 3.4 of [G4], which says that

$$|F(\lambda_1, \lambda_2)| \leq C \int_N (1 + |\lambda_1 x_1| + |\lambda_2 x_2|)^{-\frac{1}{2}} |f(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.1}$$

Here N is a small neighborhood of the origin on which the resolution of singularities algorithm of [G5] applies, and we henceforth assume $\phi(x_1, x_2)$ is supported on N. Thus for $i = 1, 2$ we have

$$|F(\lambda_1, \lambda_2)| \leq C \int_N (1 + |\lambda_i x_i|)^{-\frac{1}{2}} |f(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.2}$$

Suppose $|\lambda_i| > 2$. Splitting the integral (3.2) at $|x_i| = \frac{1}{|\lambda_i|}$, equation (3.2) becomes

$$|F(\lambda_1, \lambda_2)| \leq C \int_{\{(x_1,x_2)\in N: |x_i| < \frac{1}{|\lambda_i|}\}} |f(x_1, x_2)|^{-\rho}\, dx_1\, dx_2$$

$$+C \frac{1}{|\lambda_i|^{\frac{1}{2}}} \int_{\{(x_1,x_2)\in N: |x_i| \geq \frac{1}{|\lambda_i|}\}} {x_i}^{-\frac{1}{2}} |f(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.3}$$

By Lemma 1.2, one can replace $|f(x_1, x_2)|$ by $f^*(x_1, x_2)$ in (3.3). In the two integrals of the resulting expression, we first integrate in the variable that is not x_i, inserting the right-hand inequality of Lemma 2.1. The result is

$$|F(\lambda_1, \lambda_2)| \leq C \int_0^{\frac{1}{|\lambda_i|}} {x_i}^{-\epsilon_i} |\ln x_i|^{d_i}\, dx_i + C \frac{1}{|\lambda_i|^{\frac{1}{2}}} \int_{\frac{1}{|\lambda_i|}}^{\frac{1}{2}} x_i^{-\frac{1}{2}-\epsilon_i} |\ln x_i|^{d_i}\, dx_i$$

Integrating the two terms in (3.3) and using that $\epsilon_i > \frac{1}{2}$ we obtain the desired estimate

$$|F(\lambda_1, \lambda_2)| \leq C |\lambda_i|^{\epsilon_i - 1} |\ln \lambda_i|^{d_i} \tag{3.4}$$

This is $(2.7a)-(2.7b)$ when $|\lambda_i|>2$. When $|\lambda_i|<2$, one obtains $(2.7a)-(2.7b)$ simply by taking absolute values and integrating. Thus we have proved $(2.7a)-(2.7b)$ and the proof of part a) of Theorem 2.2 is complete.

Moving on to part b), suppose $\epsilon_i>0$, $\phi(x_1,x_2)$ is bounded below on a neighborhood of the origin and the estimate $|F(\lambda_1,\lambda_2)|\leq C(2+|\lambda_i|)^{\epsilon'-1}$ holds, where $\epsilon'<\epsilon_i<1$, and we will reach a contradiction. Without loss of generality we take $i=1$. Let $\psi(x)$ be a smooth function on $\mathbf{R}$ whose Fourier transform is a nonnegative compactly supported function equal to 1 on a neighborhood of the origin. Since $\epsilon'<\epsilon_1<1$ and $\epsilon_1>0$, we may let $\eta>0$ be such that $0<\eta+\epsilon'<\epsilon_1$. For a large K we look at

$$I_K=\int F(\lambda_1,0)\psi\left(\frac{\lambda_1}{K}\right)|\lambda_1|^{-\eta-\epsilon'}\,d\lambda_1 \tag{3.5}$$

Since $|F(\lambda_1,0)|\leq C(2+|\lambda_1|)^{\epsilon'-1}$, we have that

$$|I_K|\leq C\int(2+|\lambda_1|)^{\epsilon'-1}|\lambda_1|^{-\eta-\epsilon'}\,d\lambda_1 \tag{3.6}$$

Because $\eta>0$, the integrand in (3.6) is integrable for large $|\lambda_1|$, and because $\eta+\epsilon'<1$ the integrand in (3.6) is integrable for small $|\lambda_1|$. Hence the I_K are uniformly bounded in K. On the other hand

$$I_K=\int_{\mathbf{R}^2}\phi(x_1,x_2)|f(x_1,x_2)|^{-\rho}e^{-i\lambda_1x_1}\psi\left(\frac{\lambda_1}{K}\right)|\lambda_1|^{-\eta-\epsilon'}\,d\lambda_1\,dx_1\,dx_2 \tag{3.7}$$

Performing the λ_1 integral in (3.7) leads to

$$I_K=\int_{\mathbf{R}^2}\phi(x_1,x_2)|f(x_1,x_2)|^{-\rho}K^{1-\eta-\epsilon'}\xi(Kx_1)\,dx_1\,dx_2 \tag{3.8}$$

Here ξ is the Fourier transform of $\psi(\lambda_1)|\lambda_1|^{-\eta-\epsilon'}$. Since the Fourier transform of $\psi(\lambda_1)$ is nonnegative and the Fourier transform of $|\lambda_1|^{-\eta-\epsilon'}$ is of the form $c|x_1|^{\eta+\epsilon'-1}$, $\xi(x_1)$ is of the form $c\tilde{\xi}(x_1)$ where $\tilde{\xi}(x_1)$ is nonnegative and decays as $|x_1|^{\eta+\epsilon'-1}$ as $|x_1|\rightarrow\infty$. Thus we can rewrite (3.8) as

$$|I_K|=|c|\int_{\mathbf{R}^2}\phi(x_1,x_2)|f(x_1,x_2)|^{-\rho}K^{1-\eta-\epsilon'}\tilde{\xi}(Kx_1)\,dx_1\,dx_2 \tag{3.8$'$}$$

Since $\phi(x_1,x_2)$ is nonnegative and is positive on a neighborhood of the origin, there is a constant C and a neighborhood N of the origin such that

$$I_K\geq C\int_N|f(x_1,x_2)|^{-\rho}K^{1-\eta-\epsilon'}\tilde{\xi}(Kx_1)\,dx_1\,dx_2 \tag{3.9}$$

Shrinking N if necessary and assuming N is a union of dyadic rectangles on which Lemma 1.2 holds, we therefore have

$$I_K\geq C'\int_N|f^*(x_1,x_2)|^{-\rho}K^{1-\eta-\epsilon'}\tilde{\xi}(Kx_1)\,dx_1\,dx_2 \tag{3.10}$$

Performing the x_2 integration and using Lemma 2.1, for some $a > 0$ we therefore have

$$I_K \geq C'' \int_{-a}^{a} K^{1-\eta-\epsilon'} \tilde{\xi}(Kx_1) x_1^{-\epsilon_1} |\ln x_1|^{d_1} \, dx_1 \tag{3.11}$$

In particular, for any $b > 0$ we have

$$I_K \geq C'' \int_{b}^{a} K^{1-\eta-\epsilon'} \tilde{\xi}(Kx_1) x_1^{-\epsilon_1} |\ln x_1|^{d_1} \, dx_1 \tag{3.12}$$

Taking limits as $K \rightarrow \infty$ and using that $\tilde{\xi}(x_1)$ decays as $|x_1|^{\eta+\epsilon'-1}$, we get that for any b that

$$\sup_K I_K \geq C'' \int_{b}^{a} x_1^{\eta+\epsilon'-\epsilon_1-1} |\ln x_1|^{d} \, dx_1 \tag{3.13}$$

Since $\sup_K I_K$ is finite, we must therefore have that $\eta + \epsilon' - \epsilon_1 > 0$, contradicting the choice of η. Hence we have arrived at a contradiction and the proof of part b) of Theorem 2.2 is complete, thereby completing the proof of the whole theorem.

In the proof of Theorem 2.3 we will use the following lemma.

Lemma 3.1. Given any multiindex (a, b) there is a neighborhood N of the origin and a constant $C_{a,b,f,N}$ such that on N one has

$$|\partial_{x_1}^{a} \partial_{x_2}^{b} f(x_1, x_2)| \leq C_{a,b,f,N} \frac{1}{|x_1|^{a} |x_2|^{b}} f^*(x_1, x_2) \tag{3.14}$$

If each $f_e(x_1, x_2)$ has no zeroes on $(\mathbf{R} - \{0\})^2$, then there is in addition a neighborhood N' of the origin and a constant $c_{f,N'}$ such that $|f(x_1, x_2)| \geq c_{f,N'} f^*(x_1, x_2)$ on N'. In other words, $|f(x_1, x_2)| \sim f^*(x_1, x_2)$ on N'.

Proof. As mentioned in the proof of Lemma 1.2, Lemma 2.1 of [G2] implies that for any real analytic function $g(x_1, x_2)$ on a neighborhood of the origin with $g(0, 0) = 0$, there is an inequality $|g(x_1, x_2)| \leq Cg^*(x_1, x_2)$ on a neighborhood of the origin. Applying this to any $\partial_{x_1}^{a} \partial_{x_2}^{b} f$ equal to zero at origin gives (3.14) for that (a, b). If $\partial_{x_1}^{a} \partial_{x_2}^{b} f(0, 0) \neq 0$ the inequality is immediate, so (3.14) holds in all cases.

Suppose now that each $f_e(x_1, x_2)$ has no zeroes on $(\mathbf{R} - \{0\})^2$. We divide a neighborhood N of the origin into wedges A_i and B_i as follows. Each A_i is of the form $\{(x_1, x_2) \in N : \frac{1}{K}|x_1|^{m_i} < |x_2| < K|x_1|^{m_i}\}$ for some large K and where e_i is a compact edge of $N(f)$. Each B_i is of the form $\{(x_1, x_2) \in N : K|x_1|^{m_i} < |x_2| < \frac{1}{K}|x_1|^{m_{i+1}}\}$ for compact edges e_i and e_{i+1} of $N(f)$, or is of the form $\{(x_1, x_2) \in N : K|x_1|^{m_{n-1}} < |x_2|\}$, or is of the form $\{(x_1, x_2) \in N : |x_2| < \frac{1}{K}|x_1|^{m_1}\}$.

In the setting of Lemma 2.1 of [G2], the A_i and B_i are the sets denoted by W_{ij}. For the A_i, Lemma 2.1 of [G2] says that given any fixed K and any $\delta > 0$, there is a neighborhood V_i of the origin such that on $A_i \cap V_i$ we have

$$|f(x_1, x_2) - f_e(x_1, x_2)| < \delta |x_1|^{v_1^i} |x_2|^{v_2^i} \tag{3.15}$$

In addition, using that that each $f_e(x_1,x_2)$ has no zeroes on $(\mathbf{R}-\{0\})^2$, the mixed homogeneity of $f_e(x_1,x_2)$, and the resulting fact that $f_e(1,x)$ and $f_e(1,-x)$ have no zeroes on $[\frac{1}{K},K]\cup[-K,-\frac{1}{K}]$, there is a constant c such that $|f_e(x_1,x_2)| > c|x_1|^{v_1^i}|x_2|^{v_2^i}$ on A_i. Hence choosing $\delta = \frac{c}{2}$, we conclude that $|f(x_1,x_2)| > \frac{c}{2}|x_1|^{v_1^i}|x_2|^{v_2^i}$ on $A_i \cap V_i$. By (2.4) we have that $|x_1|^{v_1^i}|x_2|^{v_2^i} \sim f^*(x_1,x_2)$ on $A_i\cap V_i$, so there exists a c' for which $|f(x_1,x_2)| > c'f^*(x_1,x_2)$ on $A_i \cap V_i$ as needed.

For the B_i, Lemma 2.1 of [G2] says that there is a single vertex v_j of $N(f)$ and a $d_j \neq 0$ such that given any $\delta > 0$, if K were chosen large enough there is a neighborhood U_i of the origin such that $|f(x_1,x_2) - d_j v_1^j v_2^j| < \delta|x_1|^{v_1^j}|x_2|^{v_2^j}$ on $B_i \cap U_i$. Taking $\delta < \frac{1}{2}|d_j|$, we have $|f(x_1,x_2)| > \frac{1}{2}|d_j v_1^j v_2^j|$ on $B_i \cap U_i$. By (2.4) – (2.6) we have that $|v_1^j v_2^j| \sim f^*(x_1,x_2)$ on B_i, so $|f(x_1,x_2)| > c'' f^*(x_1,x_2)$ on $B_i \cap U_i$ for some constant $c'' > 0$ as needed.

Letting N' be the intersection of all U_i and V_i, we see that on N' we have $|f(x_1,x_2)| > c_{f,N'} f^*(x_1,x_2)$ for some constant $c_{f,N'}$ as needed. This completes the proof of Lemma 3.1.

Proof of Theorem 2.3.

We write $F(\lambda_1,\lambda_2) = \sum_{j,k} F_{jk}(\lambda_1,\lambda_2)$, where

$$F_{jk}(\lambda_1,\lambda_2) = \int_{\mathbf{R}^2} \phi(x_1,x_2)\beta(2^j x_1)\beta(2^k x_2)|f(x_1,x_2)|^{-\rho} e^{-i\lambda_1 x_1 - i\lambda_2 x_2}\,dx_1\,dx_2 \tag{3.16}$$

Here $\beta(x)$ is a nonnegative smooth compactly supported function on $\mathbf{R}$ whose support does not intersect some neighborhood of 0. If j is such that $2^{-j} > |\lambda_1|^{-1}$, we integrate by parts in (3.16) in the x_1 variable N times, integrating the $e^{-i\lambda_1 x_1}$ and differentiating the rest. Each time we do so we get a $\frac{1}{i\lambda_1}$ from the integration. When the derivative lands on $\beta(2^j x_1)$ or one of its derivatives we get a factor of $C2^j$, and each time the derivative lands on $\phi(x_1,x_2)\beta(2^k x_2)$ or one of its derivatives we also get a factor of $C2^j$ due to the conditions (2.8).

As for when the derivative lands on the $|f(x_1,x_2)|^{-\rho}$ factor, by Lemma 3.1 $f(x_1,x_2)$ is of a single sign in each quadrant, so $|f(x_1,x_2)|^{-\rho}$ is $(\pm f(x_1,x_2))^{-\rho}$ on a given quadrant. Each time the x_1 derivative lands on such a factor or one of its derivatives, by (3.14) and the fact that $f(x_1,x_2) \sim f^*(x_1,x_2)$, one gets a factor bounded by $C\frac{1}{|x_1|}$. Due to the support conditions on $\beta(2^j x_1)$, this too is bounded by $C2^j$.

We conclude that the integration by parts leads to an overall factor of $C2^j$. Hence N integrations by parts results in a factor of $C2^{jN} \sim C|x_1|^{-N}$. Because of this and the fact that $f(x_1,x_2) \sim f^*(x_1,x_2)$ by Lemma 3.1, we conclude that there is a neighborhood

U of the origin such that if $\phi(x_1, x_2)$ is supported in U, then for any N we have an estimate

$$|F_{jk}(\lambda_1, \lambda_2)| \leq C_N \frac{1}{|\lambda_1|^N} \int_{U \cap \{x: 2^{-j-1} < |x_1| < 2^{-j},\, 2^{-k-1} < |x_2| < 2^{-k}\}} \frac{1}{|x_1|^N} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.17a}$$

In exactly the same way, reversing the roles of the x_1 and x_2 variables, if $2^{-k} > |\lambda_2|^{-1}$ we have

$$|F_{jk}(\lambda_1, \lambda_2)| \leq C_N \frac{1}{|\lambda_2|^N} \int_{U \cap \{x: 2^{-j-1} < |x_1| < 2^{-j},\, 2^{-k-1} < |x_2| < 2^{-k}\}} \frac{1}{|x_2|^N} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.17b}$$

If both $2^{-j} > |\lambda_1|^{-1}$ and $2^{-k} > |\lambda_2|^{-1}$, we can first do N integrations by parts in the x_1 variable followed by N integrations by parts in the x_2 variable to obtain that $|F_{jk}(\lambda_1, \lambda_2)|$ is bounded by

$$C_N \frac{1}{|\lambda_1 \lambda_2|^N} \int_{U \cap \{x: 2^{-j-1} < |x_1| < 2^{-j},\, 2^{-k-1} < |x_2| < 2^{-k}\}} \frac{1}{|x_1 x_2|^N} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.17c}$$

We will obtain our desired estimates for a given $F_{jk}(\lambda_1, \lambda_2)$ as follows. When $2^{-j} \leq |\lambda_1|^{-1}$ and $2^{-k} \leq |\lambda_2|^{-1}$ we just take absolute values in (3.16) and integrate. When $2^{-j} > |\lambda_1|^{-1}$ and $2^{-k} < |\lambda_2|^{-1}$ we use (3.17a). When $2^{-j} < |\lambda_1|^{-1}$ and $2^{-k} > |\lambda_2|^{-1}$ we use (3.17b), and when $2^{-j} \geq |\lambda_1|^{-1}$ and $2^{-k} \geq |\lambda_2|^{-1}$ we use (3.17c). We will add over all j and k to obtain the desired estimates. The value of N will be determined by our arguments.

Taking absolute values in (3.16), integrating, and adding over all j and k with $2^{-j} \leq |\lambda_1|^{-1}$ and $2^{-k} \leq |\lambda_2|^{-1}$ leads to the desired estimate

$$C \int_{\{x \in U:\, |x_1| < |\lambda_1|^{-1},\, |x_2| < |\lambda_2|^{-1}\}} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.18}$$

We next add over all (j, k) such that $2^{-j} > |\lambda_1|^{-1}$ and $2^{-k} < |\lambda_2|^{-1}$. For a given k, we add estimates (3.17a) in j. Let a denote the minimum v_1^i appearing in any of the terms $x_1^{v_1^i} x_2^{v_2^i}$ defining $f^*(x_1, x_2)$. Then $f^*(2x_1, x_2) \geq 2^a f^*(2x_1, x_2)$, and $(f^*(2x_1, x_2))^{-\rho} \leq 2^{-\rho a}(f^*(2x_1, x_2))^{-\rho}$. (If $\rho < 0$, we let a be the maximal v_1^i). Thus if N is large enough, the integrand in (3.17a) decreases by a factor of at least 4 each time j increases by 1 for fixed k. As a result, the integral decreases by a factor of at least 2 each time j increases by 1 for fixed k. Hence the sum of (3.17a) over j with $2^{-j} > |\lambda_1|^{-1}$ is bounded by a constant times what one gets in (3.17a) setting $|\lambda_1| = 2^{-j}$, namely

$$C \int_{\{x \in U:\, \frac{1}{2}|\lambda_1|^{-1} < |x_1| < |\lambda_1|^{-1},\, 2^{-k-1} < |x_2| < 2^{-k}\}} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.19}$$

We now add (3.18) over all k with $2^{-k} < |\lambda_2|^{-1}$, and we see that the sum of all terms with $2^{-j} > |\lambda_1|^{-1}$ and $2^{-k} < |\lambda_2|^{-1}$ is bounded by

$$C \int_{\{x \in U:\, \frac{1}{2}|\lambda_1|^{-1} < |x_1| < |\lambda_1|^{-1},\, |x_2| < |\lambda_2|^{-1}\}} |f^*(x_1, x_2)|^{-\rho}\, dx_1\, dx_2 \tag{3.20}$$

This too is bounded by the desired estimate (3.18), so we are done with the terms where $2^{-j} > |\lambda_1|^{-1}$ and $2^{-k} < |\lambda_2|^{-1}$. By symmetry, the same method gives this estimate for the terms where $2^{-j} < |\lambda_1|^{-1}$ and $2^{-k} > |\lambda_2|^{-1}$, using (3.17*b*) in place of (3.17*a*).

It remains to consider the terms where $2^{-j} \geq |\lambda_1|^{-1}$ and $2^{-k} \geq |\lambda_2|^{-1}$. If N is large enough, similar to the argument leading to (3.19), increasing j by 1 in the expression (3.17*c*) for a fixed k decreases the term by a factor of at least 2. Hence adding over all such j leads to a bound of a constant times the term where j is minimal, that is where 2^{-j} is within a factor of 2 of $|\lambda_1|^{-1}$. But this term is exactly (3.17*b*) with $2^{-j} = |\lambda_1|^{-1}$. Hence adding these over all k with $2^{-k} \geq |\lambda_2|^{-1}$ once again leads to the desired bound (3.18). This completes the proof of Theorem 2.3.

References.

[AGV] V. Arnold, S. Gusein-Zade, A. Varchenko, *Singularities of differentiable maps*, Volume II, Birkhauser, Basel, 1988.

[G1] M. Greenblatt *Newton polygons and local integrability of negative powers of smooth functions in the plane*, Trans. Amer. Math. Soc., **358** (2006), 657-670.

[G2] M. Greenblatt, *Oscillatory integral decay, sublevel set growth, and the Newton polyhedron*, Math. Annalen **346** (2010), no. 4, 857-895.

[G3] M. Greenblatt, *Singular integral operators with kernels associated to negative powers of real analytic functions*, J. Funct. Anal. **269** (2015), no. 11, 3663-3687.

[G4] M. Greenblatt, *Convolution kernels of 2D Fourier multipliers based on real analytic functions*, submitted.

[G5] M. Greenblatt, *Uniform bounds for Fourier transforms of surface measures in* $\mathbf{R}^3$ *with nonsmooth density*, to appear, Trans. AMS.

[V] A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Functional Anal. Appl. **18** (1976), no. 3, 175-196.