Singular Integral Operators with Kernels Associated to Negative Powers of Real-Analytic Functions

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1. Introduction and theorems in the multiplicity one case.

Let $n \geq 2$ and let $b(x)$ be a real-analytic function on a neighborhood of the origin in $\mathbf{R}^n$ with $b(0) = 0$. By resolution of singularities, there is a number $\delta_0 > 0$ such that on any sufficiently small neighborhood U of the origin, $\int_U |f|^{-\delta} = \infty$ for $\delta \geq \delta_0$, and $\int_U |f|^{-\delta} < \infty$ for $\delta < \delta_0$. The number δ_0 is sometimes referred to as the "critical integrability exponent" of f at the origin. In this paper, we consider operators of the form

$$Tf(x) = \int_{\mathbf{R}^n} f(x-y)\,\alpha(x,y)\,m(y)\,|b(y)|^{-\delta_0}\,dy \tag{1.1}$$

Here $\alpha(x,y)$ is a Schwartz function, and $m(y)$ is a bounded real-valued function on a neighborhood of the origin such that $m(y)|b(y)|^{-\delta_0}$ satisfies natural derivative and cancellation conditions deriving from $b(y)$ that allows T to be considered as a type of singular integral operator. The focus of this paper will be to determine the boundedness properties of such T on L^p spaces for $1 < p < \infty$. Most of our results will concern the L^2 situation. As we will see, the operators we will consider will generalize local singular integral operators such as local versions of Riesz transforms, and also classes of local multiparameter singular integrals.

We will see that some of our proofs immediately extend to analogues of singular Radon transforms for such singular integral operators. Namely, our results will cover some operators of the following form, where $x \in \mathbf{R}^m$ and h is a real-analytic map from a neighborhood of the origin in $\mathbf{R}^n$ into $\mathbf{R}^m$ with $h(0) = 0$.

$$T'f(x) = \int_{\mathbf{R}^n} f(x-h(y))\,\alpha(x,y)\,m(y)\,|b(y)|^{-\delta_0}\,dy \tag{1.2}$$

To help define what types of kernels we allow, we now delve into the resolution of singularities near the origin of a real-analytic function $b(x)$ with $b(0) = 0$. For this

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we use the resolution of singularities theorem of [G1], but other resolution of singularities theorems including Hironaka's famous work [H1]-[H2] can be used in similar ways.

By [G1], there is a neighborhood U of the origin such that there exist finitely many coordinate change maps $\{\beta_i(x)\}_{i=1}^M$ and finitely many vectors $\{(m_{i1},...,m_{in})\}_{i=1}^M$ of nonnegative integers such that if $\rho(x)$ is a nonnegative smooth bump function supported in U with $\rho(0) \neq 0$, then $\rho(x)$ can be written in the form $\rho(x) = \sum_{i=1}^M \rho_i(x)$ in such a way that each $\rho_i \circ \beta_i(x)$, after an adjustment on a set of measure zero, is a smooth nonnegative bump function on a neighborhood of the origin with $\rho_i \circ \beta_i(0) \neq 0$. The components of each $\beta_i(x)$ are real-analytic. In addition, β_i is a bjiection from $\{x : \rho_i \circ \beta_i(x) \neq 0, x_i \neq 0 \text{ for all } i\}$ to $\{x : \rho_i(x) \neq 0\} - Z_i$ where Z_i has measure zero, and on a connected neighborhood U_i of the support of $\rho_i \circ \beta_i(x)$ the function $b \circ \beta_i(x)$ is well-defined and "comparable" to the monomial $x_1^{m_{i1}} ... x_n^{m_{in}}$, meaning that there is a real-analytic function $c_i(x)$ with $|c_i(x)| > \epsilon > 0$ on U_i such that $b \circ \beta_i(x) = c_i(x) x_1^{m_{i1}} ... x_n^{m_{in}}$ on U_i. This decomposition is such that the Jacobian determinant of $\beta_i(x)$ can be written in an analogous form $d_i(x) x_1^{e_{i1}} ... x_n^{e_{in}}$ on U_i; again the e_{ij} are integers and $|d_i(x)| > \epsilon > 0$ on U_i.

In view of the above, one has

$$\int_{\mathbf{R}^n} |b(x)|^{-\delta} \rho(x)\, dx = \sum_{i=1}^M \int_{\mathbf{R}^n} |b(x)|^{-\delta} \rho_i(x)\, dx$$

$$= \sum_{i=1}^M \int_{\mathbf{R}^n} |b \circ \beta_i(x)|^{-\delta} (\rho_i \circ \beta_i(x)) |d_i(x) x_1^{e_{i1}} ... x_n^{e_{in}}|\, dx \tag{1.3}$$

$$= \sum_{i=1}^M \int_{\mathbf{R}^n} |c_i(x)|^{-\delta} |d_i(x) x_1^{-\delta m_{i1} + e_{i1}} ... x_n^{-\delta m_{in} + e_{in}}| (\rho_i \circ \beta_i(x))\, dx \tag{1.4}$$

Since $\rho_i \circ \beta_i(0) \neq 0$, the ith term of the sum (1.4) is finite if each $-\delta m_{ij} + e_{ij} > -1$; that is, if $\delta < \frac{e_{ij}+1}{m_{ij}}$. Thus the number δ_0 is given in terms of the resolution of singularities of $b(x)$ by $\inf_{i,j} \frac{e_{ij}+1}{m_{ij}}$. The following notion plays a major role in this paper.

Definition 1.1. The *multiplicity* of the critical integrability exponent δ_0 of $b(x)$ at the origin is the maximum over all i of the cardinality of $\{j : \frac{e_{ij}+1}{m_{ij}} = \delta_0\}$.

One example of the significance of the multiplicity is as follows. Let B_r denote $\{x \in \mathbf{R}^n : |x| < r\}$ and let m denote the multiplicity of the exponent δ_0 for $b(x)$ at the origin. It can be shown (see chapter 7 of [AGV] for details) that if $r > 0$ is sufficiently small then as $\epsilon \to 0$ one has asymptotics of the form

$$|\{x \in B_r : |b(x)| < \epsilon\}| = c_r \epsilon^{\delta_0} (\ln \epsilon)^{m-1} + o(\epsilon^{\delta_0} (\ln \epsilon)^{m-1}) \tag{1.5}$$

Here $c_r > 0$. One obtains analogous asymptotics for various oscillatory integrals associated with $b(x)$. Note that (1.5) shows that the multiplicity is independent of the which resolution of singularities process is being used.

Singular integrals associated to negative powers of multiplicity one functions.

Let $b(x)$ be a real-analytic function on a neighborhood of the origin, not identically zero, with $b(0) = 0$. Let $\rho(x)$, $\{\beta_i(x)\}_{i=1}^M$, and $\{(m_{i1}, ..., m_{in})\}_{i=1}^M$ be as above. We will define singular integrals associated to $b(x)$ as follows. For a given i, we move into the "blown-up" coordinates determined by $\beta_i(x)$ and define a type of singular integral that is of magnitude bounded by $C|x_1^{m_{i1}} ... x_n^{m_{in}}|^{-\delta_0}$, with corresponding bounds on first derivatives, which is supported on the support of $\rho_i \circ \beta_i(x)$. An appropriate cancellation condition will be assumed that will ensure that the kernels are distributions. A singular integral associated to $b(x)$ will then be defined to be a sum from $i = 1$ to $i = M$ of the blow-downs of such singular integrals into the original coordinates.

Specifically, we consider $k_i(x) = \sum_{(j_1,...,j_n)\in \mathbf{Z}^n} k_{i,j_1,...,j_n}(x)$, where for some fixed C_0 the function $k_{i,j_1,...,j_n}(x)$ is C^1, supported on $\{x : |x_l| \in [2^{-j_l}, C_0 2^{-j_l}]$ for all $l\}$, and satisfies

$$|k_{i,j_1,...,j_n}(x_1, ..., x_n)| < C_0|x_1^{m_{i1}} ... x_n^{m_{in}}|^{-\delta_0} \tag{1.6}$$

We also assume that for each $l = 1, ..., n$ we have

$$|\partial_{x_l} k_{i,j_1,...,j_n}(x_1, ..., x_n)| < C_0 \frac{1}{|x_l|}|x_1^{m_{i1}} ... x_n^{m_{in}}|^{-\delta_0} \tag{1.7}$$

The cancellation condition we assume for the multiplicity one case is that for some $\epsilon_0 > 0$, whenever i and l are such that $\frac{e_{il}+1}{m_{il}} = \delta_0$ (the minimum possible value), then where $Jac_{\beta_i}(x)$ denotes the Jacobian determinant of β_i we have

$$\left| \int_{\mathbf{R}} k_{i,j_1,...,j_n}(x_1, ..., x_n)\, Jac_{\beta_i}(x_1, ... x_n)\, dx_l \right| < C_0 2^{-\epsilon_0 j_l} \tag{1.8}$$

To ensure that our singular integrals are well-defined, we also assume that the support of $k_{i,j_1,...,j_n}(x)$ is contained in that of $\rho_i \circ \beta_i(x)$. We next make the following definition.

Definition 1.2. If $b(x)$ has multiplicity one at the origin, we define a *singular integral kernel associated to* $b(x)$ to be a function $K(x)$ of the form $K(x) = \sum_{i=1}^M \rho_i(x)k_i(\beta_i^{-1}(x))$, where k_i satisfies (1.6) − (1.8) and the support condition stated afterwards.

One can simply explicitly construct $K(x)$ satisfying Definition 1.2 for any given $b(x)$ with multiplicity one at the origin, but a familiar example can be derived from local Riesz transforms:

Example. Let $L(x)$ be the local Riesz transform kernel given by $\phi(x)\frac{x_l}{|x|^{n+1}}$ for a cutoff function $\phi(x)$ supported near the origin. Then $L(x)$ satisfies Definition 1.2 for $b(x) = x_1^2 + ... + x_n^2$. Here $\delta_0 = \frac{n}{2}$. For one can write $L(x) = \sum_{i=1}^n L_i(x)$ where $L_i(x)$ is supported on a cone centered at the x_i-axis. Then if $\beta_i(x) = (x_ix_1, ..., x_ix_{i-1}, x_i, x_ix_{i+1}, ..., x_ix_n)$, the functions $L_i \circ \beta_i(x)$ will satisfy (1.6) − (1.8) with $x_1^{m_{i1}} ... x_n^{m_{in}} = x_i^{-n}$. Nonisotropic versions of $L(x)$ will satisfy Definition 1.2 for $b(x)$ of the form $x_1^{2k_1} + ... + x_n^{2k_n}$ for positive integers $k_1, ..., k_n$.

Each $K(x)$ satisfying Definition 1.2 can be viewed in a natural way as a distribution as follows. Let $k_{iL}(x)$ denote the truncated version of $k_i(x)$ given by

$$k_{iL}(x) = \sum_{(j_1,...,j_n)\in \mathbf{Z}^n:\, j_l < L \textit{ for all } l} k_{i,j_1,...,j_n}(x) \tag{1.9}$$

Define the corresponding truncated $K_L(x)$ by $K_L(x) = \sum_{i=1}^M K_{iL}(x)$, where $K_{iL}(x) = \rho_i(x) k_{iL}(\beta_i^{-1}(x))$. Note that $K_L(x)$ is a smooth compactly supported function. If $\phi(x)$ is a Schwartz function, then one has

$$\int_{\mathbf{R}^n} K_L(x)\,\phi(x)\,dx = \sum_{i=1}^M \int_{\mathbf{R}^n} \rho_i(x)\,k_{iL}(\beta_i^{-1}(x))\,\phi(x)\,dx$$

$$= \sum_{i=1}^M \int_{\mathbf{R}^n} (\rho_i \circ \beta_i(x))\,k_{iL}(x)\,Jac_{\beta_i}(x)\,(\phi \circ \beta_i(x))\,dx \tag{1.10}$$

$$= \sum_{i=1}^M \int_{\mathbf{R}^n} \sum_{(j_1,...,j_n):\, j_l < L \textit{ for all } l} (\rho_i \circ \beta_i(x))\,k_{i,j_1,...,j_n}(x)\,Jac_{\beta_i}(x)(\phi \circ \beta_i(x))\,dx \tag{1.11}$$

Let $\psi_i(x) = (\rho_i \circ \beta_i(x))(\phi \circ \beta_i(x))$. Then $\psi_i(x)$ is a smooth compactly supported function. Note that we do use the fact from [G1] that the function $\beta_i(x)$ is defined and smooth on a neighborhood of the support of $\rho_i \circ \beta_i(x)$ so that there are no issues concerning the smoothness of $\phi \circ \beta_i(x)$ on the boundary of the support of $\rho_i \circ \beta_i(x)$. Thus we can rewrite the expression (1.11) for $\int_{\mathbf{R}^n} K_L(x)\phi(x)\,dx$ as

$$= \sum_{i=1}^M \int_{\mathbf{R}^n} \sum_{(j_1,...,j_n):\, j_l < L \textit{ for all } l} k_{i,j_1,...,j_n}(x)\,Jac_{\beta_i}(x)\,\psi_i(x)\,dx \tag{1.12}$$

If i is such that $|x_1^{m_{i1}}...\,x_n^{m_{in}}|^{-\delta_0}|Jac_{\beta_i}(x)| \sim |x_1^{m_{i1}}...\,x_n^{m_{in}}|^{-\delta_0}|x_1^{e_{i1}}...\,x_n^{e_{in}}|$ is integrable on a neighborhood of the origin, then by (1.6), the form of the ith term of (1.12) ensures that the kernel $K_{iL}(x)$ is a distribution that converges as $L \to \infty$ to a finite measure which we denote by $K_i(x)$.

Next, we show that for the i for which $|x_1^{m_1}...\,x_n^{m_n}|^{-\delta_0}|Jac_{\beta_i}(x)|$ is not integrable, the cancellation condition (1.8) ensures that such an K_{iL} too converges as L goes to infinity in the distribution sense to some $K_i(x)$. We will then define $K(x) = \sum_{i=1}^M K_i(x)$. To see why this is the case, note that since $b(x)$ has multiplicity one, for each such i there is exactly one value l_0 for which $\frac{e_{il_0}+1}{m_{il_0}} = \delta_0$, and $\frac{e_{il}+1}{m_{il}} > \delta_0$ for all other values of l. Write $\psi_i(x) = \psi_i(x_1, ...\,x_{l_0-1}, 0, x_{l_0+1}, ..., x_n) + x_{l_0}\xi_i(x_1, ..., x_n)$, with ξ_i smooth.

The ith term of (1.12) can be written as the sum of two terms. In the first, $\psi_i(x)$ is replaced by $\psi_i(x_1, ... x_{l_0-1}, 0, x_{l_0+1}, ..., x_n)$ and in the second $\psi_i(x)$ is replaced by $\xi_i(x)$ and $k_{i,j_1,...,j_n}(x)$ by $x_{l_0} k_{i,j_1,...,j_n}(x)$. The second term is handled exactly as we handled the terms for which $|x_1^{m_1} ... x_n^{m_n}|^{-\delta_0} |Jac_{\beta_i}(x)|$ is integrable since the additional x_{l_0} factor causes us to once again have absolute integrability of the limiting kernel. As for the first term, we perform the x_{l_0} integration first in the ith term of (1.12). The cancellation condition (1.8) implies that the limiting kernel of the result of this integration is similarly absolutely integrable in the remaining $n-1$ variables, and thus the limit again defines a distribution.

Thus we see that $K_i(x)$ is a well defined distribution for all i and therefore $K(x) = \sum_{i=1}^M K_i(x)$ gives a well-defined distribution. Hence if $\alpha(x,y)$ is a Schwartz function on $\mathbf{R}^{n+m}$ and $f(x)$ is a Schwartz function on $\mathbf{R}^n$ then $Tf(x) = \int_{\mathbf{R}^n} f(x-y)\alpha(x,y)K(y)\,dy$ is well-defined. If for some $1 < p < \infty$ and some constant C the operators $T_L f(x) = \int_{\mathbf{R}^n} f(x-y)\alpha(x,y)K_L(y)\,dy$ are such that $||T_L||_{L^p \to L^p} \leq C$ for all Schwartz functions f and all L, then an application of the dominated convergence theorem gives that one also has $||Tf||_{L^p(\mathbf{R}^n)} \leq C||f||_{L^p(\mathbf{R}^n)}$ for all Schwartz functions, for the same constant C.

Our first theorem is simply that T is bounded on $L^2(\mathbf{R}^n)$.

Theorem 1.1. Whenever the critical integrability exponent of $b(x)$ at the origin has multiplicity one, then there is a neighborhood U of the origin such that if $K(x)$ is supported in U, there is a constant C such that for all Schwartz functions $f(x)$ one has $||Tf||_{L^2(\mathbf{R}^n)} \leq C||f||_{L^2(\mathbf{R}^n)}$.

It turns out that it is no harder prove L^2 boundedness for singular Radon transform generalizations of T. Namely, let $K(x)$ be as above, and let $h_1(x), ..., h_m(x)$ be real-analytic functions on a neighborhood of the origin in $\mathbf{R}^n$ with $h_i(x) = 0$ for all i. Let $\alpha(x,y)$ be a Schwartz function on $\mathbf{R}^m \times \mathbf{R}^n$. Then for a Schwartz function $f(x)$ in m variables, we define $T'f(x)$ by

$$T'f(x) = \int_{\mathbf{R}^n} f(x_1 - h_1(y), ..., x_m - h_m(y))\, \alpha(x,y)\, K(y)\, dy \tag{1.13}$$

The operator T above corresponds to $m = n$ and $h_i(y) = y_i$ for all i. We have the following theorem.

Theorem 1.2. Whenever the critical integrability exponent of $b(x)$ at the origin has multiplicity one, then there is a neighborhood U of the origin such that if $K(x)$ is supported in U, there is a constant C such that for all Schwartz functions $f(x)$, one has $||T'f||_{L^2(\mathbf{R}^m)} \leq C||f||_{L^2(\mathbf{R}^m)}$

To give a rough idea of how our proofs will work, note that (1.13) can be written as

$$\sum_{i=1}^M \int_{\mathbf{R}^n} f(x_1 - h_1(y), ..., x_m - h_m(y))\, \alpha(x,y)\, \rho_i(y)\, k_i(\beta_i^{-1}(y))\, dy \tag{1.14}$$

Let T_i be the operator corresponding to the ith term of (1.14). Doing a change of variables from y to $\beta_i(y)$ in the integral of (1.14) leads to

$$T_if(x) = \int_{\mathbf{R}^n} f(x_1 - h_1 \circ \beta_i(y), ..., x_m - h_m \circ \beta_i(y))\, \alpha(x, \beta_i(y))(\rho_i \circ \beta_i(y))k_i(y)\, Jac_{\beta_i}(y)\, dy \tag{1.15}$$

T_i is a sort of singular Radon transform with kernel $\alpha(x, \beta_i(y))\, (\rho_i \circ \beta_i(y))\, k_i(y)\, Jac_{\beta_i}(y)$, which we will be able to analyze by reducing to singular Radon transform estimates the author used in [G3].

2. Theorems when the multiplicity is greater than one.

When the critical integrability exponent δ_0 has multiplicity greater than one at the origin, the coordinate changes $\beta_i(x)$ used in the multiplicity one case will lead to trying to prove L^2 boundedness of an operator that resembles a multiparameter singular Radon transform, rather than a (one-parameter) singular Radon transform. Unfortunately since the $\beta_i(x)$ here involve blowups, one often ends out with a multiparameter singular Radon transform that is not bounded on L^2. As a result, instead of trying to find a general correct notion of singular integral and prove a general result, when the multiplicity is greater than one we will focus on theorems that can be proven in the original coordinates.

Newton polyhedra and related matters.

One can often determine the criticial integrability exponent of a function at the origin and its mutliplicity through the use of Newton polyhedron of the function. We turn to the relevant definitions.

Definition 2.1. Let $b(x)$ be a real-analytic function with Taylor series $\sum_\alpha b_\alpha x^\alpha$ on a neighborhood of the origin. For each α for which $b_\alpha \neq 0$, let Q_α be the octant $\{t \in \mathbf{R}^n : t_i \geq \alpha_i \text{ for all } i\}$. The *Newton polyhedron* $N(b)$ of $b(x)$ is defined to be the convex hull of all Q_α.

A Newton polyhedron can contain faces of various dimensions in various configurations. The faces can be either compact or unbounded. In this paper, as in earlier work such as [G2] and [V], an important role is played by the following functions, associated to each compact face of $N(b)$. We consider each vertex of $N(b)$ to be a compact face of dimension zero.

Definition 2.2. Let F be a compact face of $N(b)$. Then if $b(x) = \sum_\alpha b_\alpha x^\alpha$ denotes the Taylor expansion of b like above, we define $b_F(x) = \sum_{\alpha \in F} b_\alpha x^\alpha$.

We will also use the following terminology.

Definition 2.3. Assume $N(b)$ is nonempty. Then the *Newton distance* $d(b)$ of $b(x)$ is defined to be $\inf\{t : (t, t, ..., t, t) \in N(b)\}$.

Definition 2.4. The *central face* of $N(b)$ is the face of $N(b)$ of minimal dimension intersecting the line $t_1 = t_2 = ... = t_n$.

In Definition 2.4, the central face of $N(b)$ is well-defined since it is the intersection of all faces of $N(b)$ intersecting the line $t_1 = t_2 = ... = t_n$. An equivalent definition that can be used (such as in [AGV]) is that the central face of $N(b)$ is the unique face of $N(b)$ that intersects the line $t_1 = t_2 = ... = t_n$ in its interior.

Extending results of [V], in [G2] the author showed that if the zeros of each $b_F(x)$ on $(\mathbf{R} - \{0\})^n$ are of order less than $d(b)$, then the critical integrability index δ_0 is equal to $\frac{1}{d(b)}$ and the multiplicity is equal to n minus the dimension of the central face of $N(b)$. This can be used to compute δ_0 and its multiplicity for specific examples of interest, such as in the following two examples (which are covered by [V]). Suppose $b(x) = x_1^{k_1} + ... + x_n^{k_n}$ with each k_i even. Then $\delta_0 = (\frac{1}{k_1} + ... + \frac{1}{k_n})^{-1}$ and $m = 1$. On the other hand, if $b(x) = x_1^{l_1}...x_n^{l_n}$ then $\delta_0 = \frac{1}{\max_i l_i}$ and m is equal to the number of times k that $\max_i l_i$ appears in $\{l_1, ..., l_n\}$. For in the former case the line $t_1 = ... = t_n$ intersects $N(b)$ in the interior of the $n-1$ dimensional face with equation $\frac{t_1}{k_1} + ... + \frac{t_n}{k_n} = 1$, while in the latter case the line $t_1 = ... = t_n$ intersects $N(b)$ in the $n-k$ dimensional plane determined by the equations $t_l = \max_i l_i$ for all l such that $l = \max_i l_i$.

The function $b^*(x)$

In order to understand the behavior of functions satisfying the finite-type condition of [G2], it is often helpful to consider the function $b^*(x)$ defined by

$$b^*(x) = \sum_{(v_1,...,v_n)\ a\ vertex\ of\ N(b)} |x_1^{v_1}...x_n^{v_n}| \tag{2.1}$$

By Lemma 2.1 of [G2], there is a constant C such that for all x one has $|b(x)| \leq Cb^*(x)$. In Lemma 4.1 of this paper we will see that given any $\delta > 0$ there is a $\delta' > 0$ such that $|b(x)| > \delta' b^*(x)$ on a portion of any dyadic rectangle with measure at least $1-\delta$ times that of the rectangle. Hence $|b(x)| \sim b^*(x)$ except near the zeroes of $|b(x)|$.

Next, observe that the Newton polygon of any first partial $\partial_{x_l} b(x)$ is a subset of the shift of $N(b)$ by -1 units in the x_l direction. Hence the above considerations tell us that

$$|\partial_{x_l} b(x)| \leq C\frac{1}{|x_l|}b^*(x) \tag{2.2a}$$

If $b(x) \neq 0$, we also have

$$|\partial_{x_l}(|b(x)|^{-\delta_0})| \leq C\frac{1}{|x_l|}b^*(x)|b(x)|^{-1-\delta_0} \tag{2.2b}$$

Singular integrals when the multiplicity is greater than one.

When the multiplicity is greater than one, the class of $b(x)$ where we will prove L^2 boundedness of associated singular integrals are the $b(x)$ analyzed in [G2] that were discussed above (2.1). Namely, using the terminology of Definitions 2.1-2.4, we will assume that for each compact face F of $N(b)$, each zero of each $b_F(x)$ in $(\mathbf{R}-\{0\})^n$ has order less than $d(b)$. As mentioned above, by $[G2]$ in this situation we have $\delta_0 = \frac{1}{d(b)}$. Note that our theorems do not require the multiplicity to be greater than one, and in fact the theorems here will include some multiplicity one operators not covered by Theorem 1.1.

In the situation at hand, we define a singular integral associated to $b(x)$ as follows. Let $\alpha(x,y)$ be a Schwartz function on $\mathbf{R}^{n+n}$. We will consider kernels of the form $\alpha(x,y)K(y)$, where $K(y)$ is as follows. We assume that $K(y)$ can be written as $\sum_{(j_1,...,j_n)\in\mathbf{Z}^n} K_{j_1,...,j_n}(y)$ such that for some fixed C_1 the function $K_{j_1,...,j_n}(y)$ is supported on $\{x : |x_l| \in [2^{-j_l}, C_1 2^{-j_l}]$ for all $l\}$, is C^1 on $\{x : b(x) \neq 0\}$, and satisfies the estimates

$$|K_{j_1,...,j_n}(y_1,...,y_n)| < C_1|b(y_1,...,y_n)|^{-\delta_0} \tag{2.3}$$

Motivated by (2.2b), we also assume that if $b(y_1,...,y_n) \neq 0$ then for each l we have

$$|\partial_{y_l} K_{j_1,...,j_n}(y_1,...,y_n)| < C_1 \frac{1}{|y_l|} b^*(y_1,...,y_n)|b(y_1,...,y_n)|^{-1-\delta_0} \tag{2.4}$$

We further assume the cancellation conditions that for each l we have

$$\int_{\mathbf{R}} K_{j_1,...,j_n}(y_1,...,y_n)\, dy_l = 0 \tag{2.5}$$

In Lemma 4.2, we will see that in the settings of our theorems (Theorems 2.1 and 2.2) each $K_{j_1,...,j_n}(y_1,...,y_n)$ is integrable, so (2.5) makes sense if we assume it holds whenever $K_{j_1,...,j_n}(y_1,...,y_n)$ is integrable in the y_l variable for fixed values of the other y variables.

We will also assume that $K_{j_1,...,j_n}(y_1,...,y_n)$ is identically zero when $[2^{-j_1}, C_1 2^{-j_1}]\times ...$ $\times[2^{-j_n}, C_1 2^{-j_n}]$ is not contained in a certain neighborhood of the origin to be determined by our arguments.

Some motivation for our definition of a singular integral associated to $b(x)$ is the fact that for traditional multiparameter singular integrals, often a sufficient and necessary condition for L^p boundedness is that the kernel be expressible as a dyadic sum of terms satisfying standardized estimates as well as a cancellation condition. We refer to Theorem 2 of Lecture 2 of [N] for an example of a theorem of this nature. We also refer to the standard references [FS] and [NS] for more general information about multiparameter singular integrals.

Example 1. Let $b(x) = x_1^{a_1}...x_n^{a_n}$ for nonnegative integers $a_1,...,a_n$ with at least one a_i being nonzero. Then $\delta_0 = \frac{1}{\max_i a_i}$ here. If $\phi(x)$ is a cutoff function supported on a sufficiently small neighborhood of the origin, $K(x) = (-1)^{sgn(x_1)+...+sgn(x_n)}\phi(x_1^2,...,x_n^2)|b(x)|^{-\delta_0}$ will satisfy (2.3) − (2.5). In particular, $K(x) = (-1)^{sgn(x_1)+...+sgn(x_n)}\phi(x_1^2,...,x_n^2)\frac{1}{|x_1...\,x_n|}$ qualifies.

Example 2. Let $f(x_1,...,x_n)$ be any real-analytic function with $f(0,...,0) = 0$, and let $b(x) = f(x_1^2,...,x_n^2)$. Then if $\phi(x)$ is a cutoff function supported on a sufficiently small neighborhood of the origin, $K(x) = (-1)^{sgn(x_1)+...+sgn(x_n)}\phi(x_1^2,...,x_n^2)|b(x)|^{-\delta_0}$ will satisfy (2.3) − (2.5).

For a fixed value of x, the function $\alpha(x,y)K(y)$ can be viewed in a natural way as a distribution in the y variable as follows. This will resemble the discussion following (1.9). Let $K_L(y) = \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y)$ and let $\phi(y)$ be a Schwartz function. Then

$$\int_{\mathbf{R}^n} \alpha(x,y)\,K_L(y)\,\phi(y)\,dy = \int_{\mathbf{R}^n} \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y)\,\alpha(x,y)\,\phi(y)\,dy_1...dy_n \tag{2.6}$$

Let $\sigma_x(y) = \alpha(x,y)\,\phi(y)$. Then we may write $\sigma_x(y) = \sigma_x(0,y_2,...,y_n) + y_1\xi_x(y_1,...,y_n)$ for some smooth $\xi_x(y_1,...,y_n)$. Then the right-hand side of (2.6) can be rewritten as

$$\int_{\mathbf{R}^n} \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y_1,...,y_n)\,\sigma_x(0,y_2,...,y_n)\,dy_1...dy_n$$

$$+\int_{\mathbf{R}^n} \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y_1,...,y_n)\,y_1\,\xi_x(y_1,...,y_n)\,dy_1...dy_n \tag{2.7}$$

Because of the cancellation condition (2.5) in the y_1 variable, the first integral of (2.7) is zero. We next similarly write $\xi_x(y_1,...,y_n) = \xi_x(y_1,0,y_3,...,y_n) + y_2\tilde{\xi}_x(y_1,...,y_n)$ and insert it into (2.7), obtaining

$$\int_{\mathbf{R}^n} \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y_1,...,y_n)\,y_1y_2\,\tilde{\xi}_x(y_1,...,y_n)\,dy_1...dy_n \tag{2.8}$$

Going through all the y_l variables in this way, we see that $\int_{\mathbf{R}^n} \alpha(x,y)\,K_L(y)\,\phi(y)\,dy$ is equal to an expression

$$\int_{\mathbf{R}^n} \sum_{j_l<L\ for\ all\ l} K_{j_1,...,j_n}(y_1,...,y_n)\,y_1y_2...y_n\,\eta_x(y_1,...,y_n)\,dy_1...dy_n \tag{2.9}$$

Here $\eta_x(y_1,...,y_n)$ is smooth in both the x and y variables. We will see in Lemma 4.2 that the condition on the order of the zeroes of the functions $b_F(x)$ on $(\mathbf{R}-\{0\})^n$ implies that the integral of $|b(x)|^{-\delta_0}$ over any dyadic rectangle in U is uniformly bounded. Thus (2.3) implies that $K_{j_1,...,j_n}(y)\,y_1y_2...y_n$ is absolutely integrable over U. Hence $\alpha(x,y)K(y)$ is naturally a distribution in y when $K(y)$ is supported in U if its action on $\phi(y)$ is given by

$$\langle\alpha(x,y)K(y),\phi(y)\rangle = \int_{\mathbf{R}^n} \sum_{(j_1,...,j_n)\in\mathbf{Z}^n} K_{j_1,...,j_n}(y_1,...,y_n)\,y_1y_2...y_n\,\eta_x(y_1,...,y_n)\,dy_1...dy_n \tag{2.10}$$

One can then use (2.10) to define $Tf(x) = \int_{\mathbf{R}^n} f(x-y)\alpha(x,y)K(y)\,dy$ for Schwartz functions f, and then examine boundedness of such integral operators on L^p spaces. We have the following theorem in this regard for $p = 2$.

Theorem 2.1. Suppose each polynomial $b_F(x)$ only has zeroes of order less than $d(b)$ on $(\mathbf{R} - \{0\})^n$. Suppose also that there is a $C_0 > 0$ and a neighborhood U of the origin such that for each l, there is a set $Z \subset \mathbf{R}^{n-1}$ of measure zero such that the function $\partial_{x_l} b(x)$ has at most C_0 zeroes in U on any line parallel to the x_l coordinate axis whose projection onto the plane $x_l = 0$ is not in Z. Then there is an $R > 0$ such that if each $K_{j_1,...,j_n}(y)$ satisfies (2.3) $-$ (2.5) and is supported on $|y| < R$, then there is a constant C such that $||Tf||_{L^2(\mathbf{R}^n)} \leq C||f||_{L^2(\mathbf{R}^n)}$ for all Schwartz functions $f(x)$.

The condition concerning zeroes on lines parallel to the coordinates axes is needed for technical reasons in the proof. Note that this condition holds whenever $b(x)$ is a polynomial, and it is not hard to see that it always holds in two variables, using the Weierstrass Preparation Theorem for example. The author does not know if it holds for all real-analytic functions, so it is included as an assumption in Theorem 2.1 (and in Theorem 2.2 below).

For $p \neq 2$, we have a weaker statement. To motivate the statement of the theorem, in (4.2) and the line afterwards we will see that if each polynomial $b_F(x)$ is nonvanishing on $(\mathbf{R} - \{0\})^n$, then there are constants C_1 and C_2 such that

$$C_1 b^*(x) < |b(x)| < C_2 b^*(x) \tag{2.11}$$

Hence in this situation (2.4) becomes

$$|\partial_{y_l} K_{j_1,...,j_n}(y_1,...,y_n)| < C_1' \frac{1}{|y_l|}(b^*(y_1,...,y_n))^{-\delta_0} \tag{2.12}$$

For the L^p theorem, we need bounds on derivatives of higher order in order to apply the Marcinkiewicz multiplier theorem. Hence we assume that each $K_{j_1,...,j_n}(y_1,...,y_n)$ is a C^{n+1} function and there is a constant C such that for any multiindex α with $0 \leq |\alpha| \leq n+1$ we have

$$|\partial^\alpha K_{j_1,...,j_n}(y_1,...,y_n)| \leq C \frac{1}{|y_1|^{\alpha_1}...|y_n|^{\alpha_n}}(b^*(y_1,...,y_n))^{-\delta_0} \tag{2.13}$$

The condition (2.13) is motivated by the fact that by iterating (2.2a), the bounds (2.13) hold for $|b(x)|^{-\delta_0}$ whenever each $b_F(x)$ is nonvanishing on $(\mathbf{R} - \{0\})^n$.

Our L^p theorem is as follows.

Theorem 2.2. Suppose each polynomial $b_F(x)$ is nonvanishing on $(\mathbf{R} - \{0\})^n$. Suppose also that there is a $C_0 > 0$ and a neighborhood U of the origin such that for each l, there is a set $Z \subset \mathbf{R}^{n-1}$ of measure zero such that the function $\partial_{x_l} b(x)$ has at most C_0 zeroes in U on any line parallel to the x_l coordinate axis whose projection onto the plane $x_l = 0$ is not in Z. Then there is an $R > 0$ such that if each $K_{j_1,...,j_n}(y)$ satisfies $(2.3), (2.13), (2.5)$ and is supported on $|y| < R$, then if $1 < p < \infty$ there is a constant C such that $||Tf||_{L^p(\mathbf{R}^n)} \leq C||f||_{L^p(\mathbf{R}^n)}$ for all Schwartz functions $f(x)$.

Going back to the examples preceding (2.6), in the first example where $b(x) = x_1^{a_1}...x_n^{a_n}$, each kernel $K(x) = (-1)^{sgn(x_1)+...+sgn(x_n)}\phi(x_1^2,...,x_n^2)|b(x)|^{-\delta_0}$ will be covered by Theorems 2.1 and 2.2. As for the second example where $b(x) = f(x_1^2,...,x_n^2)$, the maximum order of any zero of any $b_F(x)$ on $(\mathbf{R}-\{0\})^n$ is the same as the maximum order of any $f_F(x)$ on $(\mathbf{R}^+)^n$. So when this quantity is less than $d(b) = 2d(f)$, $K(x)$ will fall under the conditions of Theorem 2.1. When each $f_F(x)$ is nonvanishing on $(\mathbf{R}^+)^n$, then $K(x)$ will fall under the conditions of Theorem 2.2 as well.

3. Proofs of theorems when the multiplicity is equal to one.

Since Theorem 1.1 is a special case of Theorem 1.2, we prove Theorem 1.2.

Proof of Theorem 1.2.

Theorem 1.2 will follow if we can prove each that for each i there is a constant C such that $||T_if||_{L^2(\mathbf{R}^m)} \leq C||f||_{L^2(\mathbf{R}^m)}$ for all Schwartz functions $f(x)$. Here T_i is as in (1.15). Let m_{ij} and e_{ij} be the monomial exponents of $b \circ \beta_i(x)$ and $Jac_{\beta_i}(x)$ as before. If i is such that $\delta_0 < \frac{1+e_{ij}}{m_{ij}}$ for each j, then the kernel of T_i is absolutely integrable and L^2 boundedness is immediate. Thus it suffices to consider only the i for which there is some l for which $\delta_0 = \frac{1+e_{il}}{m_{il}}$. Since we are assuming $b(x)$ has multiplicity one, there will only be one such l for each such i. Writing $h \circ \beta_i(y) = (h_1 \circ \beta_i(y), ..., h_m \circ \beta_i(y))$, (1.15) can be rewritten as

$$T_if(x) = \int_{\mathbf{R}^n} f(x - h \circ \beta_i(y))\, \alpha(x, \beta_i(y))\, (\rho_i \circ \beta_i(y))\, k_i(y)\, Jac_{\beta_i}(y)\, dy \tag{3.1}$$

It is more convenient for our proofs that there be no nonzero λ such that $\lambda \cdot (h \circ \beta_i(y))$ is a linear function of y. This can be accomplished as follows. If there is a nonzero λ such that $\lambda \cdot (h \circ \beta_i(y))$ is the zero function, then on each hyperplane orthogonal to λ, the operator T_i restricts to an operator of the same type as T_i here, except the ambient space is of one lower dimension. Repeating as necessary, we may assume that $\lambda \cdot (h \circ \beta_i(y))$ is never the zero function for any nonzero λ. It is worth mentioning that we are using the fact that $h \circ \beta_i(y)$ extends to a connected neighborhood of the support of $\rho_i \circ \beta_i(y)$ to ensure we don't have different functions on different connected components to worry about.

One can further ensure that $\lambda \cdot (h \circ \beta_i(y))$ is never linear by letting $(y_1, ..., y_n) = (z_1^3, ..., z_n^3)$ with the corresponding change from $\beta_i(y_1, ..., y_n)$ to $\tilde{\beta}_i(z) = \beta_i(z_1^3, ..., z_n^3)$. The exponents m_{ij} and e_{ij} can change, but $(1.6)-(1.8)$ and the other properties from resolution of singularities that we are using will still hold. Thus in the following, without loss of generality we will always assume that we are working in a situation where $\lambda \cdot (h \circ \beta_i(y))$ is not linear for any nonzero λ.

Let $k_{iL}(x) = \sum_{(j_1,...,j_n):j_l<L \; for \; all \; l} \; k_{i,j_1,...,j_n}(x)$ as in (1.9), and let T_{iL} be defined by

$$T_{iL}f(x) = \int_{\mathbf{R}^n} f(x - h\circ\beta_i(y))\,\alpha(x,\beta_i(y))\,(\rho_i\circ\beta_i(y))\,k_{iL}(y)\,Jac_{\beta_i}(y)\,dy \tag{3.2}$$

In order to prove Theorem 1.2, it suffices to show that T_{iL} is bounded on L^2 with bounds uniform in L. We can reduce to the case where $(\rho_i\circ\beta_i(y))\alpha(x,\beta_i(y))$ is replaced by a function of y (i.e. the operator is translation-invariant) through the following lemma.

Lemma 3.1. Define U_{iL} by

$$U_{iL}f(x) = \int_{\mathbf{R}^n} f(x - h\circ\beta_i(y))\,k_{iL}(y)\,Jac_{\beta_i}(y)\,dy \tag{3.3}$$

If there is a constant C depending on $b(x)$ (and the resolution of singularities procedure we are using on it), $h_1(x),...,h_m(x)$, and the constant C_0 of $(1.6)-(1.8)$, such that $||U_{iL}f||_p \leq C||f||_p$ for all Schwartz f and all L, then there is a constant C' such that $||T_{iL}f||_p \leq C'||f||_p$ for all Schwartz f and all L.

Proof. Let $\gamma(x,y)$ be the Schwartz function $(\rho_i\circ\beta_i(y))\,\alpha(x,\beta_i(y))$. We use the Fourier inversion formula in the x variable and write

$$\gamma(x,y) = \int_{\mathbf{R}^m} \hat{\gamma}(t,y)\,e^{it_1x_1+...+it_mx_m}\,dt \tag{3.4}$$

Here $\hat{\gamma}(t,y)$ refers to the Fourier transform in the x variable only. If f and g are Schwartz functions then $\int_{\mathbf{R}^n} T_{iL}f(x)g(x)\,dx$ is equal to

$$\int_{\mathbf{R}^m}\left[\int_{\mathbf{R}^m}\left(\int_{\mathbf{R}^n} f(x-h\circ\beta_i(y))\,\hat{\gamma}(t,y)\,k_{iL}(y)\;\;Jac_{\beta_i}(y)\,dy\right)\right.$$
$$\left.\times(e^{it_1x_1+...+it_mx_m}g(x))\,dx\right]dt \tag{3.5}$$

Stated another way, let U_{iLt} denote the operator

$$U_{iLt}f(x) = \int_{\mathbf{R}^n} f(x - h\circ\beta_i(y))\,\hat{\gamma}(t,y)\,k_{iL}(y)\,Jac_{\beta_i}(y)\,dy \tag{3.6}$$

Then $\int_{\mathbf{R}^n} T_{iL}f(x)g(x)\,dx$ is equal to

$$\int_{\mathbf{R}^m}\int_{\mathbf{R}^m} U_{iLt}f(x)\times(e^{it_1x_1+...+it_mx_m}g(x))\,dx\,dt \tag{3.7}$$

Since $\hat{\gamma}(t,y)$ is Schwartz, under the assumptions of this lemma there is a constant K such that

$$||U_{iLt}||_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \leq K\frac{1}{1+|t|^{m+1}} \tag{3.8}$$

Thus by (3.7) and Hölder's inequality we have

$$\left|\int_{\mathbf{R}^n} T_{iL}f(x)g(x)\,dx\right| \leq K||f||_p||g||_{p'}\int_{\mathbf{R}^m}\frac{1}{1+|t|^{m+1}}\,dt \tag{3.9}$$

Thus the T_{iL} are bounded on L^p uniformly in L and we are done with the proof of Lemma 3.1.

We now proceed to proving uniform bounds on the U_{iL}. Taking Fourier transforms, we get

$$\widehat{U_{iL}}(\lambda) = \hat{f}(\lambda)\int_{\mathbf{R}^n} e^{i\lambda\cdot(h\circ\beta_i(y))}\,k_{iL}(y)\,Jac_{\beta_i}(y)\,dy \tag{3.10}$$

Hence in order to prove Theorem 1.2, it suffices to show that there is a constant C such that $|B_{iL}(\lambda)| \leq C$ for each i, L and λ, where

$$B_{iL}(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda\cdot(h\circ\beta_i(y))}\,k_{iL}(y)\,Jac_{\beta_i}(y)\,dy \tag{3.11}$$

Without loss of generality, to simplify notation in the following we will assume that the l for which $\delta_0 = \frac{1+e_{il}}{m_{il}}$ is $l=1$. Next, we write the factor $k_{iL}(y)Jac_{\beta_i}(y)$ in (3.11) as $\sum_{m<L} p_{imL}(y)$, where we add over the dyadic pieces in the $y_2,...,y_n$ variables to form $p_{imL}(y)$:

$$p_{imL}(y) = \sum_{(j_2,...,j_n):j_l<L\ for\ all\ l>1} k_{i,m,j_2,...,j_n}(y)\,Jac_{\beta_i}(y) \tag{3.12}$$

Then (1.6) implies the estimates

$$|p_{imL}(y)| \leq C|y_1|^{-\delta_0 m_{i1}+e_{i1}}...|y_n|^{-\delta_0 m_{in}+e_{in}} \tag{3.13}$$

Similarly, (1.6) – (1.7) implies that for each l we have

$$|\partial_{y_l}\,p_{imL}(y)| \leq C\frac{1}{|y_l|}|y_1|^{-\delta_0 m_{i1}+e_{i1}}...|y_n|^{-\delta_0 m_{in}+e_{in}} \tag{3.14}$$

The cancellation condition (1.8) gives

$$\left|\int_{\mathbf{R}} p_{imL}(y_1,...,y_n)\,dy_1\right| < C2^{-\epsilon_0 m} \tag{3.15}$$

Note that $-\delta_0 m_{i1}+e_{i1} = -1$ here, while the other exponents are all greater than -1. If one changes variables $y_l = z_l^N$ in (3.11) for some $l>1$, instead of having a factor $|y_l|^{-\delta_0 m_{il}+e_{il}}$ in (3.13) – (3.14) one has a factor of $|z_l|^{(-\delta_0 m_{il}+e_{il})N}$. One also gains an additional factor of $N|z_l|^{N-1}$ from the Jacobian of the coordinate change. Thus overall one has a factor of $|z_l|^{(-\delta_0 m_{il}+e_{il}+1)N-1}$. Since $-\delta_0 m_{il}+e_{il} > -1$, if N is large enough this factor will be bounded by just $|z_l|$ and thus one can remove the z_l variable from (3.13) – (3.14). Stated another way, if one changes $y_l = z_l^N$ for all $l > 1$ (making N odd to ensure it's a one-to-one map), (3.11) becomes

$$B_{iL}(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda\cdot(h\circ\beta_i)(z_1,z_2^N,...,z_n^N)}\, q_{iL}(z)\,dz \tag{3.16}$$

Here $q_{iL}(z) = \sum_{m<L} q_{imL}(z)$, where $q_{imL}(z)$ satisfies

$$|q_{imL}(z)| \leq C|z_1|^{-1} \tag{3.17}$$

$$|\partial_{z_1} q_{imL}(z)| \leq C|z_1|^{-2}, \qquad \forall l > 1 \;\; |\partial_{z_l} q_{imL}(z)| \leq C|z_1|^{-1} \tag{3.18}$$

$$\left|\int_{\mathbf{R}} q_{imL}(z_1, ..., z_n)\,dz_1\right| \leq C2^{-\epsilon_0 m} \tag{3.19}$$

Letting $f_i(z) = h\circ\beta_i(z_1, z_2^N, ..., z_n^N)$, (3.16) becomes

$$B_{iL}(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda\cdot f_i(z)} q_{iL}(z)\,dz \tag{3.20}$$

In view of the discussion above (3.2), we may assume that $\lambda\cdot f_i(z)$ is not linear for any nonzero λ. Therefore, writing $\lambda = |\lambda|\omega$ for $\omega \in S^{m-1}$, by a compactness argument on $S^{m-1}\times supp(q_{iL})$, we may restrict consideration to ω to a small neighborhood N in S^{m-1} and replace $q_{iL}(x)$ by $\sigma(x)q_{iL}(x)$ for a function $\sigma(x)$ supported on a ball $B(x_0, r_0)$ on which there is an $\epsilon > 0$ and a single directional derivative ∂_v such that $|\partial_v^\alpha(\omega\cdot f_i)(z)| > \epsilon$ on $B(x_0, r_0)$ for some $\alpha \geq 2$. We can do this in such a way that v has a positive x_1 component. Thus if $\bar{x}$ denotes the first component of x_0, for $\omega \in N$ we are attempting to bound $\int_{\mathbf{R}^{n-1}} D_{iL}(\lambda, z_2, ..., z_n)\,dz_2...dz_n$, where

$$D_{iL}(\lambda, z_2, ..., z_n) = \int_{\mathbf{R}} e^{i|\lambda|(\omega\cdot f_i)((\bar{x},z_2,...,z_n)+tv)}\sigma((\bar{x}, z_2, ..., z_n) + tv) \times q_{iL}((\bar{x}, z_2, ..., z_n) + tv)\,dt \tag{3.21a}$$

Thus to prove Theorem 1.2, it suffices for our purposes to bound $D_{iL}(\lambda, z_2, ..., z_n)$ uniformly in $L, \lambda, z_2, ..., z_n$ for $\omega \in N$. For this, it suffices to bound $\tilde{D}_{iL}(\lambda, \tilde{\lambda}, z_2, ..., z_n)$ uniformly in $L, \lambda, \tilde{\lambda}, z_2, ..., z_n$ for $\omega \in N$, where

$$\tilde{D}_{iL}(\lambda, \tilde{\lambda}, z_2, ..., z_n) = \int_{\mathbf{R}} e^{i|\lambda|(\omega\cdot f_i)((\bar{x},z_2,...,z_n)+tv)+i\tilde{\lambda}t}\sigma((\bar{x}, z_2, ..., z_n) + tv) \times q_{iL}((\bar{x}, z_2, ..., z_n) + tv)\,dt \tag{3.21b}$$

But similarly to (3.10) – (3.11), such uniform bounds for $\tilde{D}_{iL}(\lambda, \tilde{\lambda}, z_2, ..., z_n)$ follows from uniform boundedness on L^2 of the singular Radon transforms along curves in $\mathbf{R}^2$ of the form

$$U_{i\,L\,\omega\,z_2...z_n} f(x_1, x_2) = \int_{\mathbf{R}} f\big(x_1 - t, x_2 - (\omega\cdot f_i)((\bar{x}, z_2, ..., z_n) + tv)\big)$$

$$\times\sigma((\bar{x}, z_2, ..., z_n) + tv)\, q_{iL}((\bar{x}, z_2, ..., z_n) + tv)\, dt \tag{3.22}$$

Note that

$$|q_{imL}((\bar{x}, z_2, ..., z_n) + tv) - q_{imL}((\bar{x}, z_2, ..., z_n) + t(v_1, 0, ..., 0))| \leq C|t| \max_{l>1} \sup_z |\partial_{z_l} q_{imL}(z)| \tag{3.23}$$

By (3.18), since $|t| < C2^{-m}$ and $|z_1| \sim 2^{-m}$ we have

$$|q_{imL}((\bar{x}, z_2, ..., z_n) + tv) - q_{imL}\big((\bar{x}, z_2, ..., z_n) + t(v_1, 0..., 0)\big)| \leq C' \tag{3.24}$$

Hence by the cancellation condition (3.19) one has

$$\int_{\mathbf{R}} q_{imL}((\bar{x}, z_2, ..., z_n) + tv)\, dt < C''2^{-\epsilon_0 m} \tag{3.25}$$

Since $\sigma((\bar{x}, z_2, ..., z_n)) + tv) = \sigma(\bar{x}, z_2, ..., z_n) + O(|t|)$, using (3.17) one also has

$$\int_{\mathbf{R}} \sigma((\bar{x}, z_2, ..., z_n) + tv)\, q_{imL}((\bar{x}, z_2, ..., z_n) + tv)\, dt < C'''2^{-\epsilon_0 m} \tag{3.26}$$

In other words, we have a cancellation condition in (3.22) derived from (3.19). The constant C''' in (3.26) depends on $b(x), h_1(x), ..., h_m(x)$ and the constant C of (3.17) – (3.19), which in turn depends on $b(x)$ and the constant C_0 of (1.6) – (1.8).

The arguments of [G3] provide L^2 bounds for the operators $U_{i\,L\,\omega\,z_2...z_n}$ under the assumptions (3.17) – (3.19) and a lower bound on $|\partial_v^\alpha(\omega \cdot f_i)(z)|$ that are uniform in L, ω, $z_2, ..., z_n$ for $\omega \in N$. (A slightly stronger cancellation condition is assumed but (3.19) suffices). This is because the bounds obtained in [G3] are at least as strong as the bounds obtained when the convolution is over the curve (t, t^α), in which case the bounds can be expressed in terms of the constant C of (3.17) – (3.19), the constant C'' of (3.26), and the function $h \circ \beta_i(x)$. For the ball $B(x_0, r_0)$ on which $\sigma(x)$ is supported, how small r_0 needs to be for the uniform bounds to hold will also be uniform in the various parameters but may be smaller than the r_0 we originally selected. However, this can be corrected by writing $\sigma(x)$ as a finite sum of bump functions with smaller support if needed. This completes the proof of Theorem 1.2.

4. Proof of theorems when the multiplicity is greater than one.

We start with some facts from [G1] - [G2] which will help us understand the distribution function of $b(x)$ and related properties of integrals of $|b(x)|^{-\delta_0}$. The constructions in [G1] are slightly better for our purposes so we bring our attention to them. By Lemmas 3.2' - 3.5 of [G1], if U is a sufficiently small neighborhood of the origin, up to a set of measure zero one may write $U = \cup_{i=1}^N U_i$ as a finite union of open sets such that the following hold. Each U_i is contained in one of the 2^n octants determined by coordinate hyperplanes. For each i, there is some integer $1 \leq k_i \leq n$ and a function $\gamma_i : \mathbf{R}^n \to \mathbf{R}^n$

such that each component of $\gamma_i(x)$ is plus or minus a monomial and $\gamma_i^{-1}(U_i)$ satisfies the following. If $k_i < n$, then there are cubes $(0, \eta_i)^{k_i}$ and $(0, \eta_i')^{k_i}$ with $\eta_i' > \eta_i$ and bounded open sets $O_i \subset O_i'$ whose closures are a subset of $\{x \in \mathbf{R}^{n-k_i} : x_l > 0 \text{ for all } l\}$, such that

$$(0, \eta_i)^{k_i} \times O_i \subset \gamma_i^{-1}(U_i) \subset (0, \eta_i')^{k_i} \times O_i' \tag{4.1a}$$

If $k_i = n$, then there are cubes $(0, \eta_i)^{k_i}$ and $(0, \eta_i')^{k_i}$ with $\eta_i' > \eta_i$ such that

$$(0, \eta_i)^{k_i} \subset \gamma_i^{-1}(U_i) \subset (0, \eta_i')^{k_i} \tag{4.1b}$$

In either case, there is a monomial $m_i(x_1, ..., x_{k_i})$ and constants C_i, C_i' such that on $\gamma_i^{-1}(U_i)$ one has

$$C_i m_i(x_1, ..., x_{k_i}) < |b^* \circ \gamma_i(x)| < C_i' m_i(x_1, ..., x_{k_i}) \tag{4.2}$$

If $k_i = n$, then (4.2) holds with $b^* \circ \gamma_i(x)$ replaced by $b \circ \gamma_i(x)$. If $k_i < n$, then on $\gamma_i^{-1}(U)$ the function $b \circ \gamma_i(x)$ can be expressed as $m_i(x_1, ..., x_{k_i}) g_i(x_1, ..., x_n)$ where $m_i(x_1, ..., x_{k_i})$ is a monomial and where $g_i(x_1, ..., x_n)$ satisfies the following. One may write $\gamma_i^{-1}(U_i) = \cup_{j=1}^{M_i} V_{ij}$ such that for each i and j there is an $\epsilon > 0$, a compact face F_i of $N(b)$, and a directional derivative $\partial_{v_{ij}}$ in the last $n - k_i$ variables, such that $|\partial_{v_{ij}}^{a_{ij}} g_i(x_1, ..., x_n)| > \epsilon$ on V_{ij} for some $a_{ij} \geq 0$ which is at most the maximum order of any zero of $b_{F_i}(x_1, ..., x_n)$ on $(\mathbf{R} - \{0\})^n$. When $a_{ij} = 0$, we interpret $\partial_{v_{ij}}^{a_{ij}} g_i(x)$ to just mean $g_i(x)$.

For a given $\epsilon > 0$, the following lemma explicitly bounds the measure of the portion of a dyadic rectangle where $|b(x)/b^*(x)| < \epsilon$ in terms of the maximum order of the zeroes of the $b_F(x)$ on $(\mathbf{R} - \{0\})^n$.

Lemma 4.1. Suppose $p > 0$ is an integer such that the zeroes of each $b_F(x)$ on $(\mathbf{R} - \{0\})^n$ are all of order at most p. Then there is a neighborhood U of the origin and a constant $C > 0$ such that if $R \subset U$ is a set of the form $\{x \in \mathbf{R}^n : 2^{-j_l} < |x_l| < 2^{-j_l+1}\}$ for integers j_l, then

$$|\{x \in R : |b(x)/b^*(x)| < \epsilon\}| < C\epsilon^{\frac{1}{p}}|R| \tag{4.3}$$

Proof. It suffices to show for each i an estimate of the form $|\{x \in R \cap U_i : |b(x)/b^*(x)| < \epsilon\}| < C\epsilon^{\frac{1}{p}}|R|$. Since the components of $\gamma_i(x)$ are all monomials, the absolute value of the Jacobian of $\gamma_i(x)$ is of the form $c_i x_1^{e_{i1}} ... x_n^{e_{in}}$ for some integers $e_{i1}, ..., e_{in}$ and some $c_i > 0$. Viewing $|\{x \in R \cap U_i : |b(x)/b^*(x)| < \epsilon\}|$ as the integral of 1 over $\{x \in R \cap U_i : |b(x)/b^*(x)| < \epsilon\}$ and changing coordinates via $\gamma_i(x)$, one obtains

$$|\{R \cap U_i : |b(x)/b^*(x)| < \epsilon\}| = \int_{\{x \in \gamma_i^{-1}(R) \cap \gamma_i^{-1}(U_i) :\, |b \circ \gamma_i(x)/b^* \circ \gamma_i(x)| < \epsilon\}} c_i x_1^{e_{i1}} ... x_n^{e_{in}}\, dx \tag{4.4}$$

Note that by (4.2) and the following paragraph, one has $|b \circ \gamma_i(x)/b^* \circ \gamma_i(x)| > C' g_i(x)$ for some constant C' (We can include the $k_i = n$ situation here by defining $g_i(x) = 1$). Thus in order to bound (4.4) by an expression of the form $C\epsilon^{\frac{1}{p}}|R|$, it suffices to show the following estimate of the following form for each i and j.

$$\int_{\{x \in \gamma_i^{-1}(R) \cap V_{ij} :\, g_i(x) < \epsilon\}} x_1^{e_{i1}} ... x_n^{e_{in}}\, dx < C\epsilon^{\frac{1}{p}}|R| \tag{4.5}$$

If the multiindex a_{ij} in the paragraph after (4.2) is zero, then $g_i(x)$ is bounded below, and thus (4.5) reduces to showing that $\int_{\gamma_i^{-1}(R)\cap V_{ij}} x_1^{e_{i1}}...x_n^{e_{in}}\,dx$ is bounded by a constant times $|R|$, which follows immediately from changing back into the original coordinates using γ_i. Thus it suffices to assume $a_{ij} \geq 1$. Note that this only occurs if $k_i < n$. Since the final $n - k_i$ variables are bounded below on V_{ij}, it suffices to prove a bound

$$\int_{\{x\in\gamma_i^{-1}(R)\cap V_{ij}:\, g_i(x)<\epsilon\}} x_1^{e_{i1}}...x_{k_i}^{e_{ik_i}}\,dx < C\epsilon^{\frac{1}{p}}|R| \tag{4.6}$$

We now integrate the left-hand side of (4.6) starting with the v_{ij} direction. Since $a_{ij} \leq p$, by the measure version of the Van der Corput lemma (see [C] for details), the integral in the v_{ij} direction is at most $Cx_1^{e_{i1}}...x_{k_i}^{e_{ik_i}}\epsilon^{\frac{1}{p}}$. If we next perform the integration in the remaining $n-k_i-1$ directions of last $n-k_i$ variables (if any exist), then if π_i denotes the projection on $\mathbf{R}^n$ onto the first k_i variables, we obtain

$$\int_{\{x\in\gamma_i^{-1}(R)\cap V_{ij}:\, g_i(x)<\epsilon\}} x_1^{e_{i1}}...x_{k_i}^{e_{ik_i}}\,dx \leq C\epsilon^{\frac{1}{p}}\int_{\pi_i(\gamma_i^{-1}(R)\cap V_{ij})} x_1^{e_{i1}}...x_{k_i}^{e_{ik_i}}\,dx_1...dx_{k_i} \tag{4.7}$$

$$= C\epsilon^{\frac{1}{p}}\int_{\pi_i(\gamma_i^{-1}(R)\cap V_{ij})\times[1,2]^{n-k_i}} x_1^{e_{i1}}...x_{k_i}^{e_{ik_i}}\,dx_1...dx_{k_i} \tag{4.8a}$$

$$= C'\epsilon^{\frac{1}{p}}\int_{\pi_i(\gamma_i^{-1}(R)\cap V_{ij})\times[1,2]^{n-k_i}} x_1^{e_{i1}}...x_n^{e_{in}}\,dx_1...dx_n \tag{4.8b}$$

Because the last $n-k_i$ coordinates of the points in U_i are bounded above and below away from zero, there is a constant $C_0 > 1$ such that if $(x_1, ..., x_n) \in \pi_i(\gamma_i^{-1}(R)\cap V_{ij})\times[1,2]^{n-k_i}$ then there is a point $(y_1, ..., y_n) \in \gamma_i^{-1}(R) \cap V_{ij}$ such that $\frac{1}{C_0} < \frac{y_l}{x_l} < C_0$ for each l. This property is preserved under monomial maps (perhaps with a different constant C_1), so the image of $\pi_i(\gamma_i^{-1}(R)\cap V_{ij}) \times [1,2]^{n-k_i}$ under γ_i is a subset of a corresponding dilate of $\gamma_i(\gamma_i^{-1}(R) \cap V_{ij})$, which in turn is a subset of the dilate of R. Denote this dilate by R^*. Changing coordinates in (4.8) back into the original coordinates via γ_i, we see that

$$C\epsilon^{\frac{1}{p}}\int_{\pi_i(\gamma_i^{-1}(R)\cap V_{ij})\times[1,2]^{n-k_i}} x_1^{e_{i1}}...x_n^{e_{in}}\,dx_1...dx_n \leq C''\epsilon^{\frac{1}{p}}\int_{R^*} 1\,dx$$

$$= C'''\epsilon^{\frac{1}{p}}|R| \tag{4.9}$$

This is the desired estimate (4.6) and we are done.

We also will make use of the following result.

Lemma 4.2. Suppose the zeroes of each $b_F(x)$ on $(\mathbf{R} - \{0\})^n$ are all of order less than $d(b)$. Then there is a neighborhood U of the origin and constants $C, \eta > 0$ such that if $\epsilon > 0$ and $R \subset U$ is a set of the form $\{x \in \mathbf{R}^n : 2^{-j_l} < |x_l| < 2^{-j_l+1}\}$, then $\int_{\{x\in R:\, |b(x)|<\epsilon|b^*(x)|\}} |b(x)|^{-\delta_0} < C\epsilon^{\eta}$. In particular, since there is a constant C' such that

$|b(x)| \leq C' b^*(x)$ on any such R, there is a constant C'' such that $\int_R |b(x)|^{-\delta_0} < C''$ for such $R \subset U$.

Proof. Since the terms of $b^*(x)$ are absolute values of monomials, there is a constant $c > 1$ and an $x_0 \in R$ such that $c\, b^*(x_0) \geq b^*(x) \geq \frac{1}{c} b^*(x_0)$ on R. Hence it suffices to prove an estimate of the form $\int_{\{x \in R:\, |b(x)| < \epsilon |b^*(x_0)|\}} |b(x)|^{-\delta_0} < C\epsilon^\eta$. By the relation between L^p norms and distribution functions, applied to $\frac{1}{|b(x)|}$, one has

$$\int_{\{x \in R:\, |b(x)| < \epsilon |b^*(x_0)|\}} |b(x)|^{-\delta_0} = \delta_0 \int_0^\infty t^{\delta_0 - 1} \big|\{x \in R : |b(x)| < \min(\epsilon |b^*(x_0)|, \frac{1}{t})\}\big|\, dt \tag{4.10}$$

It is natural to break up (4.10) into two pieces, the first where $t < \frac{1}{\epsilon b^*(x_0)}$ and the second where $t \geq \frac{1}{\epsilon b^*(x_0)}$. Then the right-hand side of (4.10) becomes

$$\delta_0 \int_0^{\frac{1}{\epsilon b^*(x_0)}} t^{\delta_0 - 1} |\{x \in R :\, |b(x)| < \epsilon |b^*(x_0)|\}|\, dt + \delta_0 \int_{\frac{1}{\epsilon b^*(x_0)}}^\infty t^{\delta_0 - 1} \big|\{x \in R : |b(x)| < \frac{1}{t}\}\big|\, dt \tag{4.11}$$

Performing the first integral in the first term of (4.11) results in

$$\epsilon^{-\delta_0} |b^*(x_0)|^{-\delta_0} |\{x \in R :\, |b(x)| < \epsilon |b^*(x_0)|\}| \tag{4.12}$$

By Lemma 4.1, (4.12) is bounded by $C|b^*(x_0)|^{-\delta_0} \epsilon^{\frac{1}{p} - \delta_0} |R|$ for some $p < d(b) = \frac{1}{\delta_0}$. Hence we have

$$\int_{\{x \in R:\, |b(x)| < \epsilon |b^*(x_0)|\}} |b(x)|^{-\delta_0} \leq C|b^*(x_0)|^{-\delta_0} \epsilon^{\frac{1}{p} - \delta_0} |R| + \delta_0 \int_{\frac{1}{\epsilon b^*(x_0)}}^\infty t^{\delta_0 - 1} \big|\{x \in R : |b(x)| < \frac{1}{t}\}\big|\, dt \tag{4.13}$$

Note that $\{x \in R : |b(x)| < \frac{1}{t}\} \subset \{x \in R : |b(x)| < \frac{c_0}{t b^*(x_0)} b^*(x)\}$, so by Lemma 4.1 for some constant C_0 we have

$$\int_{\{x \in R:\, |b(x)| < \epsilon |b^*(x_0)|\}} |b(x)|^{-\delta_0} \leq C|b^*(x_0)|^{-\delta_0} \epsilon^{\frac{1}{p} - \delta_0} |R| + C_0 \int_{\frac{1}{\epsilon b^*(x_0)}}^\infty t^{\delta_0 - 1} \left(\frac{1}{t b^*(x_0)}\right)^{\frac{1}{p}} |R|\, dt \tag{4.14}$$

Note that the exponent $\delta_0 - 1 - \frac{1}{p}$ is less than -1 since $p < d(b) = \frac{1}{\delta_0}$. Hence integrating the second term on the right of (4.14) leads to the following for some constant C_1.

$$\int_{\{x \in R:\, |b(x)| < \epsilon |b^*(x_0)|\}} |b(x)|^{-\delta_0} \leq C|b^*(x_0)|^{-\delta_0} \epsilon^{\frac{1}{p} - \delta_0} |R| + C_1 |b^*(x_0)|^{-\delta_0} \epsilon^{\frac{1}{p} - \delta_0} |R| \tag{4.15}$$

Since $|R| \sim |x_1 ... x_n|$ for any $(x_1, ..., x_n) \in R$, in order to prove Lemma 4.2 with $\eta = \frac{1}{p} - \delta_0$, it suffices to show that there is a constant C_2 such that for any x we have

$$|x_1 ... x_n| (b^*(x))^{-\delta_0} \leq C_2 \tag{4.16}$$

Since $\delta_0 = \frac{1}{d(b)}$ in the case at hand, (4.16) is equivalent to the statement that

$$b^*(x) \geq C_3 |x_1 ... x_n|^{d(b)} \tag{4.17}$$

Since $(d(b), ..., d(b))$ is on the Newton polyhedron $N(b)$, there are nonnegative α_i with $\alpha_1 + ... + \alpha_k = 1$ such that each component of $(d(b), ..., d(b))$ is greater than or equal to that of $\alpha_1 v_1 + ... + \alpha_k v_k$ for some vertices $v_1, ..., v_k$ of $N(b)$. Hence $|x_1 ... x_n|^{d(b)} \leq |x^{v_1}|^{\alpha_1} ... |x^{v_k}|^{\alpha_k}$. So by the generalized AM-GM inequality one has $|x_1 ... x_n|^{d(b)} \leq \sum_{i=1}^k \alpha_i |x^{v_i}| \leq b^*(x)$ as needed. This completes the proof of Lemma 4.2.

Similar to the multiplicity one case, in order to show $||Tf||_p \leq C||f||_p$ for all Schwartz f for a given $1 < p < \infty$, it suffices to show the that if $K(y)$ is supported on a sufficiently small neighborhood of the origin there is a constant C such that $||T_L f||_p \leq C||f||_p$ for all Schwartz f and each L, where $T_L f(x) = \int_{\mathbf{R}^n} f(x-y)\, \alpha(x,y)\, K_L(y)\, dy$. Here $K_L(y) = \sum_{j_l < L \ for\ all\ l} K_{j_1,...,j_n}(y)$ as in (2.6). As in Lemma 3.1 for the multiplicity one case, we may also replace $\alpha(x,y)$ by just 1. Thus we focus our attention on U_L given by

$$U_L f(x) = \int_{\mathbf{R}^n} f(x-y) K_L(y)\, dy \tag{4.18}$$

Our goal will be to prove U_L is bounded on L^p with a norm independent of L under the hypotheses of Theorem 2.1 or 2.4. The next two lemmas provide bounds on the $|\widehat{K_{j_1,...,j_n}}(\xi)|$ that allow us to prove such uniform bounds.

Lemma 4.3. Under the assumptions of Theorem 2.1, there is a constant $C > 0$ such that if l is such that $2^{-j_l}|\xi_l| \leq 1$, then

$$|\widehat{K_{j_1,...,j_n}}(\xi)| \leq C 2^{-j_l} |\xi_l|$$

Proof. $\widehat{K_{j_1,...,j_n}}(\xi)$ is given by

$$\widehat{K_{j_1,...,j_n}}(\xi) = \int_{\mathbf{R}^n} K_{j_1,...,j_n}(x) e^{-i\xi_1 x_1 - ... - i\xi_n x_n}\, dx \tag{4.19}$$

Since the integral of $K_{j_1,...,j_n}(x)$ in the x_l variable is equal to zero by (2.5), one can subtract $K_{j_1,...,j_n}(x) e^{\sum_{k \neq l} -i\xi_k x_k}$ from the integrand in (4.19) without changing the integral, so we have

$$\widehat{K_{j_1,...,j_n}}(\xi) = \int_{\mathbf{R}^n} K_{j_1,...,j_n}(x)(e^{-i\xi_l x_l} - 1) e^{\sum_{k \neq l} -i\xi_k x_k}\, dx \tag{4.20}$$

Since $|\xi_l x_l| \sim 2^{-j_l}|\xi_l| \leq C$ when $K_{j_1,...,j_n}(x) \neq 0$, in (4.20) one has that $(e^{-i\xi_l x_l} - 1) \leq C|x_l \xi_l| < C' 2^{-j_l} |\xi_l|$ and we get

$$|\widehat{K_{j_1,...,j_n}}(\xi)| \leq C' 2^{-j_l} |\xi_l| \int_{\mathbf{R}^n} |K_{j_1,...,j_n}(x)|\, dx \tag{4.21}$$

Using Lemma 4.2 we obtain the desired estimate

$$|\widehat{K_{j_1,...,j_n}}(\xi)| \leq C''2^{-j_l}|\xi_l| \tag{4.22}$$

Lemma 4.4. Under the assumptions of Theorem 2.1, there are constants $\rho, C > 0$ such that if l is such that $2^{-j_l}|\xi_l| \geq 1$, then

$$|\widehat{K_{j_1,...,j_n}}(\xi)| \leq C\frac{1}{(2^{-j_l}|\xi_l|)^{\rho}} \tag{4.23}$$

Proof. Let $\sigma_1(x)$ be a smooth increasing nonnegative function on $\mathbf{R}^+$ with $\sigma_1(x) = 1$ for $|x| < 1$ and $\sigma_1(x) = 0$ for $|x| > 2$. Let $\sigma_2(x) = 1 - \sigma_1(x)$. For a constant $\rho_0 > 0$ to be determined by our arguments, for any fixed x_0 in the dyadic rectangle corresponding to $(j_1, ..., j_n)$ we write

$$\widehat{K_{j_1,...,j_n}}(\xi) = \int_{\mathbf{R}^n} \sigma_1\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)K_{j_1,...,j_n}(x)e^{-i\xi_1x_1-...-i\xi_nx_n}\,dx$$

$$+\int_{\mathbf{R}^n} \sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)K_{j_1,...,j_n}(x)e^{-i\xi_1x_1-...-i\xi_nx_n}\,dx \tag{4.24}$$

The first term of (4.24) is bounded by

$$\int_{\{x:|b(x)|\leq 2(2^{-j_l}|\xi_l|)^{-\rho_0}\ b^*(x_0)\}} |K_{j_1,...,j_n}(x)| \tag{4.25}$$

Using (2.3) and Lemma 4.1, we see that this term is at most $C(2^{-j_i}|\xi_i|)^{-\frac{\rho_0}{d}}$ for some d, which gives the bound of the right-hand side of (4.23).

Proceeding to the second term of (4.24), we integrate by parts, integrating the $e^{-i\xi_1x_1-...-i\xi_nx_n}$ factor in the x_l variable and differentiating the remaining factors. The resulting term is given by

$$\frac{1}{i\xi_l}\int_{\mathbf{R}^n} \partial_{x_l}\left[\sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)K_{j_1,...,j_n}(x)\right]e^{-i\xi_1x_1-...-i\xi_nx_n}\,dx \tag{4.26}$$

If the x_l derivative in (4.26) lands on the $K_{j_1,...,j_n}(x)$ factor, one obtains a term which by (2.4) is bounded by

$$C_1\frac{1}{|\xi_l|}\int_{\mathbf{R}^n} \sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)\frac{1}{|x_l|}b^*(x)|b(x)|^{-1-\delta_0}\,dx \tag{4.27}$$

Due to the σ_2 factor in (4.27), on the support of the integrand of (4.27) we have $|b(x)| \geq (2^{-j_l}|\xi_l|)^{-\rho_0}b^*(x_0)$. Thus (4.27) is bounded by

$$C_1\frac{1}{|\xi_l|}\int_{R_{j_1,...,j_n}} \sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)\frac{1}{|x_l|}b^*(x)(2^{-j_l}|\xi_l|)^{\rho_0(1+\delta_0)}b^*(x_0)^{-1-\delta_0}\,dx \tag{4.28}$$

Here $R_{j_1,\ldots,j_n}$ denotes the (expanded) dyadic rectangle-like set on which $K_{j_1,\ldots,j_n}$ is supported. Since $\sigma_2(t) \leq 1$ for all t, $|x_l| \sim 2^{-j_l}$ on $R_{j_1,\ldots,j_n}$, and $b^*(x)$ is within a constant factor of $b^*(x_0)$ on $R_{j_1,\ldots,j_n}$, (4.28) is bounded by

$$C_2\left(\frac{1}{2^{-j}|\xi_l|}\right)(2^{-j_l}|\xi_l|)^{\rho_0(1+\delta_0)}\int_{R_{j_1,\ldots,j_n}}(b^*(x))^{-\delta_0}\,dx \tag{4.29}$$

By Lemma 4.2 (which applies to negative powers of the smaller function $|b(x)|$), we see that the above is bounded by

$$C_3\left(\frac{1}{2^{-j}|\xi_l|}\right)(2^{-j_l}|\xi_l|)^{\rho_0(1+\delta_0)} \tag{4.30}$$

Thus so long as ρ_0 is chosen so that $\rho_0(1+\delta_0) < \frac{1}{2}$ for example, this term of (4.26) satisfies the bounds needed in this lemma.

We now bound the term where the derivative in (4.26) lands on the $\sigma_2((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)})$ factor. Observe that

$$\partial_{x_l}\left(\sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)\right) = \pm\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{\partial_{x_l}b(x)}{b^*(x_0)}\right)\sigma_2'\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right) \tag{4.31}$$

Since $|b(x)| \geq (2^{-j_l}|\xi_l|)^{-\rho_0}b^*(x_0)$ in the support of the σ' factor, by (2.3), on the support of the integrand of this term of (4.26) we have

$$|K_{j_1,\ldots,j_n}(x)| \leq C_4(2^{-j_l}|\xi_l|)^{\rho_0\delta_0}|b^*(x_0)|^{-\delta_0} \tag{4.32}$$

As a result, the absolute value of the term of (4.26) in question is bounded by

$$C_4|\xi_l|^{-1}(2^{-j_l}|\xi_l|)^{\rho_0\delta_0}|b^*(x_0)|^{-\delta_0}\int_{R_{j_1,\ldots,j_n}}\left|\partial_{x_l}\left(\sigma_2\left((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}\right)\right)\right|dx \tag{4.33}$$

We first integrate in the x_l variable in (4.33). By the hypotheses of Theorem 2.1 concerning zeroes of $\partial_{x_l}b(x)$, for any fixed value of the remaining $n-1$ variables (outside a set of measure zero) the domain of integration in the x_l variable can be written as the union of boundedly many intervals on which $\partial_{x_l}(\sigma_2((2^{-j_l}|\xi_l|)^{\rho_0}\frac{|b(x)|}{b^*(x_0)}))$ does not change sign. Thus on each of these intervals this derivative integrates back to the function. Since σ_2 is bounded this means the x_l integrals in (4.33) are uniformly bounded in the remaining variables. Thus doing the x_l integral first and then integrating over the remaining variables shows that (4.33) is bounded by

$$C_5|\xi_l|^{-1}(2^{-j_l}|\xi_l|)^{\rho_0\delta_0}|b^*(x_0)|^{-\delta_0}2^{\sum_{i\neq l}-j_i} \tag{4.34}$$

Since $b^*(x) \sim b^*(x_0)$ on $R_{j_1,\ldots,j_n}$, (4.34) is bounded by

$$C_6|\xi_l|^{-1}(2^{-j_l}|\xi_l|)^{\rho_0\delta_0}2^{j_l}\int_{R_{j_1,\ldots,j_n}}|b^*(x)|^{-\delta_0}\,dx \tag{4.35}$$

As in (4.30), the integral in (4.35) is uniformly bounded and we obtain the bound

$$C_7(2^{-j_l}|\xi_l|)^{\rho_0\delta_0}\frac{1}{2^{-j_1}|\xi_l|} \tag{4.36}$$

So as long as $\rho_0\delta_0 < 1$, we see from (4.36) that the term of (4.26) under consideration is also is bounded by the right-hand side of (4.23). We have now shown that the first term of (4.24) and each term of (4.26) all are bounded by the right-hand side of (4.23) and thus we are done with the proof of Lemma 4.4.

Proof of Theorem 2.1.

We will prove L^2 boundedness of U_L uniformly in L by bounding the Fourier transform $\widehat{K_L}(\xi)$ uniformly in L and ξ. Since $\widehat{K_L}(\xi) = \sum_{(j_1,\dots,j_n)\in\mathbf{Z}^n: j_l < L \text{ for all } l} \widehat{K_{j_1,\dots,j_n}}(\xi)$, we have the bound

$$|\widehat{K_L}(\xi)| \leq \sum_{(j_1,\dots,j_n)\in\mathbf{Z}^n: j_l < L \text{ for all } l} |\widehat{K_{j_1,\dots,j_n}}(\xi)| \tag{4.37}$$

We use the better of the two estimates from Lemmas 4.2 and 4.3 in each term of (4.37) then add the result. Let $(k_1,\dots,k_n)$ be the vector of integers such that for each l, 2^{k_l} is the nearest power of 2 to $|\lambda_l|$. For any M the number of $(j_1,\dots,j_n)$ such that $\max_l |j_l - k_l| = M$ is bounded by CM^{n-1}, and for each such $(j_1,\dots,j_n)$ Lemma 4.3 or 4.4 gives a bound $|\widehat{K_{j_1,\dots,j_n}}(\xi)| \leq C'2^{-\rho_1 M}$ for some $\rho_1 > 0$. Hence in (4.37) the sum over all terms with $\max_l |j_l - k_l| = M$ is bounded by $C''M^{n-1}2^{-\rho_1 M}$. Adding over all M gives a uniform bound and we are done.

Proof of Theorem 2.2.

We will make use of the Marcinkiewicz multiplier theorem (see Theorem 6' on p. 109 of [S]), which implies that L^p bounds on U_L that are uniform in L will follow if we can show that there is a constant C such that for each multiindex α with $|\alpha| \leq n$ and each L we have the estimate

$$|\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}\partial^\alpha\widehat{K_L}(\xi_1,\dots,\xi_n)| \leq C \tag{4.38}$$

Returning to the x variables, this will follow as in the proof of Theorem 2.1 if we can show that for each multiindex α with $0 \leq |\alpha| \leq n$ the kernel $\partial^\alpha(x^\alpha K_{j_1,\dots,j_n}(x))$ satisfies the conditions of Lemmas 4.3 and 4.4. But the fact that (2.13) holds for $K_{j_1,\dots,j_n}(x)$ immediately implies that (2.13) also holds for $\partial^\alpha(x^\alpha K_{j_1,\dots,j_n}(x))$. The cancellation condition (2.5) also holds for $\partial^\alpha(x^\alpha K_{j_1,\dots,j_n}(x))$; if the x_l variable is not represented in α it can be shown by multiplying (2.5) through by x^α and then applying ∂^α under the integral sign, while if the x_l variable is represented in α, then the integral (2.5) is zero simply because one is integrating the derivative of a compactly supported C^1 function. Hence each kernel $\partial^\alpha(x^\alpha K_{j_1,\dots,j_n}(x))$ satisfies Lemmas 4.3 and 4.4 and Theorem 2.2 follows.

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