

A Constructive Elementary Method for Local Resolution of Singularities

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Abstract

Building on the local resolution of singularities methods of [G1]-[G3], we provide a constructive local resolution of singularities method with several features beyond those of [G1]-[G3]. The methods of this paper have had applications to exponential sums and oscillatory integrals [G4] and in proving theorems on singular integrals with singularities on a real-analytic variety [G5]. Our arguments are elementary classical analysis arguments that are quite different from traditional resolution of singularities techniques. A number of illustrative examples of our methods are given.

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1. Introduction and statements of results

In this paper we provide a constructive local resolution of singularities method, building on the local resolution of singularities methods of [G1]-[G3]. It applies to all functions with a convergent power series over an arbitrary local field K of characteristic zero. By the classification of such fields, this means $K = \mathbf{R}, \mathbf{C}$, the p -adics \mathbf{Q}_p , or a finite field extension of \mathbf{Q}_p . The algorithm of this paper is an entirely elementary and self-contained classical analysis argument, using only the implicit function theorem and elementary facts about power series and Newton polyhedra, and the methods are quite different from traditional resolution of singularities methods. The results of this paper have found applications to exponential sums in analytic number theory [G4] and singular integrals whose singularities lie on an analytic variety [G5].

The blowups used in this paper will all be traditional blowups along $n - 2$ dimensional coordinate hyperplanes in appropriate coordinate systems. The reason for this is that such blowups correspond to the simplest permissible linear transformations on the Newton polyhedron of the function whose zero set is being resolved, and linear transformations on such Newton polyhedra play a major role in our arguments. Thus such blowups form natural building blocks for the more complicated linear transformations we will need. They are also relatively easily made compatible with partitions of unity, as we will see.

One important difference between this paper and the earlier papers [G1]-[G3] is that with the algorithm in its current form, we are able to use standard blowups rather

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than the monomial maps of [G1]-[G3] that force one to deal with technical issues related to such monomial maps not necessarily being one to one; these issues make the p -adic case especially difficult. This is the chief reason why this paper covers the situation of an arbitrary local field of characteristic zero, which is not true of [G1]-[G3], and this is why the methods of this paper have provided applications to exponential sums and oscillatory integrals in [G4].

Another noteworthy difference between this paper and [G1]-[G3] is that the complicated iterated domain subdivisions using Newton polyhedra of [G1]-[G3] have now been reduced to a short one-stage argument, given in section 2. In general, when dealing with oscillatory integral estimates and similar matters, the divisions into subdomains with respect to the Newton polyhedron provided in sections 2 and 3 are more natural to use than the corresponding parts of [G1]-[G3]. This was exploited in [G5].

The coordinate changes used in the resolution process will be of the following types.

Definition 1.1. An *invertible affine linear map* is a map $Ax + b$ on K^n , where A is linear and invertible and where $b \in K^n$.

Definition 1.2. A *blowup* is a map $b : K^n \rightarrow K^n$ of the form $b(x) = (b_1(x), \dots, b_n(x))$ where there exist j, k such that $b_i(x) = x_i$ for all $i \neq j$ and $b_j(x) = x_j x_k$.

Definition 1.3. A *quasitranslation* is a function on a neighborhood of the origin in K^n of the form

$$g(x) = (x_1, \dots, x_{i-1}, x_i + a(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n)$$

Here $a(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ has a convergent power series on a neighborhood of the origin in K^{n-1} with $a(0) = 0$.

Our local resolution of singularities theorem is as follows.

Theorem 1.1. Let K be a local field of characteristic zero. Let $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha}$ be a function with a convergent power series on an open $U_0 \subset K^n$ containing the origin such that $f(0) = 0$ with at least one f_{α} being nonzero. Let K_0 denote $K - \{0\}$. There is a bounded open U with $0 \in U \subset U_0$ such that if $\phi(x)$ is a smooth function supported in U then outside a set of measure zero $\phi(x)$ can be written as $\sum_{i=1}^N \phi_i(x)$, each $\phi_i(x)$ supported in U , where for each i there is a bounded open set $W_i \subset K^n$ and a function $\alpha_i : W_i \rightarrow U$ satisfying the following.

a) Each α_i is a finite composition of invertible affine linear maps, blowups, and quasitranslations such that the restriction of α_i to $W_i \cap K_0^n$ is an analytic diffeomorphism onto its image.

b) Let $V_i = \{x \in W_i : \phi_i \circ \alpha_i(x) \neq 0\}$. Then V_i is an open subset of $W_i \cap K_0^n$ with $cl(V_i) \subset W_i$. One can change $\phi_i \circ \alpha_i(x)$ on $W_i - K_0^n$ in such a way that the resulting function

is smooth. Furthermore, α_i is an analytic diffeomorphism from V_i to $\{x \in U : \phi_i(x) \neq 0\}$.

c) On W_i , $f \circ \alpha_i(x)$ and the Jacobian determinant of $\alpha_i(x)$ can both be written in the form $c_i(x)m_i(x)$, where $c_i(x)$ is nonvanishing and $m_i(x)$ is a monomial.

Note that in the case where K is a p -adic field, after adjusting on $W_i - K_0^n$, the functions $\phi_i \circ \alpha_i(x)$ are locally constant.

Theorem 1.1 directly implies a more traditionally stated local resolution of singularities theorem involving open sets covering an open set containing the origin:

Theorem 1.2. Let $f(x)$ and U_0 be as in Theorem 1.1. There are bounded open sets $V_i \subset K^n$, $i = 1, \dots, N$ such that for each i there is a function $\alpha_i : V_i \rightarrow U_0$ satisfying the following.

a) Each α_i is a finite composition of invertible affine linear maps, blowups, and quasitranslations which is an analytic diffeomorphism from V_i onto its image.

b) Up to a set of measure zero, $\cup_{i=1}^N \alpha_i(V_i)$ contains an open set containing the origin.

c) Each α_i extends to an open W_i containing $cl(V_i)$ such that on W_i , both $f \circ \alpha_i(x)$ and the Jacobian determinant of $\alpha_i(x)$ can be written in the form $c_i(x)m_i(x)$, where $c_i(x)$ is nonvanishing and $m_i(x)$ is a monomial.

Theorem 1.2 follows immediately from Theorem 1.1 by letting $\phi(x)$ be any function equal to 1 on a neighborhood of the origin and then letting V_i be as in Theorem 1.1.

Theorems 1.1 and 1.2, being local results, are of course weaker than Hironaka's famous work [H1]-[H2] and their analytic counterparts. It should be pointed out there have been other elementary proofs of resolution of singularities theorems over \mathbf{R} and \mathbf{C} , the most well-known being [BM1] [Pa1]-[Pa2] [Su]. We also refer to the paper [CoGrPr] for further elementary resolution of singularities results, modeled on [J] [Pa2] [Su]. It should also be mentioned that there have also been a number of significant simplifications and extensions of Hironaka's proof in recent years, such as [BM2] [EH] [EV1]-[EV2] [K] [W]. While these papers are in the language of modern algebraic geometry and thus are not classical analysis arguments, they are quite accessible and give quite general results of an algebraic nature. The arguments of this paper are quite different from those of the above papers as we describe now.

This paper will proceed as follows. (The following is an oversimplification, but will give the reader an idea of the overall argument). Let $f(x)$ have a convergent power series on a neighborhood of the origin, not identically zero, such that $f(0) = 0$. In section 2, we will subdivide a small neighborhood of the origin into a finite collection of "wedges" A_{ij} . To each wedge A_{ij} there corresponds a unique vertex or compact face of the Newton

polyhedron of $f(x)$ of dimension i such that the terms of $f(x)$'s Taylor expansion whose exponents are on that face dominate the overall Taylor expansion of $f(x)$ in a certain way that is made precise in Theorem 2.1.

In section 3, we will take a small cube F centered at the origin and write $F = \cup_{k=1}^q F_k$ such that after a sequence of blowups, F_k becomes another cube E_k . These blowups are done such that (Lemma 3.1) the functions x^v for v a vertex of the Newton polyhedron of f become, in the blown up coordinates, of the form x^w where the w are now ordered lexicographically. This has the effect that the intersection of each E_k with each blown-up A_{ij} is either empty or is a set A'_{ijk} comparable in a certain sense to a set of the form $C \times D$, where C is a cube centered at the origin of dimension $n - i$ and where D is an open set of dimension of i which does not intersect the coordinate hyperplanes. If $i = 0$ then a nonempty A'_{ijk} is just comparable to a cube C of dimension n . The exact statements (and what "comparable" means here) are given in Lemmas 3.2 and 3.2'. We will need A'_{ijk} to be of this type of regular shape for our induction to proceed.

We will also see (Lemma 3.5) that the above operations can be made compatible with partitions of unity; if $\phi(x)$ is smooth and supported on a small enough neighborhood of the origin, one can write $\phi(x) = \sum_{i=1}^N \phi_i(x)$ such that each $\phi_i(x)$ transforms under one of the above sequences of blowups to a smooth function that is approximately supported on one of the A'_{ijk} .

In section 4, we will prove Theorem 1.1. The beginning of the proof is similar to how Hironaka-based proofs work. We induct on the dimension n , and for a given n we induct on the order m of the zero of $f(x)$ at the origin. We rotate coordinates so that $\frac{\partial^m f}{\partial x_n^m}(0) \neq 0$ and $\frac{\partial^l f}{\partial x_n^l}(0) = 0$ for $l < m$. We then do a quasitranslation after which $\frac{\partial^{m-1} f}{\partial x_n^{m-1}}(x_1, \dots, x_{n-1}, 0) = 0$ for all x_1, \dots, x_{n-1} , and then use the induction hypothesis on the dimension n , applied to each $\frac{\partial^l f}{\partial x_n^l}(x_1, \dots, x_{n-1}, 0)$ with $l < m$, and reduce to the case where $f(x)$ is of the form

$$f(x) = a(x_1, \dots, x_n)x_n^m + \sum_{p < m-1} a_p(x_1, \dots, x_{n-1})m_p(x_1, \dots, x_{n-1})x_n^p \quad (1.1)$$

Here each $m_p(x_1, \dots, x_{n-1})$ is a monomial and $a(x_1, \dots, x_n)$ and each $a_p(x_1, \dots, x_{n-1})$ are nonvanishing near the origin.

The argument now diverges from Hironaka-type arguments such as [H1] [H2] [BM1] [BM2]. With $f(x)$ now of the form (1.1), we divide a neighborhood of the origin into the above-mentioned F_k and perform the associated blowups converting F_k to E_k . Using the results of Section 3, we will see that because of the way the A_{ij} are defined, the fact that $\frac{\partial^m f}{\partial x_n^m} \neq 0$ and the fact that there are no terms in the Taylor expansion of $f(x)$ for which x_n appears to degree $m - 1$ imply that on each A'_{ijk} we can factor out a monomial after which we are left with a function $g(x)$ whose zeroes have order at most $m - 1$.

Then induction on m will give the result; $g(x)$ will satisfy Theorem 1.1 and our methods will be such that the monomial being factored out also will blow up to a function of the desired form $c(x)m(x)$ under the coordinate changes on $g(x)$, where $m(x)$ is a monomial and $c(x)$ is nonvanishing. Thus we end out with a sequence of blowups after which $f(x)$ is also of this form. An inspection of the argument then reveals that the Jacobian of the composition of all coordinate changes is also of the desired form $c(x)m(x)$. Furthermore, the overall argument will be performed so as to be compatible with partitions of unity as required by Theorem 1.1.

It should be pointed out that while applying the induction hypothesis on the dimension on coefficient functions such as in (1.1) frequently occurs in resolution of singularities algorithms that are related to Hironaka's work, the divisions into domains with respect to the Newton polyhedron in sections 2 and the subsequent arguments of section 3 are different and are important parts of the proof of Theorem 1.1.

Such divisions have antecedents, such as in the use of toric resolution of singularities in the analysis of oscillatory integrals such as in [V], as well as the various Phong-Stein-Sturm papers on oscillatory integral operators in two dimensions such as [PS] and [PSSt]. The divisions we will be making will be more precise than the type used in toric resolution of singularities and furthermore will be done in the domain of the function being resolved rather than in terms of cones and so on. The nature of these divisions will help enable us to prove a local resolution of singularities theorem that applies to all functions, rather than only functions satisfying a nondegeneracy condition such as in [V].

One aspect of our arguments worth mentioning is that unlike in traditional resolution of singularities methods, we do not require the centers of the blowups to be contained in the singular set or the zero set of the function being resolved. The classical analysis applications of this paper so far have not required it, and not making this stipulation gives us more flexibility in our proofs.

2. Subdivisions according to the Newton polyhedron.

We start with some relevant definitions.

Definition 2.1. Let $f(x)$ be a function such that $f(x)$ has a convergent power series expansion $\sum_{\alpha} f_{\alpha} x^{\alpha}$ on a neighborhood of the origin in K^n . For any α for which $f_{\alpha} \neq 0$, let Q_{α} be the orthant $\{t \in \mathbf{R}^n : t_i \geq \alpha_i \text{ for all } i\}$. Then the *Newton polyhedron* $N(f)$ of $f(x)$ is defined to be the convex hull of all Q_{α} .

A Newton polyhedron can contain faces of various dimensions in various configurations. These faces can be either compact or unbounded. In this paper, as in earlier work such as [G3] and [V], an important role is played by the following functions, defined for compact faces of the Newton polyhedron. A vertex is always considered to be a compact face of dimension zero.

Definition 2.2. Suppose F is a compact face of $N(f)$. Then if $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha}$ denotes the Taylor expansion of f like above, define $f_F(x) = \sum_{\alpha \in F} f_{\alpha} x^{\alpha}$.

Definition 2.3. $v(f)$ denotes the set of vertices of $N(f)$.

We now describe how we subdivide a neighborhood of the origin according to the Newton polyhedron of a function. Let U be an open set in K^n containing the origin and let $f : U \rightarrow K$ be a function with a (nontrivial) convergent power series on U such that $f(0, \dots, 0) = 0$. As above, let $N(f)$ denote the Newton polyhedron of f , and let $v(f)$ denote the set of vertices of $N(f)$. We list the compact faces of $N(f)$ of dimension i as $\{F_{ij}\}_{j=1}^{k_i}$; a vertex of $N(f)$ is considered to be a face of dimension zero. In the following, we will make use of constants C_1, \dots, C_n such that $C_1 > N$ and each $C_{i+1} > C_i^N$ for a large N dictated by our arguments that will depend only on $N(f)$.

Letting $K_0 = K - \{0\}$ as before, we will write $E = \{x \in K_0^n : 0 < |x_l| < C_n^{-1} \text{ for all } l\}$ as the union, modulo a set of measure zero, of regions A_{ij} . On a given A_{ij} , the terms $f_{\alpha} x^{\alpha}$ for $\alpha \in F_{ij}$ will dominate the Taylor expansion of $f(x)$ in a sense to be made precise by Theorem 2.1 and Lemma 2.2. We will start by defining A_{n-1j} successively in j . Once the A_{ij} have been defined for a given i , we will proceed to define the A_{i-1j} successively in j , and so on, until we have defined the A_{0j} . For a given i , it does not matter in what order we list the faces F_{ij} here as long as we have fixed some enumeration.

The definition of the A_{ij} is as follows. Once all $A_{i'j'}$ have been defined for $i' > i$ and for $i' = i, j' < j$, we define A_{ij} to be the set of points x in E such that the following three conditions hold.

- a) $\sup_{v \in v(f) \cap F_{ij}} |x^v| = \sup_{v \in v(f)} |x^v|$.
- b) $\inf_{v \in v(f) \cap F_{ij}} |x^v| \geq C_i^{-1} \sup_{v \in v(f)} |x^v|$ ($= C_i^{-1} \sup_{v \in v(f) \cap F_{ij}} |x^v|$).
- c) $x \notin A_{i'j'}$ for any (i', j') with $i' > i$ or with $i' = i$ and $j' < j$.

Condition c) is omitted when defining $A_{n-1,1}$. Note that in the case where $i = 0$ condition b) is tautologically satisfied, so that the x in A_{0j} are those x for which a specific $|x^v|$ is maximal and which were not selected to be in any previous A_{ij} .

We next prove a lemma which will be important in the proof of the main theorem of this section, Theorem 2.1.

Lemma 2.0. Suppose F_{ij} is a face of $N(f)$ and F is either a face of $N(f)$ properly containing F_{ij} or is $N(f)$ itself. Let w be a point in the interior of F . Then there is a $\mu_0 > 0$ depending on w such that if the number N in the definition of the C_i were chosen sufficiently large (also depending on w), then we have $|x^w| < C_{i+1}^{-\mu_0} \inf_{v \in v(f) \cap F_{ij}} |x^v|$ for all $x \in A_{ij}$.

Proof. The argument is slightly different when F is bounded from when it is unbounded, so we will consider the two cases separately, starting with the unbounded case.

Case 1. F is unbounded.

When F is unbounded there is some direction $c = (c_1, \dots, c_n)$ tangent to F such that each component c_i is nonnegative, with at least one positive. Since w is in the interior of F , we may replace c by a small multiple of itself and say that $w - c \in F$. But since $w - c \in N(f)$, one can necessarily write

$$w - c = \sum_{v_k \in v(f)} \alpha_k v_k + d \quad (2.1)$$

Here each α_k is nonnegative, $\sum_k \alpha_k = 1$, and all the components of d are nonnegative. Therefore

$$|x^w| = |x^{c+d}| \prod_{v_k \in v(f)} |x^{v_k}|^{\alpha_k} \quad (2.2)$$

If $x \in A_{ij}$ then each factor $|x^{v_k}|$ is at most $C_i \inf_{v \in v(f) \cap F_{ij}} |x^v|$ by condition b) of the definition of A_{ij} . As a result, since $\sum_k \alpha_k = 1$ equation (2.2) implies

$$|x^w| \leq C_i |x^{c+d}| \inf_{v \in v(f) \cap F_{ij}} |x^v| \quad (2.3)$$

Since at least one component of $c + d$ is positive, if the diameter C_n^{-1} of the original E were chosen sufficiently small, for some $\mu_0 > 0$ we have

$$C_i |x^{c+d}| < C_{i+1}^{-\mu_0}$$

This gives $|x^w| < C_{i+1}^{-\mu_0} \inf_{v \in v(f) \cap F_{ij}} |x^v|$ as needed. Thus we are done with the lemma in the case where F is unbounded.

Case 2. F is bounded.

We now move onto the case where F is a bounded, and thus compact face of $N(f)$. Let (i', j') be such that $F = F_{i'j'}$. Since w is in the interior of F , one can write

$$w = \sum_{v_k \in v(f) \cap F_{i'j'}} \alpha_k v_k \quad (2.4)$$

Here the α_k are positive numbers summing to one. Since $i' > i$ the point x was not chosen in to be in $A_{i'j'}$ when it was defined. Part a) of the definition of $A_{i'j'}$ holds since $x \in A_{ij}$ and $F_{ij} \subset F = F_{i'j'}$, and part c) holds since x made it to A_{ij} . As a result, part b) of the definition fails; there is some $v_{k_0} \in v(f) \cap F_{i'j'}$ such that

$$|x^{v_{k_0}}| < (C_{i'})^{-1} \sup_{v \in v(f) \cap F_{i'j'}} |x^v|$$

$$= (C_{i'})^{-1} \sup_{v \in v(f) \cap F_{ij}} |x^v| \quad (2.5)$$

The last equality follows from part a) of the definitions of A_{ij} and $A_{i'j'}$. Note that (2.4) can be rewritten as

$$|x^w| = \prod_{v_k \in v(f) \cap F_{i'j'}} |x^{v_k}|^{\alpha_k} \quad (2.6)$$

So in view of (2.5) we have

$$|x^w| < (C_{i'})^{-\frac{1}{\alpha_{k_0}}} \left(\sup_{v \in v(f) \cap F_{ij}} |x^v| \right)^{\alpha_{k_0}} \prod_{v \in v(f), v \neq v_{k_0}} \left(\sup_{v \in v(f) \cap F_{ij}} |x^v| \right)^{\alpha_k} \quad (2.7)$$

Since the α_k sum to 1 the right-hand side of (2.7) is

$$(C_{i'})^{-\frac{1}{\alpha_{k_0}}} \sup_{v \in v(f) \cap F_{ij}} |x^v| \quad (2.8)$$

Since $x \in A_{ij}$, the above is at most

$$\leq (C_{i'})^{-\frac{1}{\alpha_{k_0}}} C_i \inf_{v \in v(f) \cap F_{ij}} |x^v| \quad (2.9)$$

Since $i' > i$, if the constants C_1, C_2, \dots, C_n were chosen to be increasing fast enough then for a uniform constant μ_0 we have $(C_{i'})^{-\frac{1}{\alpha_{k_0}}} C_i \leq C_{i'}^{-\mu_0} \leq C_{i+1}^{-\mu_0}$. Thus (2.9) completes the proof of Lemma 2.0. \square

We now come to the chief result of this section, which gives a useful description of the way in which the terms $|x^v|$ for $v \in v(f) \cap F_{ij}$ dominate on A_{ij} . It is the generalization of Lemma 2.0 of [G3] to arbitrary K , with a streamlined proof.

Theorem 2.1. If the number N in the definition of the C_i is chosen sufficiently large, the following hold for all $i > 0$.

a) If $x \in E$ such that the following two conditions hold, then $x \in A_{ij}$.

- 1) $\inf_{v \in v(f) \cap F_{ij}} |x^v| \geq C_i^{-1} \sup_{v \in v(f)} |x^v|$.
- 2) $\sup_{v \in v(f): v \notin F_{ij}} |x^v| < C_{n-1}^{-1} \inf_{v \in v(f) \cap F_{ij}} |x^v|$.

b) There is a $\mu > 0$ depending only on $N(f)$ such that if $x \in A_{ij}$ then the following two conditions hold.

- 1) $\inf_{v \in v(f) \cap F_{ij}} |x^v| \geq C_i^{-1} \sup_{v \in v(f)} |x^v|$.
- 2) $\sup_{v \in v(f): v \notin F_{ij}} |x^v| \leq C_{i+1}^{-\mu} \inf_{v \in v(f) \cap F_{ij}} |x^v|$.

In the case where $i = 0$, we just have parts 2) of both a) and b).

Proof. Part a) is straightforward: If $x \in E$ satisfies the conditions of part a), then if $F_{i'j'}$ is such that $i' > i$ or $i' = i$ and $j' < j$ then there is at least one vertex v' such that $v' \in F_{i'j'}$ but $v' \notin F_{ij}$. As a result, for all $v \in F_{ij}$ one has $|x^{v'}| < C_{n-1}^{-1}|x^v| \leq C_{j'}^{-1}|x^v|$. So condition b) of the definition of $A_{i'j'}$ will not be satisfied. Thus $x \notin A_{i'j'}$. Conditions 1) and 2) now ensure that x will be selected to be a member of A_{ij} . This completes the proof of part a).

We proceed to the more difficult part b). Note that condition 1) automatically holds by construction, so our goal is to show that condition 2) holds for all $x \in A_{ij}$. Let F_{ij} be any compact face of $N(f)$, let $v \in v(f)$ be on F_{ij} and let $v' \in v(f)$ not on F_{ij} . Our goal is to show that there is a $\mu > 0$ independent of (i, j) such that $|x^{v'}| < C_{i+1}^{-\mu}|x^v|$ whenever $x \in A_{ij}$.

Let F denote the face of $N(f)$ of minimal dimension containing F_{ij} and v' ; in the event there is no such face we define F to be all of $N(f)$ which will serve as a substitute. Let G denote the convex hull of F_{ij} and v' . I claim that G intersects the interior of F . To see this, let $\{w \in \mathbf{R}^n : a_j \cdot w = b_j\}$ be supporting hyperplanes of $N(f)$ transverse to F such that the boundary of F is given by $\{w \in F : a_j \cdot w = b_j \text{ for some } j\}$; in the event that $F = N(f)$ we take these to be all supporting hyperplanes of $N(f)$. Note that for each j , $\{w \in G : a_j \cdot w = b_j\}$ is either all of G or a subset of G of measure zero. It will not be all of G , for if it were then G would be a subset of $\{w \in F : a_j \cdot w = b_j\}$, a face of $N(f)$ of lower dimension than F , and this would contradict the minimality of the dimension of F . We conclude that each $\{w \in G : a_j \cdot w = b_j\}$ is a subset of G of measure zero. Thus so is their union. In particular there exists some $w \in G$ with $a_j \cdot w \neq b_j$ for each j . This is equivalent to w being in the interior of F as needed.

From now on we view w as fixed for a given F_{ij} and v' ; we will be applying Lemma 2.0 and we want the constants μ_0 and N to depend on $N(f)$ only and not the choices of w .

Let $x \in A_{ij}$. Without loss of generality we may assume that $w \notin F_{ij} \cup \{v'\}$ since $F_{ij} \cup \{v'\}$ is a subset of the convex hull of F_{ij} and $\{v'\}$ of measure zero. So w is in the interior of the segment connecting v' to some $a \in F_{ij}$ and we may write $w = \beta v' + (1 - \beta)a$ for some $0 < \beta < 1$. Thus we have

$$|x^{v'}| = |x^w|^{\frac{1}{\beta}} |x^a|^{-\frac{1-\beta}{\beta}} \quad (2.10)$$

Using Lemma 2.0, this is at most

$$C_{i+1}^{-\frac{\mu_0}{\beta}} \left(\inf_{v \in v(f) \cap F_{ij}} |x^v|^{\frac{1}{\beta}} \right) |x^a|^{-\frac{1-\beta}{\beta}} \quad (2.11)$$

Since $a \in F_{ij}$, one may write a as a convex combination $\sum_{v_k \in v(f) \cap F_{ij}} \gamma_k v_k$. Here the γ_k are nonnegative numbers whose sum is 1. Thus we have

$$|x^a| = \prod_{v_k \in v(f) \cap F_{ij}} |x^{v_k}|^{\gamma_k} \quad (2.12)$$

Because each $|x^{v_k}|$ in the above product is at least $\inf_{v \in v(f) \cap F_{ij}} |x^v|$, we have $|x^a| \geq \inf_{v \in v(f) \cap F_{ij}} |x^v|$. Substituting this back into (2.11) gives

$$|x^{v'}| \leq C_{i+1}^{-\frac{\mu_0}{\beta}} \left(\inf_{v \in v(f) \cap F_{ij}} |x^v| \right)^{\frac{1}{\beta}} \left(\inf_{v \in v(f) \cap F_{ij}} |x^v| \right)^{-\frac{1-\beta}{\beta}} \quad (2.13)$$

$$= C_{i+1}^{-\frac{\mu_0}{\beta}} \inf_{v \in v(f) \cap F_{ij}} |x^v| \quad (2.14)$$

Since v' was an arbitrary vertex of $N(f)$ not on F_{ij} , setting μ to be the minimal $\frac{\mu_0}{\beta}$ we get part b) of Theorem 2.1 and we are done. \square

The next lemma is a way of describing how on A_{ij} , the terms $f_\alpha x^\alpha$ for $\alpha \in F_{ij}$ dominate the Taylor expansion of $f(x)$. For the case where $K = \mathbf{R}$, it is Lemma 2.1 of [G3]. The proof there transfers over word for word to general K , so we will not reprove it here.

Lemma 2.2. For any positive integer d , there is a constant $E_{d,f}$ depending on d and $f(x)$ and a positive constant η depending on $N(f)$ such that on A_{ij} we have

$$\sum_{\alpha \notin F_{ij}} |f_\alpha| |\alpha|^d |x^\alpha| < E_{d,f} (C_{i+1})^{-\eta} \sup_{v \in v(f)} |x^v|$$

Examples.

Example 1. Suppose $f(x, y)$ is a function of two real variables. Let e be an edge of $N(f)$. Then e joins two vertices of $N(f)$, which we denote by $v_1 = (\alpha_1, \beta_1)$ and $v_2 = (\alpha_2, \beta_2)$, where $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$. Let $-\frac{1}{m}$ be the slope of this edge. Then $\alpha_1 + m\beta_1 = \alpha_2 + m\beta_2$. So whenever $|y| = |x|^m$ one has that $|x^{\alpha_1} y^{\beta_1}| = |x^{\alpha_2} y^{\beta_2}|$. More generally, $|x^{\alpha_1} y^{\beta_1}|$ and $|x^{\alpha_2} y^{\beta_2}|$ are within a factor of C_1 of each other if (x, y) is in the domain B_e defined by

$$B_e = \{(x, y) : C_1^{-\frac{1}{\beta_2 - \beta_1}} |x|^m < |y| < C_1^{\frac{1}{\beta_2 - \beta_1}} |x|^m\}$$

Furthermore, for any other vertex $v = (\alpha, \beta)$ of $N(f)$, if $|y| = C|x|^m$ then $|x^\alpha y^\beta| = C^\beta |x|^c$ for some $c > \alpha_1 + m\beta_1 = \alpha_2 + m\beta_2$. Thus if we are in a small enough neighborhood of the origin, then on B_e each such $|x^\alpha y^\beta|$ will be less than both $C_1^{-1} |x^{\alpha_1} y^{\beta_1}|$ and $C_1^{-1} |x^{\alpha_2} y^{\beta_2}|$. Hence if E denotes the square $\{(x, y) \in \mathbf{R}^2 : 0 < |x|, |y| < C_2^{-1}\}$ for large enough C_2 , we may take our A_{1j} to be the sets $B_e \cap E$ for the various edges e of $N(f)$.

As for the A_{0j} , between any two A_{1j} corresponding to intersecting edges there will be an A_{0j} of the form $\{(x, y) \in E : C_1^{\frac{1}{\beta_2 - \beta_1}} |x|^m < |y| < C_1^{-\frac{1}{\beta_2' - \beta_1'}} |x|^{m'}\}$. There will be two additional A_{0j} 's, one of the form $\{(x, y) \in E : C_1^{\frac{1}{\beta_2 - \beta_1}} |x|^m < |y|\}$ and the other of the form $\{(x, y) \in E : |y| < C_1^{-\frac{1}{\beta_2 - \beta_1}} |x|^m\}$, corresponding the uppermost and lowermost

vertices of $N(f)$ respectively. We are excluding the case where $N(f)$ has only one vertex, in which case there are no A_{1j} 's and exactly one A_{0j} , consisting of all of E .

Example 2. Suppose now we are in three dimensions, again in the real case. Let F be a two-dimensional face of $N(f)$. If $n = (n_1, n_2, n_3)$ denotes a vector normal to this face with each $n_i > 0$, then for $v \in v(f)$, $n \cdot v$ is maximized for $v \in F$. In particular, on a curve $\{(c_1 t^{n_1}, c_2 t^{n_2}, c_3 t^{n_3}) : t > 0\}$, the functions $|x^v|$ for $v \in F$ dominate the $|x^v|$ for $v \notin F$ as $t \rightarrow 0$, while the x^v for $v \in F$ remain within a fixed factor of each other. Thus by Theorem 2.1 the A_{2j} corresponding to F will be a "horn" approximately consisting of a union of such curves with positive c_1, c_2 , and c_3 within some power of C_2 of each other, along with its 7 reflected horns about the coordinate planes.

Now let e be a one-dimensional edge of $N(f)$, joined by two vertices v_1 and v_2 . By Theorem 2.1, the corresponding A_{1j} will contain points in a small cube centered at the origin for which $|x^{v_1}|$ and $|x^{v_2}|$ are within a factor of C_1 of each other, such that $|x^v|$ for vertices v not on e are of magnitude less than $C_2^{-\mu}$ times the smaller of $|x^{v_1}|$ or $|x^{v_2}|$. It will contain all points in this cube for which $|x^{v_1}|$ and $|x^{v_2}|$ are within C_1 of each other, and for which $|x^v|$ for vertices v not on e are of magnitude less than C_2^{-1} times the smaller of $|x^{v_1}|$ or $|x^{v_2}|$. In the event that there are no vertices other than v_1 and v_2 , then A_{1j} consists of all points in the cube for which $|x^{v_1}|$ and $|x^{v_2}|$ are within C_1 of each other, similar to the situation for the A_{1j} in the 2-dimensional case. It can be shown that for the vertices not on e one may restrict consideration to vertices v not on e which are on two-dimensional compact faces of $N(f)$ containing e .

Geometrically, the A_{1j} again can be viewed as horns in the upper right quadrant along with 7 reflections about the coordinate planes, but this time the horns will be not be contained in any bundle of curves in the sense the A_{2j} were.

Lastly if v is a vertex of $N(f)$, then by Theorem 2.1 the corresponding A_{0j} will consist of points in a small cube centered at the origin for which $|x^v| > C_1^\mu |x^{v'}|$ for all other vertices v' of $N(f)$, and will contain all points in this cube for which $|x^v| > C_2 |x^{v'}|$ for all vertices v' of $N(f)$. Similar to above it can be shown that one can restrict to v' on edges of $N(f)$ containing v . In the event that v is the only vertex of $N(f)$, then A_{0j} will be an entire cube centered at the origin, and there will be no other A_{ij} .

Example 3. Consider the case in n dimensions where $N(f)$ has a vertex on each coordinate axis at some height h , the same for each axis, and no other vertices. Then the A_{ij} are as follows. For each nonempty $I \subset \{1, \dots, n\}$ there is a face F_{ij} for $i = |I|$ which is the convex hull of the vertices on the x_i axis for $i \in I$. By Theorem 2.1, the corresponding A_{ij} consists of points in a small cube centered at the origin for which the $|x_i|$ for $i \in I$ are within a factor of $C_i^{\frac{1}{h}}$ of the maximal $|x_i|$, achieved for some $i \in I$, and such that $|x_i|$ for $i \notin I$ are less than $C_{i+1}^{-\frac{\mu}{h}} |x_i|$ for all $i \in I$. Furthermore, A_{ij} contains all the points in this cube for which $|x_i|$ for $i \in I$ are within a factor of $C_i^{\frac{1}{h}}$ of the maximal $|x_i|$, and such that

$|x_i|$ for $i \notin I$ are less than $C_{n-1}^{-\frac{k}{h}}|x_i|$ for all $i \in I$.

3. Performing the blowups; the A'_{ijk} in the new coordinates.

Our first lemma gives a special case of resolution of singularities which will be used as a building block for the resolution of singularities algorithm of this paper. Let $K_0 = K - \{0\}$ as before. In our first lemma, one starts with the cube $F = \{x \in K_0^n : |x_l| \leq 1 \text{ for all } l\}$, divides it into sets $F_1 = \{x \in F : |x_j| \leq |x_k|\}$ and $F_2 = \{x \in F : |x_j| > |x_k|\}$ for some variables x_j and x_k . One then blows up the sets F_1 and F_2 , changing variables $x_j = x_j x_k$ in F_1 and $x_k = x_k x_j$ in F_2 , leaving the other variables unchanged. The result are two cubes F'_1 and F'_2 (not necessarily closed this time), which we then blow up in an analogous fashion. This will be repeated as needed to obtain the lemma.

Lemma 3.1. Let $F = \{x \in K_0^n : |x_l| \leq 1 \text{ for all } l\}$, and let $m_1(x), \dots, m_p(x)$ be distinct monomials. Then there is a way of starting with F and then doing a finite sequence of blowups in the above sense after which one has cubes E_1, \dots, E_q (now not necessarily closed) such that in the blown-up coordinates of a given cube E_i , the monomials $m_1(x), \dots, m_p(x)$ are new monomials $\tilde{m}_1(x), \dots, \tilde{m}_p(x)$ whose exponents are linearly ordered. That is, if $\tilde{m}_i(x) = x^\alpha$ and $\tilde{m}_j(x) = x^\beta$, then either $\beta_k \geq \alpha_k$ for each k or $\beta_k \leq \alpha_k$ for each k .

Proof. We can write the collection of all possible $\frac{m_i(x)}{m_j(x)}$ as $\{\frac{x^{\alpha_{ij}}}{x^{\beta_{ij}}} : i, j = 1, \dots, p\}$ where for a given i and j the monomials $x^{\alpha_{ij}}$ and $x^{\beta_{ij}}$ have no variable in common. In order to prove the lemma, we need to perform a sequence of blowups after which on a given E_k , the blown up $\frac{x^{\alpha_{ij}}}{x^{\beta_{ij}}}$ becomes either a monomial or a reciprocal of a monomial. If one accomplishes this for one $\frac{x^{\alpha_{ij}}}{x^{\beta_{ij}}}$, this function will still be a monomial or a reciprocal of a monomial respectively after any future blowups. So it suffices to prove we can do this for a single $\frac{x^{\alpha_{ij}}}{x^{\beta_{ij}}}$ as we can then just repeat the process for each $\frac{x^{\alpha_{ij}}}{x^{\beta_{ij}}}$ one after another.

So we suppress the indices i and j and focus our attention on a single function $\frac{x^\alpha}{x^\beta}$, for which we will show there exists the desired sequence of blowups. Writing $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, we let $a = \max_m \alpha_m$, and let $b = \max_m \beta_m$. The proof is first by induction on $c = \max(a, b)$. Given c we induct on the number M of variables x_i appearing to the c power in x^α or x^β (To be precise, in our arguments either $M > 1$ and we will decrease M by 1, or $M = 1$ and we decrease the value of c .) The induction starts with the trivial $(c, M) = (1, 1)$ case.

Case 1. a = b. Suppose $a = b = c$. Let x_j be a variable appearing in x^α to the c th power, and let x_k be a variable appearing in x^β to the c th power. If one performs a blowup in the x_j and x_k variables, one obtains two regions. On the first region x_j becomes $x_j x_k$ with the other variables remaining fixed, and on the second region x_k becomes $x_j x_k$ with the other variables remaining fixed. On the first region the x_j^c in the numerator becomes $x_j^c x_k^c$, and the new x_k^c appearing in the numerator can be cancelled with the x_k^c appearing in the denominator. Similarly, in the second region, an extra x_j^c appears in the denominator which can be cancelled with the x_j^c in the numerator. Thus in either case,

one of the variables appearing to the c th power in the overall ratio $\frac{x^\alpha}{x^\beta}$ disappears and the expression otherwise is unchanged. Thus we have decreased M by 1 and we may now apply the induction hypothesis. Thus we are done with the argument for when $a = b$.

Case 2. $a \neq b$. Our proof is symmetric in α and β , so without loss of generality we assume that $a > b$. Given our $(c, M)(= (a, M))$, we induct on the number N of indices i for which x_i appears in x^β to any power. This final induction starts with the $N = 0$ case where $x^\beta = 1$ and the result is immediate.

Let (a, M, N) be the case we are currently trying to prove. Let x_j be a variable appearing to the a th power in x^α , and let x_k be any variable appearing in x^β . Define d to be the number such that x_k appears to the d power in x^β , where $d < a$. We perform a blowup in the x_j and x_k variables. We obtain two domains. On the first domain, the x_j^a in the numerator becomes $x_j^a x_k^a$. The x_k^a in the numerator can be cancelled with the x_k^d the denominator, resulting in an x_k^{a-d} in the numerator and there is no longer the variable x_k appearing in the denominator. Thus N is decreased and by induction hypothesis we are done for the first domain. On the second domain, the x_k^d in the denominator becomes $x_j^d x_k^d$, and the x_j^a in the numerator can be cancelled with the x_j^d in the denominator, obtaining a x_j^{a-d} in the numerator in place of x_j^a , with the function $\frac{x^\alpha}{x^\beta}$ otherwise unchanged. Thus if $M > 1$, M becomes $M - 1$, and if $M = 1$ $c = a$ gets decreased. Thus by induction hypothesis in the second domain we again are done. This concludes the proof. \square

Blowing up the domains A_{ij}

We now let $f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}$ be a the power series of an analytic function on a neighborhood of the origin. We apply Lemma 3.1 to the monomials $\{x^v : v \text{ is a vertex of the Newton polygon } N(f)\}$, obtaining the resulting collection of cubes, which we denote by E_1, \dots, E_q . Let γ_k denote the composition of the blowups used in creating E_k . Then there is a linear function L_k such that the coordinate change γ_k converts these x^v into the monomials $x^{L_k(v)}$ and furthermore the exponents $\{L_k(v) : v \text{ is a vertex of } N(f)\}$ are linearly ordered under the lexicographic ordering by Lemma 3.1.

We next examine what happens to the domains A_{ij} from Section 2 under these sequences of blowups. Define $A_{ijk} = A_{ij} \cap \gamma_k(E_k)$. Since the A_{ij} were (up to a set of measure zero) a subdivision of the cube $E = \{x \in K_0^n : |x_l| < C_n^{-1} \text{ for all } l\}$, the A_{ijk} are also a subdivision of E . Define $A'_{ijk} = \gamma_k^{-1}(A_{ijk}) = \gamma_k^{-1}(A_{ij}) \cap E_k$, and define $G_k = \gamma_k^{-1}(E) \cap E_k$. Then for a fixed k , the A'_{ijk} are a subdivision of G_k . Note that if we write $\gamma_k(x) = (\gamma_{k1}(x), \dots, \gamma_{kn}(x))$, then we have

$$G_k = \{x \in E_k : |\gamma_{kl}(x)| < C_n^{-1} \text{ for all } l\}$$

Next, we examine how Theorem 2.1 translates in the blown up coordinates for a given k . $G_k \subset E_k \subset \{x \in K_0^n : |x_l| \leq 1 \text{ for all } l\}$, so since the exponents of the transformed monomials $x^{L_k(v)}$ are linearly ordered, there is a corresponding ordering of the functions

$x^{L_k(v_i)}$; for a given p and q either $|x^{L_k(v_p)}| \leq |x^{L_k(v_q)}|$ throughout G_k , or $|x^{L_k(v_q)}| \leq |x^{L_k(v_p)}|$ throughout G_k .

This makes translating Theorem 2.1 in the blown up coordinates rather straightforward. As in section 2, we let $v(f)$ denote the set of vertices of $N(f)$. Since the functions $\{|x^{L_k(v)}| : v \in v(f)\}$ are linearly ordered on G_k in the sense of the above paragraph, there are vertices v', v'', v''' of $N(f)$ such that throughout G_k one has the following

$$\begin{aligned} |x^{L_k(v')}| &= \sup_{v \in v(f)} |x^{L_k(v)}| \\ |x^{L_k(v'')}| &= \inf_{v \in v(f) \cap F_{ij}} |x^{L_k(v)}| \\ |x^{L_k(v''')}| &= \sup_{v \in v(f) \cap F_{ij}^c} |x^{L_k(v)}| \end{aligned}$$

Note that $L_k(v') \leq L_k(v'')$ under the lexicographical ordering, with strict inequality if and only if $i > 0$. Note also that by Theorem 2.1 b) part 2, if (i, j, k) is such that A'_{ijk} is nonempty then we have $L_k(v'') < L_k(v''')$. Thus for $i > 0$ we may define the nonconstant monomial $p_{ijk}(x) = x^{L_k(v'') - L_k(v')}$ and for all i we may define the nonconstant monomial $q_{ijk}(x) = x^{L_k(v''') - L_k(v'')}$. Then Theorem 2.1 translates into the following.

Lemma 3.2. If the number N in the definition of the C_i is chosen sufficiently large, then for all $i > 0$ and all j and k we have (up to a set of measure zero)

$$\begin{aligned} \{x \in G_k : |p_{ijk}(x)| \geq C_i^{-1} \text{ and } |q_{ijk}(x)| < C_{n-1}^{-1}\} &\subset A'_{ijk} \\ A'_{ijk} &\subset \{x \in G_k : |p_{ijk}(x)| \geq C_i^{-1} \text{ and } |q_{ijk}(x)| < C_{i+1}^{-\mu}\} \end{aligned} \quad (3.1a)$$

If $i = 0$, then for all j and k we have (up to a set of measure zero)

$$\{x \in G_k : |q_{0jk}(x)| < C_{n-1}^{-1}\} \subset A'_{0jk} \subset \{x \in G_k : |q_{0jk}(x)| < C_1^{-\mu}\} \quad (3.1b)$$

Since each G_k is a subset of the unit cube, if $x \in G_k$ such that $|p_{ijk}(x)| \geq C_i^{-1}$, then $|x_r| \geq C_i^{-\frac{1}{\deg(p_{ijk})}}$ for all r such that x_r appears in $p_{ijk}(x)$. This fact will be highly relevant for our arguments. For example, it implies that we can assume that not all variables appear in $p_{ijk}(x)$. For if they did, then on A'_{ijk} we would have that $|x_r| > C_i^{-\frac{1}{\deg(p_{ijk})}}$ for all r . Then assuming C_{i+1} was chosen large enough relative to C_i , the right-hand side of (3.1a) implies that A'_{ijk} is empty.

Thus when $i > 0$, permuting the variables if necessary we may let p be such that the variables x_1, \dots, x_p do not appear in $p_{ijk}(x)$ and the variables x_{p+1}, \dots, x_n do appear in $p_{ijk}(x)$. We rename our variables now, letting $y_l = x_l$ for $1 \leq l \leq p$, and $z_l = x_{l+p}$ for $1 \leq l \leq n - p$. We correspondingly write $q_{ijk}(x)$ as $s_{ijk}(y)t_{ijk}(z)$ and $p_{ijk}(x)$ as $p_{ijk}(z)$.

For consistency of notation, when $i = 0$ we rename each x_l variable y_l , and then rename the function $q_{0jk}(x)$ as $s_{0jk}(y)$. Then Lemma 3.2 can be recast as follows.

Lemma 3.2'. If $i > 0$, then we have

$$\begin{aligned} \{(y, z) \in G_k : |s_{ijk}(y)| < C_{n-1}^{-1} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} &\subset A'_{ijk} \\ A'_{ijk} &\subset \{(y, z) \in G_k : |s_{ijk}(y)| < C_{i+1}^{-\mu} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} \end{aligned} \quad (3.2a)$$

If $i = 0$ then we have

$$\{y \in G_k : |s_{0jk}(y)| < C_{n-1}^{-1}\} \subset A'_{0jk} \subset \{y \in G_k : |s_{0jk}(y)| < C_1^{-\mu}\} \quad (3.2b)$$

Proof. The $i = 0$ case is identical to that of Lemma 3.2, so we assume $i > 0$. (3.1a) can be rewritten as

$$\begin{aligned} \{(y, z) \in G_k : |s_{ijk}(y)t_{ijk}(z)| < C_{n-1}^{-1} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} &\subset A'_{ijk} \\ A'_{ijk} &\subset \{(y, z) \in G_k : |s_{ijk}(y)t_{ijk}(z)| < C_{i+1}^{-\mu} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} \end{aligned} \quad (3.3)$$

Since $C_i^{-\frac{1}{\deg(p_{ijk})}} \leq |z_l| \leq 1$ for all l on A'_{ijk} , on A'_{ijk} one has $C_i^{-\frac{\deg(t)}{\deg(p)}} \leq |t_{ijk}(z)| \leq 1$. Thus (3.3) implies

$$\begin{aligned} \{(y, z) \in G_k : |s_{ijk}(y)| < C_{n-1}^{-1} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} &\subset A'_{ijk} \\ A'_{ijk} &\subset \{(y, z) \in G_k : |s_{ijk}(y)| < C_i^{\frac{\deg(t)}{\deg(p)}} C_{i+1}^{-\mu} \text{ and } |p_{ijk}(z)| \geq C_i^{-1}\} \end{aligned} \quad (3.4)$$

If the constants C_i were chosen to increase rapidly enough, one has $C_i^{\frac{\deg(t)}{\deg(p)}} C_{i+1}^{-\mu} < C_{i+1}^{-\frac{\mu}{2}}$. Thus replacing μ by $\frac{\mu}{2}$ if necessary equation (3.2a) follows and we are done. \square

Note that the right-hand sides of (3.2a) – (3.2b) ensure that there is at least one y_l variable appearing in $s_{ijk}(y)$ (i.e. no $s_{ijk}(y)$ is ever just 1).

Lemma 3.3. For each (i, j, k) with $i > 0$ there is an α_{ijk} such that each $x^{L_k(v)}$ for a vertex $v \in F_{ij}$ can be written as $y^{\alpha_{ijk}} z^\eta$ for some η , and such that each $x^{L_k(v)}$ for a vertex $v \notin F_{ij}$ can be written as $y^{\alpha'_{ijk}} z^\eta$ with $\alpha'_{ijk} > \alpha_{ijk}$ under the lexicographic ordering.

Proof. Since the z variables were defined to be the variables for which the minimal $|x^{L_k(v)}|$ for a vertex $v \in F_{ij}$ differs from the maximal $|x^{L_k(v)}|$ over all v (and therefore over all $v \in F_{ij}$ by part a) of the definition of the A_{ij}), all $x^{L_k(v)}$ for $v \in F_{ij}$ will be the same in the y variables. In other words, there's a fixed α_{ijk} such that each $x^{L_k(v)}$ for $v \in F_{ij}$ can be written as $y^{\alpha_{ijk}} z^{L'(v)}$ for some linear map L' . Furthermore, since $s_{ijk}(y)t_{ijk}(z)$ was defined as the ratio of the minimal $|x^{L_k(v)}|$ for $v \in F_{ij}$ (now known to be of the form $y^{\alpha_{ijk}} z^\eta$ for some η) to the maximal $|x^{L_k(v)}|$ for $v \notin F_{ij}$ and $s_{ijk}(y)$ has at least one y_l

variable appearing by above, each $x^{L_k(v)}$ for $v \notin F_{ij}$ will also contain this y_l variable and therefore can be written as $y^{\alpha'_{ijk}} z^{L'(v)}$ with $\alpha'_{ijk} > \alpha_{ijk}$ under the lexicographic ordering. This completes the proof. \square

Example. The two-dimensional case.

Suppose $f(x, y)$ is a function of two variables. Then by Lemma 3.2', a given A'_{1jk} satisfies

$$\begin{aligned} \{(x, y) \in G_k : |s_{1jk}(x)| < C_1^{-1} \text{ and } |p_{1jk}(y)| \geq C_1^{-1}\} &\subset A'_{1jk} \\ A'_{1jk} &\subset \{(x, y) \in G_k : |s_{1jk}(x)| < C_2^{-\mu} \text{ and } |p_{1jk}(y)| \geq C_1^{-1}\} \end{aligned} \quad (3.5)$$

The structure of the G_k provides restrictions on what an A'_{1jk} satisfying (3.5) can be. Since we are in two dimensions, up to a set of measure zero G_k is given by

$$G_k = \{(x, y) : |x|, |y| < 1, |\gamma_{k1}(x, y)| < C_2^{-1}, |\gamma_{k2}(x, y)| < C_2^{-1}\} \quad (3.6)$$

Here $\gamma_{k1}(x, y)$ and $\gamma_{k2}(x, y)$ are monomials and are the components of the composition of blowups $\gamma_k(x, y)$. If either $\gamma_{k1}(x, y)$ or $\gamma_{k2}(x, y)$ is a function of y only, then $G_k \subset \{(x, y) : |y| > C_2^{-\frac{1}{m}}\}$ for some m , and if the C_i were chosen to be increasing fast enough any set A'_{1jk} satisfying (3.5) would be empty. So γ_{k1} and γ_{k2} must depend on x as well as possibly y .

Since $p_{1jk}(y)$ is a monomial, if y is such that $|p_{1jk}(y)| > C_1^{-1}$, then $|y| > C_1^{-\frac{1}{\deg(p)}}$. Then a condition $|x^a y^b| < C_2^{-1}$ with $a > 0$, $b \geq 0$ implies that $|x| < C_2^{-\frac{1}{a}} C_1^{\frac{b}{a \deg(p)}}$. Assuming the C_i were chosen to increase fast enough this means that on $G_k \cap A'_{1jk}$ for some a we have $|x| < C_2^{-\frac{1}{2a}}$. In this case the condition in (3.5) that $|s_{1jk}(x)| < C_1^{-1}$ or $C_2^{-\mu}$ are superfluous, and so whenever $i = 1$ we have

$$A'_{1jk} = \{(x, y) \in G_k : |p_{1jk}(y)| \geq C_1^{-1}\} \quad (3.7)$$

Since $p_{1jk}(y)$ is just a monomial, this is the same as

$$A'_{1jk} = \{(x, y) \in G_k : |y| \geq C_1^{-\frac{1}{\deg(p)}}\} \quad (3.7')$$

Note that by above, $|x| < C_2^{-\frac{1}{2a}}$ on A'_{1jk} .

We now look at the sets A'_{0jk} . By Lemma 3.2' we have

$$\{(x, y) \in G_k : |s_{0jk}(x, y)| < C_1^{-1}\} \subset A'_{0jk} \subset \{(x, y) \in G_k : |s_{0jk}(x, y)| < C_1^{-\mu}\} \quad (3.8)$$

If $s_{0jk}(x, y)$ contains both the x and y variables, then by (3.6) and the left-hand side of (3.8) the set A'_{0jk} will be all of G_k . If $s_{0jk}(x, y)$ is a function of just one variable, say y ,

there are a few possibilities. If one or both of γ_{k1} and γ_{k2} is a function of y only, then by (3.6) A'_{0jk} will be all of G_k like above. If one or both are functions of x only, then $G_k \subset \{(x, y) : |x| < C_2^{-d}, |y| < 1\}$ for some $d > 0$, so by (3.8) A'_{0jk} will be contained in the small box $\{(x, y) : |x| < C_2^{-d}, |y| < C_1^{-\frac{\mu}{\deg(s_{0jk})}}\}$. If γ_{k1} and γ_{k2} are both functions of x and y , then A'_{0jk} is a short but wide region of $|y|$ height at most $C_1^{-\frac{\mu}{\deg(s_{0jk})}}$ but which extends in the x -direction all the way to $|x| = 1$.

Note that due to the forms we have determined for the various A'_{ijk} , if $G_k - A'_{0jk}$ is nonempty it must consist of a single $A'_{1j'k'}$.

In preparation for our next lemma, for a given face F_{ij} of $N(f)$ write $f(x) = f_{F_{ij}}(x) + g_{ij}(x)$ (cf Definition 2.2). Recalling that γ_k was defined to be the composition of blowups corresponding to E_k , we let $u_{ijk} = f_{F_{ij}} \circ \gamma_k$ and let $v_{ijk} = g_{ij} \circ \gamma_k$. It is a consequence of Lemma 3.3 that when $i > 0$ each monomial of the polynomial $u_{ijk}(y, z)$ is of the form $y^{\alpha_{ijk}} z^\eta$ and each monomial of the Taylor series of $v_{ijk}(y, z)$ at the origin is of the form $y^{\alpha'_{ijk}} z^\eta$ for some $\alpha'_{ijk} \geq \alpha_{ijk}$. When $i = 0$ the same holds if we interpret z^η to be 1. In particular, when $i > 0$ we can write $u_{ijk}(y, z) = y^{\alpha_{ijk}} U_{ijk}(z)$ and $v_{ijk}(y, z) = y^{\alpha_{ijk}} V_{ijk}(y, z)$ for analytic functions U_{ijk} and V_{ijk} , so that $f \circ \gamma_k(y, z) = y^{\alpha_{ijk}} F_{ijk}(y, z)$ where $F_{ijk}(y, z) = U_{ijk}(z) + V_{ijk}(y, z)$. When $i = 0$ the analogous statement is that $f \circ \gamma_{0jk}(y)$ can be written in the form $y^{\alpha_{0jk}} F_{0jk}(y)$ for some analytic $F_{0jk}(y)$ with nonzero constant term.

Our next lemma shows that if o_{ij} is the maximum order of any zero of $f_{F_{ij}}(x)$ on K_0^n , then if $i > 0$ on A'_{ijk} we have corresponding lower bounds on some derivative of $f \circ \gamma_k(y, z)$ in the z variables of order at most o_{ij} , with an analogous statement in the $i = 0$ case.

Lemma 3.4. Assuming the constants C_i were chosen to be increasing sufficiently quickly the following hold.

a) Suppose $i > 0$. Let o_{ij} be the maximum order of any zero of $f_{F_{ij}}(x)$ on K_0^n . Let $\pi(y, z) = z$ denote projection onto the z coordinates, and let $B_{ijk} = \pi(A'_{ijk})$. Then B_{ijk} can be written as a finite union $B_{ijk} = \cup_l B_{ijkl}$ such that for each l there is a directional derivative $\sum_{m=1}^{n-p} \beta_m \partial_{z_m}$ with each β_m rational and $\sum_m |\beta_m| = 1$, a $0 \leq q \leq o_{ij}$, and a constant $\delta_{ijkl} > 0$ such that for all $(y, z) \in A'_{ijk}$ with $z \in B_{ijkl}$ we have

$$\left| \left(\sum_m \beta_m \partial_{z_m} \right)^q F_{ijk}(y, z) \right| \geq \delta_{ijkl} \quad (3.9a)$$

Equivalently, $\left| \left(\sum_m \beta_m \partial_{z_m} \right)^q (f \circ \gamma_k)(y, z) \right| \geq \delta_{ijkl} |y^{\alpha_{ijk}}|$. Here α_{ijk} is as in Lemma 3.3.

b) Suppose $i = 0$, so that F_{0j} is a vertex v of $N(f)$. Let α_{0jk} be such that x^v becomes $y^{\alpha_{0jk}}$ in the blown-up coordinates. Then there is a constant $\delta_{0jk} > 0$ such that for all $y \in A'_{0jk}$ we have

$$|F_{0jk}(y)| \geq \delta_{0jk} \quad (3.9b)$$

Equivalently, $|f \circ \gamma_k(y)| \geq \delta_{0jk} |y^{\alpha_{0jk}}|$.

Proof. When $i > 0$, the condition that all zeroes of $f_{F_{ij}}(x)$ on K_0^n have order at most o_{ij} implies that all zeroes of $f_{F_{ij}} \circ \gamma_k(y, z) = u_{ijk}(y, z) = y^{\alpha_{ijk}} U_{ijk}(z)$ on K_0^n also have order at most o_{ij} . Thus the same is true for $U_{ijk}(z)$. As is well known (see Ch 8 of [St] for details), the space of partial derivative operators ∂^α of order q is spanned by a finite list of directional derivative operators $(\sum_m \beta_m \partial_{z_m})^q$, where each β_m is rational. Without loss of generality, we may assume $\sum_m |\beta_m| = 1$. Thus when $i > 0$ one can write B_{ijk} as the finite union $\cup_l B_{ijkl}$ such that for each l there is a directional derivative $\sum_m \beta_m \partial_{z_m}$, a $0 \leq q \leq o_{ij}$, and a $\delta_{ijkl} > 0$ such that for all $z \in B_{ijkl}$ we have

$$|(\sum_m \beta_m \partial_{z_m})^q U_{ijk}(z)| > 2\delta_{ijkl} \quad (3.10)$$

As a result, for (y, z) with $z \in B_{ijkl}$ we have

$$|(\sum_m \beta_m \partial_{z_m})^q u_{ijk}(y, z)| \geq 2\delta_{ijkl} |y^{\alpha_{ijk}}| \quad (3.11)$$

If $i = 0$, (3.11) still holds if we take $q = 0$ since $u_{0jk}(y)$ is just a monomial $U_{0jk} y^{\alpha_{0jk}}$ here.

Write the Taylor expansion of $v_{ijk}(y, z)$ as $\sum_{\gamma, \delta} g_{\gamma\delta} y^\gamma z^\delta$ (Only the y variable appears if $i = 0$). When applying the derivative $(\sum_m \beta_m \partial_{z_m})^q$ to a given monomial $g_{\gamma\delta} y^\gamma z^\delta$, one obtains at most n^q terms, each having a coefficient with magnitude at most $(|\gamma| + |\delta|)^q |g_{\gamma\delta}|$. Since each of the $|z_m|$ are bounded below by C_i^{-1} on A'_{ijk} , the magnitude of the corresponding term is at most $C_i^q (|\gamma| + |\delta|)^q |g_{\gamma\delta} y^\gamma z^\delta|$. Thus the magnitude of the sum of all of these at most n^q terms is at most $C_i^q n^q (|\gamma| + |\delta|)^q |g_{\gamma\delta} y^\gamma z^\delta| \leq C_i^{o_{ij}} n^{o_{ij}} (|\gamma| + |\delta|)^{o_{ij}} |g_{\gamma\delta} y^\gamma z^\delta|$.

Translating this back into the original x -coordinates, if $f_\alpha x^\alpha$ denotes the term of $f(x)$'s Taylor expansion corresponding to $g_{\gamma\delta} y^\gamma z^\delta$, on A'_{ijk} we have

$$|(\sum_m \beta_m \partial_{z_m})^q g_{\gamma\delta} y^\gamma z^\delta| \leq C_i^{o_{ij}} n^{o_{ij}} (|\gamma| + |\delta|)^{o_{ij}} |f_\alpha x^\alpha|$$

Because the transition from x to (y, z) coordinates is via the monomial map γ_k , there exists a constant D_k such that $|\gamma| + |\delta| \leq D_k |\alpha|$. Therefore on A'_{ijk} we have

$$|(\sum_m \beta_m \partial_{z_m})^q g_{\gamma\delta} y^\gamma z^\delta| \leq (C_i D_k n)^{o_{ij}} |\alpha|^{o_{ij}} |f_\alpha x^\alpha| \quad (3.12)$$

Since the $g_{\gamma\delta} y^\gamma z^\delta$ are the terms of $v_{ijk}(y, z)$'s Taylor expansion, by definition of $v_{ijk}(y, z)$ the terms $f(x)$'s Taylor expansion corresponding to the $g_{\gamma\delta} y^\gamma z^\delta$ are exactly those with $\alpha \notin F_{ij}$. Thus adding (3.12) over all (γ, δ) we get

$$|(\sum_m \beta_m \partial_{z_m})^q v_{ijk}(y, z)| \leq (C_i D_k n)^{o_{ij}} \sum_{\alpha \notin F_{ij}} |\alpha|^{o_{ij}} |f_\alpha| |x^\alpha| \quad (3.13)$$

By Lemma 2.2, for some $\eta > 0$ the right-hand side of (3.13) is bounded by $C_i^{o_{ij}}$ times a constant depending on f times $C_{i+1}^{-\eta} \sup_{v \in v(f)} |x^v|$. Note that since $x \in A_{ij}$, this supremum $|x^v|$ will occur for some $v \in F_{ij}$, so $|x^v|$ can be written as $|y^{\alpha_{ijk}} z^\xi|$ for some ξ . This in turn is at most $|y^{\alpha_{ijk}}|$ since each $|z_m| \leq 1$. So for a constant M depending on f we have

$$\left| \left(\sum_m \beta_m \partial_{z_m} \right)^q v_{ijk}(y, z) \right| \leq M C_i^{o_{ij}} C_{i+1}^{-\eta} |y^{\alpha_{ijk}}|$$

By (3.10), δ_{ijkl} is bounded below by the minimum absolute value of certain derivatives of $U_{ijk}(z)$ on $\{z \in K^{n-p} : C_i^{-1} \leq |z_l| \leq 1 \text{ for all } l\}$. So as long as the constants C_i were chosen to be increasingly sufficiently quickly, for $(y, z) \in A'_{ijk}$ we have for all l that

$$\left| \left(\sum_m \beta_m \partial_{z_m} \right)^q v_{ijk}(y, z) \right| \leq \delta_{ijkl} |y^{\alpha_{ijk}}| \quad (3.14)$$

Combining (3.14) with (3.11) gives (3.9a) – (3.9b) and we are done. \square

Remark. By Lemma 3.3 and the fact that the $|z_m|$ are bounded below, in (3.9a) – (3.9b) we can replace $y^{\alpha_{ijk}}$ or $y^{\alpha_{0jk}}$ by $m \circ \gamma_k(y, z)$ or $m \circ \gamma_k(y)$ respectively, where $m(x) = x^v$ for any vertex v of $N(f)$ on F_{ij} . Note the constants δ_{ijkl} may change.

The final lemma of this section shows how one can form partitions of unity into functions that can be blownup by the γ_k 's into smooth functions supported on sets that are approximately subsets of A'_{ijk} , and on whose support (3.9a) or (3.9b) holds.

Lemma 3.5. Let $\phi(x) \in C_c(E_0)$, where E_0 is the cube $\{x \in K_0^n : |x_l| < C_n^{-1} \text{ for all } l\}$ which was partitioned modulo a set of measure zero into the A_{ij} in section 2. Then one can write $\phi(x) = \sum_{ijkl} \phi_{ijkl}(x)$, where each ϕ_{ijkl} is supported in E_0 and each $\rho_{ijkl} = \phi_{ijkl} \circ \gamma_k$ when restricted to K_0^n is a smooth function which extends to a smooth function on all of K^n such that the following hold.

a) Suppose $i > 0$. Let $Y = \{y : |y_l| < 2 \text{ for all } l\}$, $Y' = \{y : |y_l| < \frac{1}{2} C_n^{-1} \text{ for all } l\}$, and $Z = \{z : \frac{1}{2} C_i^{-1} < |z_m| < 2 \text{ for all } m\}$. For some nonempty open $Z' \subset Z$ we have

$$Y' \times Z' \subset \text{supp}(\rho_{ijkl}) \subset Y \times Z \quad (3.15)$$

Furthermore, for some $0 \leq p \leq o_{ij}$, on $\text{supp}(\rho_{ijkl})$ equation (3.9a) holds, possibly with a different constant.

b) Suppose $i = 0$. Let $Y = \{y : |y_l| < 2 \text{ for all } l\}$, $Y' = \{y : |y_l| < \frac{1}{2} C_n^{-1} \text{ for all } l\}$. Then

$$Y' \subset \text{supp}(\rho_{0jkl}) \subset Y \quad (3.16)$$

Furthermore, on $\text{supp}(\rho_{0jkl})$ equation (3.9b) holds, possibly with a different constant.

Proof. Let $\psi(x)$ be a smooth nonnegative function on K such that $\psi(x)$ is supported on $|x| < 1 + \epsilon$ such that $\psi(x) = 1$ for $|x| < 1 - \epsilon$, where ϵ will be dictated by our arguments. In

the case where K is a p -adic field we can take $\psi(x)$ to just be the characteristic function of $\{x : |x| \leq 1\}$. We consider the sequence of blowups that occurred when the proof of Lemma 3.1 was applied to the vertices of $N(f)$, creating the cubes E_k that were used in defining the A_{ijk} . Suppose the first blowup was in the x_p and x_q variables. For $\phi(x) \in C_c(E_0)$, we write $\phi(x) = \phi_1(x) + \phi_2(x)$, where $\phi_1(x) = \psi(\frac{x_q}{x_p})\phi(x)$ and where $\phi_2(x) = (1 - \psi(\frac{x_q}{x_p}))\phi(x)$. When $x_p = 0$ we set $\phi_1(x) = 0$ and $\phi_2(x) = \phi(x)$.

Doing the blowup turning x_q into $x_p x_q$ while leaving the other variables fixed turns $\phi_1(x)$ into the smooth function $\psi(x_q)\phi(x_1, \dots, x_{q-1}, x_p x_q, x_{q+1}, \dots, x_n)$, and similarly doing the blowup converting x_p into $x_p x_q$ while leaving the other variables fixed turns $\phi_2(x)$ into the smooth function $(1 - \psi(\frac{1}{x_p}))\phi(x_1, \dots, x_{p-1}, x_p x_q, x_{p+1}, \dots, x_n)$. Note both functions are supported on $B(0, 1 + 2\epsilon) = \{x \in K^n : |x_l| < 1 + 2\epsilon \text{ for all } l\}$. We next perform the analogous procedure on the blown-up $\phi_1(x)$ and $\phi_2(x)$ with respect to the second blowup of the resolution process of Lemma 3.1, obtaining four smooth functions, each supported in the same cube $B(0, 1 + 2\epsilon)$. After finitely many steps we are done, and we obtain a finite collection $\rho_1(x), \dots, \rho_q(x)$ of smooth functions such that each $\rho_k(x)$ is supported on a set slightly larger than E_k , and such that outside a set of measure zero $\phi(x) = \sum_{k=1}^q \rho_k \circ \gamma_k^{-1}(x)$.

By continuity, (3.9a) – (3.9b) will still hold (possibly with different constants) if $A'_{ijk} = \gamma_k^{-1}(A_{ij}) \cap E_k$ is enlarged to $A''_{ijk} = \gamma_k^{-1}(A_{ij}) \cap B(0, 1 + 2\epsilon)$ for small enough ϵ , with corresponding slightly larger B''_{ijkl} . So we can do a partition of unity and write $\rho_k = \sum_{ijkl} \rho'_{ijkl}$ such that (3.9a) or (3.9b) holds on the support of ρ'_{ijkl} . Furthermore, if ϵ is small enough, then (3.15) and (3.16) will hold as a consequence of Lemma 3.2'. (For the left-hand sides, we are using that the G_k contain $B(0, C_n^{-1})$ since G_k is defined via monomial conditions $|\gamma_{kl}(x)| < C_n^{-1}$). Thus if we let ρ_{ijkl} be ρ'_{ijkl} times the characteristic function of K_0^n , then $\phi(x) = \sum_{ijkl} \rho_{ijkl} \circ \gamma_k^{-1}(x)$ gives the desired decomposition of $\phi(x)$ and we are done. \square

4. Proof of Theorem 1.1.

We now commence the proof of Theorem 1.1. The analytic diffeomorphism conditions stipulated by Theorem 1.1 will follow from the Jacobian determinant condition in part c) because all the maps we will use will be analytic diffeomorphisms on their domains except where the Jacobian determinant vanishes. Thus we need not discuss these conditions in the proof.

Let $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha}$ be a function represented by a (nontrivial) power series convergent on some neighborhood of the origin in K^n , with $f(0) = 0$. We will prove Theorem 1.1 by induction on the dimension n . Since the $n = 1$ case is trivial, we always assume $n \geq 2$.

Let α be a multiindex with $|\alpha|$ minimal such that f_{α} is nonzero. Let $m = |\alpha|$. As mentioned in the proof of Lemma 3.4, the derivatives of order m are spanned by

directional derivatives with rational coefficients, so we may let $\partial_v = \sum_k \beta_k \partial_{x_k}$, $\beta_k \in \mathbf{Q}$, be a directional derivative such that $\partial_v^m f(0) \neq 0$. We may do a linear change of variables with rational coefficients and assume that v is the x_n direction. Hence without loss of generality we may assume that

$$\partial_{x_n}^m f(0) \neq 0 \quad (4.1a)$$

Since m is minimal, for $l < m$ we have

$$\partial_{x_n}^l f(0) = 0 \quad (4.1b)$$

For our given n , we will prove Theorem 1.1 by induction on the m for which (4.1a) – (4.1b) hold. Namely, we assume that either $m = 1$ or that $m > 1$ and we know Theorem 1.1 for all $m' < m$. We will actually be inductively proving a slightly stronger statement than Theorem 1.1; in addition to showing $f \circ \alpha_i(x)$ and the Jacobian of $\alpha_i(x)$ are of the form $c_i(x)m_i(x)$, $c_i(x)$ nonvanishing and $m_i(x)$ a monomial, we will also show that for each coordinate function $g_j(x) = x_j$ for $j < n$ the functions $g_j \circ \alpha_i(x)$ are also of this form.

By the implicit function theorem applied to $\partial_{x_n}^{m-1} f$ (the proof for the case $K = \mathbf{R}$ works equally well for arbitrary K), there exists some function $g(x_1, \dots, x_{n-1})$ with a convergent power series on a neighborhood of the origin in K^{n-1} such that $g(0) = 0$ and

$$\partial_{x_n}^{m-1} f(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) = 0 \quad (4.2)$$

Hence if we let $F(x) = f(x_1, \dots, x_{n-1}, x_n + g(x_1, \dots, x_{n-1}))$ we have

$$\partial_{x_n}^{m-1} F(x_1, \dots, x_{n-1}, 0) = 0 \quad (4.3)$$

Although we are not emphasizing it here, in the language of resolution of singularities we are taking a surface resembling a hypersurface of maximal contact and shifting it to the hyperplane $x_n = 0$. Next, we write $F(x)$ as

$$F(x) = h_m(x)x_n^m + \sum_{p < m-1} h_p(x_1, \dots, x_{n-1})x_n^p \quad (4.4)$$

There is no $p = m - 1$ term in (4.4) due to the condition (4.3), and $h_m(0) \neq 0$ due to the condition (4.1a). We also have $h_p(0) = 0$ for $p < m$ due to (4.1b). Note that if $m = 1$, the sum in (4.4) is empty, and we have $F(x) = h_1(x)x_n$. Thus we are done with the $m = 1$ case to start off the induction.

We now use the induction hypothesis on the dimension n and apply Theorem 1.1 to $z(x_1, \dots, x_{n-1}) = \prod_{k=1}^{n-1} x_k \times \prod_{p < m-1} h_p(x_1, \dots, x_{n-1})$. (If an $h_p(x_1, \dots, x_{n-1})$ is the zero function we exclude it from the product). Then if $\psi(x_1, \dots, x_{n-1})$ is a cutoff function supported on a sufficiently small neighborhood of the origin of K^{n-1} , $\psi(x_1, \dots, x_{n-1})$ can be written as a finite sum $\sum_i \psi_i(x_1, \dots, x_{n-1})$ satisfying the conditions of Theorem 1.1. So if $\phi(x_1, \dots, x_n)$ is a cutoff function supported on a sufficiently small neighborhood of the origin of K^n , one can take a $\psi(x_1, \dots, x_{n-1})$ equal to 1 on a neighborhood of the

support of ϕ , write $\psi = \sum_i \psi_i$ in the above fashion, and then induce a decomposition $\phi = \sum_i \phi_i$ where $\phi_i = \phi\psi_i$. If α_i is one of the functions arising from Theorem 1.1 applied to $z(x_1, \dots, x_{n-1})$ here, then by part b) of Theorem 1.1, $\phi_i \circ \alpha_i(x) = (\phi \circ \alpha_i(x))(\psi_i \circ \alpha_i(x))$ can be adjusted on the coordinate hyperplanes $\{x_l = 0\}$ for $l < n$ to become a smooth function whose support is contained in an open set where $z \circ \alpha_i(x_1, \dots, x_{n-1})$ is of the desired form $c_i(x_1, \dots, x_{n-1})m_i(x_1, \dots, x_{n-1})$, where $c_i(x_1, \dots, x_{n-1})$ is nonvanishing and $m_i(x_1, \dots, x_{n-1})$ is a monomial.

It is easy to show that if a product of functions is of this form, then so is each member of that product. Thus given the definition of $z(x_1, \dots, x_{n-1})$, the functions $x_k \circ \alpha_i$ for $k < n$ and each $h_p(x_1, \dots, x_{n-1})$ is also of the desired form $c_i(x)m_i(x)$ on a neighborhood of the support of $\phi_i(x)$. Correspondingly, switching indices from i to q , we write each $h_p \circ \alpha_q(x_1, \dots, x_{n-1})$ in the form $a_{pq}(x_1, \dots, x_{n-1})m_{pq}(x_1, \dots, x_{n-1})$, where $m_{pq}(x_1, \dots, x_{n-1})$ is either a monomial or the zero function, and each $a_{pq}(x)$ is nonvanishing on a neighborhood of the support of $\phi_q \circ \alpha_q(x)$. We therefore can write

$$F \circ \alpha_q(x) = a_{mq}(x)x_n^m + \sum_{p < m-1} a_{pq}(x_1, \dots, x_{n-1})m_{pq}(x_1, \dots, x_{n-1})x_n^p \quad (4.5)$$

Here $a_{pq}(x)$ also is nonvanishing on a neighborhood of the support of $\phi_q \circ \alpha_q$. In order to prove Theorem 1.1, it suffices to show that for each q one can write $\phi_q \circ \alpha_q(x)$ as a sum $\sum_j \eta_j(x)$ of cutoff functions for which $F \circ \alpha_q(x)$ and each $x_k \circ \alpha_q(x)$ for $k < n$ satisfies Theorem 1.1 with these $\eta_j(x)$. Since each $x_k \circ \alpha_q(x)$ is of the canonical form $c(x)m(x)$, $m(x)$ a monomial and $c(x)$ nonvanishing, this follows from showing that $F \circ \alpha_q(x)$ and each coordinate function x_k for $k < n$ satisfies Theorem 1.1 with these $\eta_j(x)$, which is what we will prove.

Claim. In order to prove the inductive step of Theorem 1.1 it suffices to prove that for any point x_0 in the support of any $\phi_q \circ \alpha_q(x)$ there is an open set U_{x_0} centered at x_0 such that Theorem 1.1 holds simultaneously for $F \circ \alpha_q(x + x_0)$ and each function x_k for $k < n$.

Proof. Suppose we have proven the above. We can use a partition of unity to write $\phi_q \circ \alpha_q(x)$ as a sum $\sum_j \eta_j(x)$ of cutoff functions such that each η_j is supported in one of the U_{x_0} , so that we may apply Theorem 1.1 to $F \circ \alpha_q(x + x_0)$ and the functions x_k for $k < n$ using the cutoff functions $\eta_i(x + x_0)$. This gives most what we are trying to prove; what remains to do is first proving that the composition of the coordinate functions x_k for $k < n$ with the shift by x_0 and then with the coordinate changes from applying the induction hypothesis to $F \circ \alpha_q(x + x_0)$ are of the form $c(x)m(x)$, where $m(x)$ is a monomial and $c(x)$ is nonvanishing, and secondly proving that the determinant of the composition of all the maps in question is of the desired form.

Assuming the U_{x_0} are chosen sufficiently small, if k and x_0 are such that $(x_0)_k \neq 0$ then $x_k + (x_0)_k$ has no zeroes in U_{x_0} , while if $(x_0)_k = 0$ then $x_k + (x_0)_k$ is just x_k . So if one proves that all the coordinate functions in the shifted coordinates are monomialized by the coordinate changes on $F \circ \alpha_q(x + x_0)$, the same will be true of the original functions x_k

under the composition of the shift with these coordinate changes. As for the determinant, by induction hypothesis we used in $n - 1$ dimensions, the determinant of α_q is of the form $c_q(x_1, \dots, x_{n-1})m_q(x_1, \dots, x_{n-1})$, m_q a monomial and c_q nonvanishing. Similar to above, composing this with the shift by x_0 results in a function of the same form on U_{x_0} as long as U_{x_0} is sufficiently small. Since we are assuming we know Theorem 1.1 for each x_k for $k < n$ in the shifted coordinates, and that the determinant of the coordinate changes on $F \circ \alpha_q(x + x_0)$ are of the form $c_q(x)m_q(x)$ for a monomial m_q and a nonvanishing c_q , the chain rule for determinants now implies that the determinant of the composition of all coordinate changes is also of this form $c_q(x)m_q(x)$. Thus the claim is proven. \square

We now proceed to show that for x_0 in the support of $\phi_q \circ \alpha_q(x)$, Theorem 1.1 holds for $F \circ \alpha_q(x + x_0)$ and each coordinate function x_k with $k < n$. By (4.5), $\partial^{m-1}(F \circ \alpha_q)(x_0) \neq 0$ for x_0 in the support of $\phi_q \circ \alpha_q(x)$ except at those $x_0 = ((x_0)_1, \dots, (x_0)_n)$ for which $(x_0)_n = 0$. So when $(x_0)_n \neq 0$ one can apply the induction hypothesis and get that Theorem 1.1 holds for $F \circ \alpha_q(x + x_0)$ and each coordinate function x_k with $k < n$. Similarly, for $p < m - 1$ for which the sum (4.5) has a nonzero term, one has that $\partial^p(F \circ \alpha_q)(x_0) \neq 0$ unless $a_{pq}((x_0)_1, \dots, (x_0)_{n-1}) = 0$, and one again can apply the induction hypothesis to get that Theorem 1.1 holds for $F \circ \alpha_q(x + x_0)$ and each coordinate function x_k with $k < n$. Thus we need only consider x_0 such that $(x_0)_n = 0$ and each $a_{pq}((x_0)_1, \dots, (x_0)_{n-1}) = 0$. In this case, $F \circ \alpha_q(x + x_0)$ is once again of the form (4.5). So without loss of generality, we take $x_0 = 0$ in our subsequent arguments and look at $F \circ \alpha_q(x)$.

Let $g(x) = F \circ \alpha_q(x)$. Note that (4.5) implies that $N(g)$ has a special form. Namely, there is a vertex at $(0, \dots, 0, m)$, while there is at most one vertex at height h for any $h < m$. Furthermore, if $\sum_{\beta} g_{\beta} x^{\beta}$ denotes the Taylor expansion of $g(x)$ at the origin, then $g_{\beta} = 0$ for all $\beta = (\beta_1, \dots, \beta_n)$ with $\beta_n = m - 1$. In particular, there is no vertex at height $m - 1$. These imply the following important fact:

Lemma 4.1. If F is any compact face of $N(g)$, then any zero of $g_F(x)$ (cf Definition 2.2) in K_0^n has order at most $m - 1$.

Proof. Let F be a compact face of $N(g)$. Since there is at most one vertex at any given height, we may let v be the vertex of $N(g)$ such that the n th component v_n is maximal. By the above considerations, either $v_n \leq m - 2$ or $v_n = m$. In the former case $\partial_{x_n}^{v_n} g_F(x)$ is a monomial which therefore doesn't vanish on K_0^n . Hence any zero of $g_F(x)$ in K_0^n has order at most $v_n \leq m - 2$. In the latter case, since $g_{\beta} = 0$ for all β with $\beta_n = m - 1$, we have that $\partial_{x_n}^{m-1} g_F(x)$ is just x_n and therefore any zero of $g_F(x)$ in K_0^n has order at most $m - 1$. Hence Lemma 4.1 is proven. \square

We move to the next stage of our argument. Recall we have reduced to showing Theorem 1.1 for the coordinate functions x_k for $k < n$ together with $g(x) = F \circ \alpha_q(x)$ satisfying (4.5). We apply Lemma 3.5 to $g(x)$, which tells us that for $\eta(x)$ supported in a sufficiently small neighborhood of the origin we may write $\eta(x) = \sum_r \eta_r(x)$ where each $\eta_r(x)$ is one of the functions denoted by $\phi_{ijkl}(x)$ in Lemma 3.5. For each r there is a finite composition of blowups $\gamma_r(x)$ such that each $\eta_r \circ \gamma_r$ can be adjusted on a set of measure

zero to become a smooth function. Furthermore, Lemma 3.5 says that on the support of $\eta_r \circ \gamma_r$ one of two possibilities occurs. In the first possibility (the $i = 0$ case of Lemma 3.5), $g \circ \gamma_r(y)$ can be written in the form $y_1^{\alpha_1} \dots y_n^{\alpha_n} h(y)$ where by (3.9b) there is a constant $\delta > 0$ such that $|h(y)| > \delta$. In the second possibility, corresponding to $i > 0$, we can split into y and z variables such that $g \circ \gamma_r(y, z)$ can be written as $y_1^{\alpha_1} \dots y_p^{\alpha_p} h(y, z)$, where by (3.15) some z -directional derivative of $h(y, z)$ of order $\leq m - 1$ is bounded below in magnitude by some $\delta > 0$. Doing a linear change of variables in z if necessary, we may always assume it is the z_n derivative, so that $|\partial_{z_n}^a h(y, z)| > \delta$ on the support of $\eta_r \circ \gamma_r$ for some $0 \leq a \leq m - 1$. The condition that $a \leq m - 1$ comes from Lemma 4.1 and the fact that in Lemma 3.5 the order of this derivative is at most the maximum order of the zeroes of the functions $g_F(x)$ on K_0^n .

In the first case, we are already done; $g \circ \gamma_r(y)$ is of the required form, and furthermore since the components of $\gamma_r(y)$ are all monomials, the composition of each coordinate function x_k with $\gamma_r(y)$ is a monomial, as is the Jacobian determinant of γ_r . Moving on to the second case, if $a = 0$ then we are done for the same reasons as in the first case. So we assume $a > 0$. If $j_k(x)$ denotes the coordinate function x_k , then each $j_k \circ \gamma_r(y, z)$ is a monomial in y and z . By (3.15), the $|z_m|$ are bounded above and below on the support of $\eta_r \circ \gamma_r$, so on this support we can write $j_k \circ \gamma_r(y, z) = c_{kr}(y, z)m_{kr}(y)$, where $m_{kr}(y)$ is a monomial in the y variables only and where $c_{kr}(y, z)$ is nonvanishing.

We now apply the induction hypothesis. Namely, we do a partition of unity and write $\eta_r \circ \gamma_r(y, z) = \sum_s \sigma_{rs}(y, z)$, where $\sigma_{rs}(y, z)$ is a cutoff function such that the induction hypothesis applies to $h_1(y, z) = h(y + y_0, z + z_0)$ with the cutoff function $\sigma_{rs}(y + y_0, z + z_0)$. We can do this since $h_1(y, z)$ everywhere has a nonvanishing z_n derivative of order at most $m - 1$. Note that if the support of σ_{rs} is appropriately small, which we may assume, then in the expression $g \circ \gamma_r(y + y_0, z + z_0) = (y_1 + (y_0)_1)^{\alpha_1} \dots (y_p + (y_0)_p)^{\alpha_p} h_1(y, z)$ the factors $(y_l + (y_0)_l)^{\alpha_l}$ for $(y_0)_l \neq 0$ are never zero on the support of $\sigma_{rs}(y + y_0, z + z_0)$, while the for the remaining l 's the factor $(y_l + (y_0)_l)^{\alpha_l}$ is just $y_l^{\alpha_l}$.

Thus on the support of $\sigma_{rs}(y + y_0, z + z_0)$, the function $g \circ \gamma_r(y + y_0, z + z_0)$ can be written in the form $g_1(y, z) = c(y)m(y)h_1(y, z)$ for a monomial $m(y)$ and nonvanishing $c(y)$. We now apply the induction hypothesis to $h_1(y, z)$, simultaneously monomializing $h_1(y, z)$, all y_l variables, and all z_l variables other than $l = n$. If $\beta_j(x)$ are the resulting coordinate changes then not only is $h_1 \circ \beta_j(x)$ of the desired form $c(x)m(x)$, but each $y_l \circ \beta_j(x)$ is also of the desired form. Thus $g_1 \circ \beta_j(x)$ is of the form $c(x)m(x)$ as well. Thus $g(x)$ transforms in the desired fashion under the composition of γ_r , the shift by (x_0, y_0) and β_j .

Similarly, if $j_k \circ \gamma_r(y, z)$ is as in three paragraphs ago, then assuming the support of σ_{rs} is sufficiently small $j_k \circ \gamma_r(y + y_0, z + z_0)$ is of the form $c_{kr}(y, z)m_{kr}(y)$ for a monomial $m_{kr}(y)$ and nonvanishing $c_{kr}(y, z)$ because as we saw $j_k \circ \gamma_r(y, z)$ was of this form, although the new $m_{kr}(y)$ will no longer depend on any y_l variable for which $(y_0)_l$ is nonzero, similar to two paragraphs ago. Thus since all y_l variables transform into functions of the form $c(x)m(x)$ under the coordinate changes β_j by induction hypothesis, the composition of

$j_k \circ \gamma_r(y + y_0, z + z_0)$ with each β_j will also transform into functions of this $c(x)m(x)$ form. Thus we see that the coordinate functions x_k transform into functions of the desired form under the composition of γ_r , the shift by (y_0, z_0) , and β_j .

We have now seen that g and each coordinate function x_k transform in the desired fashion under the composition of γ_r , the shift by (y_0, z_0) , and β_j . We still must show that the determinant of the composition of the above coordinate changes is of the desired form $c(x)m(x)$. Note that the determinant of $\gamma_k(y, z)$ is a monomial since its components are monomials. Since the $|z_l|$ are bounded below, this determinant can be written as $c(y, z)m(y)$ for a monomial $m(y)$ and nonvanishing $c(y, z)$. The composition of $\gamma_k(y, z)$ with the shift by (y_0, z_0) will have the same determinant, which will also be of the form $c(y, z)m(y)$ in the shifted coordinates if the support of $\sigma_{rs}(y, z)$ is appropriately small, similar to the last paragraph as well as three paragraphs ago. By induction hypothesis, each $j_k \circ \beta_j(x)$ is of the desired form $c(x)m(x)$, as is the determinant of β_j . Plugging these facts into the chain rule for determinants gives that the composition of all coordinate changes also is of this form $c(x)m(x)$.

The proof of Theorem 1.1 is now complete; if $\phi(x)$ is a cutoff function as in the statement of Theorem 1.1, $\tau_i(x)$ for $i = 1, \dots, N$ denote all the cutoff functions in the final blown-up coordinates produced by the above argument, and $\alpha_i(x)$ denotes the composition of coordinate changes corresponding to $\tau_i(x)$, then we define $\tau'_i(x)$ to be $\tau_i(x)$ times the characteristic function of K_0^n and $\phi_i(x) = \tau'_i \circ \alpha_i^{-1}(x)$. Then the $\phi_i(x)$ satisfy the conditions of Theorem 1.1 and we are done. \square

Example 1. Suppose after being put in the form (4.5), $F \circ \alpha_q(x)$ has nondegenerate principal part in the sense of Varchenko [V], meaning that for each face compact face G of $N(F \circ \alpha_q)$ the polynomial $(F \circ \alpha_q)_G(x)$ has zeroes of order at most 1 on K_0^n . Then the rest of the proof of Theorem 1.1 proceeds as follows.

One first does the constructions of section 2, subdividing a neighborhood of the origin into the sets A_{ij} on which the terms of the Taylor expansion of $F \circ \alpha_q(x)$ whose exponents lie on the face F_{ij} dominate. Next one does the constructions of section 3 on $F \circ \alpha_q(x)$ and accordingly writes a cutoff function $\phi(x)$ in the form $\phi(x) = \sum_{ijkl} \phi_{ijkl}(x)$ as in Lemma 3.5. The support of $\phi_{ijkl} \circ \gamma_k(x)$ will be contained in a set which is approximately the blowup of A_{ij} by γ_k , and by Lemma 3.5, on the support of $\phi_{ijkl} \circ \gamma_k(x)$ the function $F \circ \alpha_q \circ \gamma_k(x)$ can be written as $y_1^{a_1} \dots y_p^{a_p} F_{ijkl}(y, z)$, where on the support of $\rho_{ijkl}(x)$ either $F_{ijkl}(y, z)$ does not vanish or has nonzero gradient in the z variables wherever $F_{ijkl}(y, z) = 0$.

Now one is effectively done resolving the singularities of $f(x)$; $F_{ijkl}(y, z)$ is comparable to the monomial $y_1^{a_1} \dots y_p^{a_p}$ on a neighborhood of any (y, z) where $F_{ijkl}(y, z) \neq 0$, while by the implicit function theorem, on a neighborhood of any zero of $F_{ijkl}(y, z)$ locally one can do a quasitranslation in a z variable to convert $F_{ijkl}(y, z)$ into a function of the form $y_1^{a_1} \dots y_p^{a_p} z_i c(y, z)$ where $c(y, z)$ is nonvanishing. Note that the determinant of the composition of the two coordinate changes is also comparable to a monomial in this

situation; γ_k has monomial components and therefore its determinant is a monomial, and the quasitranslation in z_n will not change this since $|z_n|$ is bounded away from 0.

Note that if $f(x)$ started out having nondegenerate principal part, then one could jump directly to performing the constructions of sections 2 and 3 on $f(x)$, and the blown up $f \circ \gamma_k(x)$ will have the above properties. So after doing a single quasitranslation the zero set of $f \circ \gamma_k(x)$ will be resolved.

Example 2. Suppose we are in two dimensions, and $f(x, y)$ satisfies (4.1a) – (4.1b), so that $f(x, y)$ has a zero of order m at the origin and has nonvanishing m th y derivative. After a quasitranslation in the y variable $f(x, y)$ gets into the form (4.4). We do not have to do a resolution of singularities in $n - 1$ dimensions to get in the form (4.5); in two dimensions any $F(x, y)$ satisfying (4.4) is already in the form (4.5).

One now does the constructions of Section 3 on $F(x, y)$. Then given any cutoff function $\phi(x, y)$ supported on a small enough neighborhood of the origin, we can write $\phi(x, y) = \sum_{ijkl} \phi_{ijkl}(x, y)$ as in Lemma 3.5, and we look at the functions $F \circ \gamma_k(x, y)$ on the support of the $\rho_{ijkl}(x, y) = \phi_{ijkl} \circ \gamma_k(x, y)$. If $\rho_{ijkl}(x, y)$ corresponds to an A'_{ijk} with $i = 0$, then by Lemma 3.5 b), on the support of $\rho_{0jkl}(x, y)$ the function $F \circ \gamma_k(x, y)$ has already been monomialized and we are done.

If $\rho_{ijkl}(x, y)$ corresponds to an A'_{ijk} with $i = 1$, then by Lemma 3.5a), on the support of $\rho_{1jkl}(x, y)$, the function $F \circ \gamma_k(x, y)$ is of the form $x^p g(x, y)$ and $g(x, y)$ has a nonvanishing y derivative of order at most $m - 1$. Furthermore, $\rho_{1jkl}(x, y)$ is supported away from $y = 0$ in this situation. One then proceeds to resolve $g(x, y)$ and x simultaneously, which can be done using the induction hypothesis.

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