L^p Sobolev regularity of averaging operators over hypersurfaces and the Newton polyhedron

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Abstract

L^p to L^p_β boundedness results are proven for translation invariant averaging operators over hypersurfaces in Euclidean space. The operators can either be Radon transforms or averaging operators with multiparameter fractional integral kernel. In many cases, the amount $\beta > 0$ of smoothing proven is optimal up to endpoints, and in such situations this amount of smoothing can be computed explicitly through the use of appropriate Newton polyhedra.

1   Introduction and theorem statements

In this paper we consider convolution operators with hypersurface measures on $\mathbb{R}^{n+1}$ of the following form, where $\mathbf{x}$ denotes $(x_1, ..., x_n)$ and $\mathbf{t}$ denotes $(t_1, ..., t_n)$.

$$Tf(\mathbf{x}, x_{n+1}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{t}, x_{n+1} - S(\mathbf{t}))K(\mathbf{t})\, d\mathbf{t} \tag{1.1}$$

Here $S(\mathbf{t})$ is a real-analytic function on a neighborhood U of the origin and $K(\mathbf{t})$ is a function supported in U that is C^1 on $\{\mathbf{t} \in U : t_i \neq 0 \text{ for all } i\}$ which satisfies estimates as follows. Write $\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_m)$, where $\mathbf{t}_i$ denotes $(t_{i1}, ..., t_{il_i})$ such that the various t_{ij} variables comprise the whole list $t_1, ..., t_n$. Then for some $0 \leq \alpha_i < l_i$ and some $C > 0$ we assume the following estimates.

$$|K(\mathbf{t})| \leq C \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \tag{1.2a}$$

$$|\partial_{t_{ij}} K(\mathbf{t})| \leq C \frac{1}{|t_{ij}|} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \qquad \text{for all } i \text{ and } j \tag{1.2b}$$

In the parlance of harmonic analysis, operators satisfying $(1.1), (1.2a), (1.2b)$ are sometimes referred to as fractional Radon transforms or fractional singular Radon transforms. The case

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where each $\alpha_i = 0$ includes the traditional Radon transform operators. Our goal will be to prove $L^p(\mathbb{R}^{n+1})$ to $L^p_\beta(\mathbb{R}^{n+1})$ boundedness properties for T that are sharp up to endpoints and which can be computed explicitly with the help of certain Newton polyhedra, as will be described after the proof of Lemma 2.1. The author's earlier paper [G1] covers, among other things, the situation where $p = 2$ and each $\alpha_i = 0$, and our methods and results will extend those of [G1].

By the translation and rotation invariance properties of convolution operators, without loss of generality we may assume that

$$S(0, ..., 0) = 0 \qquad\qquad \nabla S(0, ..., 0) = (0, ..., 0) \qquad\qquad (1.3)$$

To avoid trivialities, we also assume S is not identically zero. In order to describe our results, we will make use of the following terminology.

Definition 1.1. *Let $f(\mathbf{t})$ be a real analytic function defined on a neighborhood of the origin in $\mathbb{R}^n$, and let $f(\mathbf{t}) = \sum_\alpha f_\alpha \mathbf{t}^\alpha$ denote the Taylor expansion of $f(\mathbf{t})$ at the origin. For any α for which $f_\alpha \neq 0$, let Q_α be the octant $\{\mathbf{t} \in \mathbb{R}^n : t_i \geq \alpha_i \text{ for all } i\}$. Then the Newton polyhedron $N(f)$ of $f(\mathbf{t})$ is defined to be the convex hull of all Q_α.*

A Newton polyhedron can contain faces of various dimensions in various configurations. These faces can be either compact or unbounded. A vertex of $N(f)$ is considered to be a face of dimension zero.

Definition 1.2. *Where $f(\mathbf{t})$ is as in Definition 1.1, we define $f^*(\mathbf{t})$ by*

$$f^*(\mathbf{t}) = \sum_{(v_1, ..., v_n) \text{ a vertex of } N(f)} |t_1^{v_1} ... t_n^{v_n}| \qquad\qquad (1.4)$$

By Lemma 2.1 of [G2], there is a constant C such that for all $\mathbf{t}$ one has $|f(\mathbf{t})| \leq Cf^*(\mathbf{t})$.

Let $d\mu$ denote the measure $\prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} dm$, where m denotes Lebesgue measure. In Lemma 2.1, we will show that there is an $r_0 > 0$, an $a_0 > 0$, and an integer d_0 satisfying $0 \leq d_0 \leq n - 1$, such that if $r < r_0$ then there are positive constants b_r and B_r such that for $0 < \epsilon < \frac{1}{2}$ we have

$$b_r \epsilon^{a_0} |\ln \epsilon|^{d_0} < \mu(\{\mathbf{t} \in (0, r)^n : S^*(\mathbf{t}) < \epsilon\}) < B_r \epsilon^{a_0} |\ln \epsilon|^{d_0} \qquad\qquad (1.5)$$

The quantity a_0 will play a key role in our results. In order to state our main theorem, we will also need the following definitions.

Definition 1.3. *Suppose F is a compact face of the Newton polyhedron $N(f)$. Then if $f(\mathbf{t}) = \sum_\alpha f_\alpha \mathbf{t}^\alpha$ denotes the Taylor expansion of f like above, define $f_F(\mathbf{t}) = \sum_{\alpha \in F} f_\alpha \mathbf{t}^\alpha$.*

Definition 1.4. *For $f(\mathbf{t})$ as above, we denote by $o(f)$ the maximum order of any zero of any $f_F(\mathbf{t})$ on $(\mathbb{R} - \{0\})^n$. We take $o(f) = 0$ if there are no such zeroes.*

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We now come to our main theorem.

Theorem 1.1. *Suppose $S(\mathbf{t})$ is a real analytic function on a neighborhood of the origin satisfying (1.3). Let $g = \min(a_0, l_1 - \alpha_1, ..., l_m - \alpha_m)$, where the α_i and l_i are as in the beginning of the paper and a_0 is as in (1.5). Then there is a neighborhood V of the origin such that if $K(\mathbf{t})$ is supported on V and satisfies $(1.2a) - (1.2b)$ then the following hold.*

1) *Let A denote the open triangle with vertices $(\frac{1}{2}, \frac{1}{\max(o(S),2)})$, $(0,0)$, and $(1,0)$, and let $B = \{(x,y) \in A : y < g\}$. Then T is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^p_\beta(\mathbb{R}^{n+1})$ if $(\frac{1}{p}, \beta) \in B$.*

2) *Suppose $g < 1$, $K(\mathbf{t})$ is nonnegative, and there exists a positive constant C_0 and a neighborhood N_0 of the origin such that $K(\mathbf{t}) > C_0 \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k}$ on $\{\mathbf{t} \in N_0 : t_i \neq 0 \text{ for all } i\}$. Then if $1 < p < \infty$ and T is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^p_\beta(\mathbb{R}^{n+1})$ we must have $\beta \leq g$.*
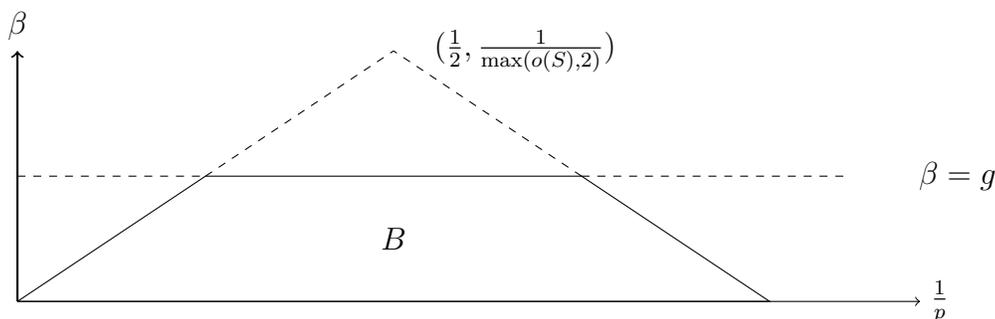
Sample boundedness region when $g < \frac{1}{\max(o(S),2)}$.

Note that when $g < \frac{1}{\max(o(S),2)}$, then the two parts of Theorem 1.1 combined say that for $\frac{1}{p} \in (\frac{\max(o(S),2)}{2}g, 1 - \frac{\max(o(S),2)}{2}g)$, the amount of L^p Sobolev smoothing given by part 1, g derivatives, is optimal except possibly missing the endpoint $\beta = g$. When $g = \frac{1}{\max(o(S),2)}$ the same is true for $p = 2$. A natural question to ask is when does the endpoint $\beta = g$ also hold. Although we will not show it here, it turns out that when $p = 2$ sometimes this endpoint estimate holds and sometimes it does not. The author does not know what happens in the $p \neq 2$ situation.

Another natural question to ask is in what situations can the boundedness region B of Theorem 1.1 be enlarged. In highly nondegenerate situations one can get a much larger boundedness region. For example, consider the most nondegenerate situation, where each $\alpha_i = 0$, each $l_i = 1$, $K(\mathbf{t})$ is smooth, and $S(\mathbf{t}) = \sum_{i=1}^{n} t_i^2$. Then by the well-known Fourier transform decay estimates for nondegenerate hypersurface measures (see p. 348 of [S]) one has that T is bounded from $L^2(\mathbb{R}^{n+1})$ to $L^2_{\frac{n}{2}}(\mathbb{R}^{n+1})$.

But even when $g \leq \frac{1}{\max(o(S),2)}$ one might ask if the set B can be enlarged within the set of $(\frac{1}{p}, \beta)$ for which $\beta < g$. By [C] [Gr] the set cannot be enlarged beyond endpoints

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when $n = 1$, but for general n this appears to be a delicate issue which may require more information than can be provided with the Newton polyhedra used in this paper.

It is not hard to show that part 2) of Theorem 1.1 does not necessarily hold when $g \geq 1$. A simple counterexample is the case where $K(\mathbf{t})$ is smooth and nonnegative with $K(0) > 0$, and where again each $l_i = 1$, each $\alpha_i = 0$, and $S(\mathbf{t}) = \sum_{i=1}^{n} t_i^2$. Then $g = 1$, but as described above T gains $\frac{n}{2}$ derivatives on $L^2(\mathbb{R}^{n+1})$.

Example 1. Suppose we are in the case where each $\alpha_i = 0$, such as in the case of smooth $K(\mathbf{t})$. Then $g = \min(a_0, l_1, ..., l_m)$. Because the triangle A of Theorem 1.1 has its upper vertex no higher than $(\frac{1}{2}, \frac{1}{2})$, if $a_0 > \frac{1}{2}$ the set B of Theorem 1.1 is just the triangle A and the theorem does not necessarily give any sharp estimates, while if $a_0 \leq \frac{1}{2}$ we have $g = a_0$ and Theorem 1.1 gives a result that is sharp up to endpoints if $a_0 \leq \frac{1}{\max(o(S),2)}$. The $p = 2$ case of these facts follow from Theorem 1.5 of [G1], which in turn generalizes portions of a paper of Varchenko [V] that covers the cases where $o(S) = 0$ or 1.

As will be explained in greater detail in the proof of Lemma 2.1, in this example a_0 is the reciprocal of the Newton distance of $S^*(\mathbf{t})$, defined as follows.

Definition 1.5. *The Newton distance $d(f)$ of a function $f(\mathbf{t})$ is defined to be* $\inf\{c : c, c, ..., c, c) \in N(f)\}$.

Example 2. Suppose $m = 1$, so that $d\mu = |\mathbf{t}|^{-\alpha_1} dm$ for some $\alpha_1 < n$. Then given that (1.3) holds, one has $S^*(\mathbf{t}) \leq \sum_{i=1}^{n} |t_i|^2 = |\mathbf{t}|^2$, and as a result if $0 < r < 1$ and $\epsilon^{\frac{1}{2}} < r$ one has

$$\mu(\{\mathbf{t} \in (0,r)^n : S^*(\mathbf{t}) < \epsilon\}) \geq \mu(\{\mathbf{t} \in (0,r)^n : |\mathbf{t}|^2 < \epsilon\})$$

$$= \int_{|\mathbf{t}| < \epsilon^{\frac{1}{2}}} |\mathbf{t}|^{-\alpha_1} d\mathbf{t} \tag{1.6}$$

The integral (1.6) is easily seen to be of the form $C\epsilon^{\frac{n-\alpha_1}{2}}$. Hence $a_0 \leq \frac{n-\alpha_1}{2}$. Since $g = \min(a_0, n - \alpha_1)$, we have that $g = a_0$ whenever $m = 1$. The value of a_0 can then be computed using Newton polyhedra as in the proof of Lemma 2.1.

Example 3. Suppose $m = n$ so that each $\mathbf{t}_k$ is one dimensional. Then we have

$$\mu(\{\mathbf{t} \in (0,r)^n : S^*(\mathbf{t}) < \epsilon\}) = \int_{\{\mathbf{t} \in (0,r)^n : S^*(\mathbf{t}) < \epsilon\}} \prod_{k=1}^{n} t_k^{-\alpha_k} d\mathbf{t} \tag{1.7}$$

We change variables from t_k to $u_k = t_k^{1-\alpha_k}$. If $S^*(\mathbf{t}) = \sum_{i=1}^{N} |t_1|^{v_{i1}} ... |t_n|^{v_{in}}$, then we denote by $S^{**}(\mathbf{u})$ the function $\sum_{i=1}^{N} |u_1|^{\frac{v_{i1}}{1-\alpha_1}} ... |u_n|^{\frac{v_{in}}{1-\alpha_n}}$. Then for an appropriate box B, (1.7) becomes of the form

$$\mu(\{\mathbf{t} \in (0,r)^n : S^*(\mathbf{t}) < \epsilon\}) = C \int_{\{\mathbf{u} \in B : S^{**}(\mathbf{u}) < \epsilon\}} 1 \, d\mathbf{u}$$

$$= Cm(\{\mathbf{u} \in B : S^{**}(\mathbf{u}) < \epsilon\}) \tag{1.8}$$

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As in Example 1, the reciprocal of the Newton distance of S^{**} gives the exponent in the growth rate of $m(\{\mathbf{u} \in B : S^{**}(\mathbf{u}) < \epsilon\})$, which by (1.8) then gives the value of a_0.

Some history.

There has been a great deal of work done on function space boundedness properties of Radon transforms and related operators, so we focus on the L^p to L^p_β boundedness questions being considered here. The case of translation-invariant Radon transforms with smooth density functions over curves in $\mathbb{R}^2$ was thoroughly analyzed in [Gr] and [C]. In the general non-translation-invariant case for curves in $\mathbb{R}^2$, again with smooth density functions, comprehensive L^p_α to L^q_β estimates that are sharp up to endpoints are proven in [Se].

For translation-invariant Radon transforms over two dimensional surfaces in $\mathbb{R}^3$, there are a number of results. If $p = 2$, the level of Sobolev space smoothing directly translates into a surface measure Fourier transform decay rate problem, and for the case of smooth density functions the stability theorems of Karpushkin [Ka1] [Ka2] combined with [V] give some sharp decay rate results. Generalizations to smooth phases then follow from [IKM]. When one has a singular density function, in [G3] the author proved some theorems corresponding to the case $m = 1$ of this paper that include results that go beyond what is proved here. Other results for singular density functions appear in [G4].

For higher dimensional hypersurfaces, if the density functions are singular enough in the sense that the α_i are close enough to l_i, then there will be an interval I containing 2 such that sharp L^p Sobolev smoothing estimates follow from the results of [St1] when $p \in I$. This extends earlier work of the author [G5]. If one lets the α_i actually equal l_i and one adds an appropriate cancellation condition one has a multiparameter singular Radon transform, and L^p boundedness results for such operators were proven in [St1] [St2], extending the results in [CNSW].

Additional results for higher dimensional hypersurfaces appear in [Cu], and in [PSe] Sobolev space estimates are proven for translation-invariant Radon transforms over curves. There have also been a number of papers connecting L^p Sobolev regularity for Radon transforms to Fourier integral operators, such as [GreSeW] [GreSe1] [GreSe2].

2   Some useful lemmas

Lemma 2.1. *There is an $r_0 > 0$, an $a_0 > 0$, and an integer d_0 satisfying $0 \leq d_0 \leq n - 1$, such that if $r < r_0$ then there are positive constants b_r and B_r such that for $0 < \epsilon < \frac{1}{2}$ equation (1.5) holds.*

Proof. Given $r > 0$, we divide $(0, r)^n$ (up to a set of measure zero) into $n!$ regions $\{A_l\}_{l=1}^{n!}$, where each A_l is a region of the form $\{\mathbf{t} \in (0, r)^n : t_{j_1} < t_{j_2} < ... < t_{j_n}\}$, where $t_{j_1}, ..., t_{j_n}$ is a permutation of the t variables. We focus our attention one one such A_l and we let u_i

denote t_{j_i}. Then in the u_i variables the function $S^*(\mathbf{t})$ becomes a function $S_l^*(\mathbf{u})$ of the form $\sum_{i=1}^N |u^{w_i}|$, where the components of a given w_i are a permutation of that of v_i in the expression $S^*(\mathbf{t}) = \sum_{i=1}^N |t^{v_i}|$. Next, observe that

$$\mu(\{\mathbf{t} \in A_l : S^*(\mathbf{t}) < \epsilon\}) = \int_{\{\mathbf{t} \in A_l : S^*(\mathbf{t}) < \epsilon\}} \prod_{k=1}^m |t_k|^{-\alpha_k} \, d\mathbf{t} \tag{2.1}$$

Note that on A_l a function $|t_k|$ is comparable in magnitude to u_{b_k} for the b_k which is maximal amongst the u_i appearing in $\mathbf{t}_k$. Thus in terms of the u variables there are positive constants C_1 and C_2 such that

$$C_1 \int_{\{\mathbf{u}:\, 0 < u_1 < \ldots < u_n < r,\, S_l^*(\mathbf{u}) < \epsilon\}} \prod_{k=1}^m u_{b_k}^{-\alpha_k} \, d\mathbf{u} < \mu(\{\mathbf{t} \in A_l : S^*(\mathbf{t}) < \epsilon\})$$

$$< C_2 \int_{\{\mathbf{u}:\, 0 < u_1 < \ldots < u_n < r,\, S_l^*(\mathbf{u}) < \epsilon\}} \prod_{k=1}^m u_{b_k}^{-\alpha_k} \, d\mathbf{u} \tag{2.2}$$

We make a change of variables on A_l, setting $u_k = \prod_{i=k}^n y_i$. In the y variables, A_l becomes the rectangular box $(0,1)^{n-1} \times (0,r)$ and on A_l, the function $S_l^*(\mathbf{u})$ becomes $S_l^{**}(\mathbf{y}) = \sum_{i=1}^N y^{W_i}$ for some multiindices W_i. Then for some β_k the integral in (2.2) becomes

$$\int_{\{\mathbf{y} \in (0,1)^{n-1} \times (0,r):\, S_l^{**}(\mathbf{y}) < \epsilon\}} \prod_{k=1}^n y_k^{-\beta_k} \, d\mathbf{y} \tag{2.3}$$

Because $K(\mathbf{t})$ is integrable on a neighborhood of the origin, each $\beta_k < 1$ here. We next change variables from y_k to $z_k = y_k^{1-\beta_k}$ in (2.3). Given $W_i = (W_{i1}, \ldots, W_{in})$, let X_i denote $(\frac{W_{i1}}{1-\beta_1}, \ldots, \frac{W_{in}}{1-\beta_n})$ and let $S_l^{***}(\mathbf{z}) = \sum_{i=1}^N z^{X_i}$. Then for some positive constant C' equation (2.3) becomes

$$C' \int_{\{\mathbf{z} \in (0,1)^{n-1} \times (0,r^{1-\beta_k}):\, S_l^{***}(\mathbf{z}) < \epsilon\}} 1 \, d\mathbf{z} \tag{2.4}$$

In summary, $\mu(\{\mathbf{t} \in A_l : S^*(\mathbf{t}) < \epsilon\})$ is comparable in magnitude to the Lebesgue measure of the set of points in $(0,1)^{n-1} \times (0,r^{1-\beta_k})$ where $S_l^{***}(\mathbf{z}) < \epsilon$. We can apply Theorem 1.2 of [G2] to $S_l^{***}(\mathbf{z})$ and say that for $r > 0$ sufficiently small there exist positive constants $c_{r,l}$, $d_{r,l}$, and a_l, and an integer $0 \le d_l \le n-1$ such that

$$c_{r,l} \epsilon^{a_l} |\ln \epsilon|^{d_l} < m(\{\mathbf{z} \in (0,r)^n : S_l^{***}(\mathbf{z}) < \epsilon\}) < d_{r,l} \epsilon^{a_l} |\ln \epsilon|^{d_l} \tag{2.5}$$

Here m denotes Lebesgue measure. By dilation invariance of the existence of such estimates, (2.5) will also hold with $(0,r)^n$ replaced by $(0,1)^{n-1} \times (0,r^{1-\beta_k})$, with different constants. Technically, Theorem 1.2 of [G2] requires that the components $\frac{W_{ij}}{1-\beta_j}$ all be integers but the proof of that theorem is valid in the current setting. It should be pointed out that (2.5) can also be shown using related considerations from [V]. Adding (2.5) over all l gives (1.5),

taking a_0 to be the minimal a_l and d_0 to be the maximal d_l over all those a_l for which $a_l = a_0$. This concludes the proof of Lemma 2.1.

By Theorem 1.2 of [G1], the quantity a_l is exactly the reciprocal of the Newton distance (Definition 1.5) of $S_l^{***}(\mathbf{z})$, and d_l is $n-1-l$, where l denotes the minimal dimension of any compact face of the Newton polyhedron of $S_l^{***}(\mathbf{z})$ containing the point where the line $z_1 = z_2 = ... = z_n$ intersects this Newton polyhedron. Here a vertex of a Newton polyhedron is taken to be a face of dimension zero. We refer to chapter 7 of [AGuV] for more information. Thus one can explicitly determine the quantities a_0 and d_0 using the Newton polyhedron of $S(\mathbf{x})$ itself, by finding $S^*(\mathbf{x})$, then doing the coordinate changes of the proof of Lemma 2.1 and finding the various Newton polyhedra of the functions $S_l^{***}(\mathbf{z})$.

We also will make use of the following lemma from [G1].

Lemma 2.2. (Lemma 3.7 of [G1]). *Suppose $f(\mathbf{x})$ is a smooth function on a neighborhood of the origin in $\mathbb{R}^n$ with $f(0) = 0$ and $\nabla f(0) = (0, ..., 0)$. Suppose $f(\mathbf{x})$ has a nonvanishing Taylor expansion at the origin, and that either $f(\mathbf{x})$ is real-analytic or a smooth function whose Newton polyhedron intersects each coordinate axis.*

Then there is a neighborhood U of the origin and constants K and η such that if R is any dyadic rectangle in U then R may be divided into at most K rectangles R_j such that if 2^{-k_i} denotes the length of R in the x_i direction, for each R_j there is an a and a single $y = (y_1, ..., y_n)$ with $|y_i| \leq 2^{-k_i}$ for all i such that on R_j we have

$$|(y \cdot \nabla)^a f(\mathbf{x})| \geq \eta f^*(R_j) \tag{2.6}$$

Here $f^*(R_j)$ denotes $\sup_{R_j} f^*(\mathbf{x})$. For any j, in (2.6) one can arrange that $0 \leq a \leq o(f)$ or that $2 \leq a \leq \max(2, o(f))$.

3   Proof of part 1 of Theorem 1.1

We embed T in an analytic family, defining T_z as follows, where $S^*(\mathbf{t})$ is as in (1.4).

$$T_z f(\mathbf{x}, x_{n+1}) = e^{z^2} \int_{\mathbb{R}^n} (\min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|))^z f(\mathbf{x} - \mathbf{t}, x_{n+1} - S(\mathbf{t})) K(\mathbf{t}) \, d\mathbf{t} \tag{3.1}$$

Note that $T_0 = T$. We will show that if $0 > s_0 > \max(-g, -\frac{1}{\max(o(S),2)})$, for z on the line $Re(z) = s_0$, if $2 < p < \infty$ one has estimates $||T_z f||_{L^p} \leq C_{s_0} ||f||_{L^p}$. We will then show that if $s_1 > \max(0, \frac{1}{\max(o(S),2)} - g)$, then on the line $Re(z) = s_1$ one has estimates $||T_z f||_{L^2_\beta} \leq C_{s_1} ||f||_{L^2}$ for $\beta = \frac{1}{\max(o(S),2)}$. Interpolating between these two estimates gives a Sobolev space estimate for $T = T_0$. Then letting p go to infinity, s_0 go to $\max(-g, -\frac{1}{\max(o(S),2)})$, and s_1 go to $\max(0, \frac{1}{\max(o(S),2)} - g)$, the statement of part 1 of Theorem 1.1 will follow.

We start by proving the L^p boundedness estimates for $p > 2$.

Lemma 3.1. *Suppose $0 > s_0 > \max(-g, -\frac{1}{\max(o(S),2)})$. Then for z on the line $Re(z) = s_0$, if $2 < p < \infty$ there exists a constant C_{s_0} such that $||T_z f||_{L^p} \leq C_{s_0}||f||_{L^p}$.*

Proof. Let $s_0 > -g$ and suppose $Re(z) = s_0$. Then $T_z f$ can be expressed as $f * \nu_z$ for some measure ν_z, and as a result for any $2 < p < \infty$, $||T_z||_{L^p \to L^p}$ is bounded by the integral of the magnitude of the density function in (3.1). So we have the following, where we write $z = s_0 + it$.

$$||T_z||_{L^p \to L^p} \leq e^{s_0^2 - t^2} \int_{\mathbb{R}^n} (\min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|))^{s_0} |K(\mathbf{t})| \, d\mathbf{t} \tag{3.2}$$

Inserting (1.2a), for some $r_0 > 0$ we have

$$||T_z||_{L^p \to L^p} \leq C e^{s_0^2} \int_{(-r_0, r_0)^n} (\min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|))^{s_0} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \tag{3.3}$$

Since $s_0 < 0$, this is in turn bounded by

$$C e^{s_0^2} \int_{(-r_0, r_0)^n} |S^*(\mathbf{t})|^{s_0} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} + C e^{s_0^2} \sum_{i=1}^n \int_{(-r_0, r_0)^n} |t_i|^{s_0} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \tag{3.4}$$

We first estimate the left term in (3.4). Assuming as we may that r_0 is sufficiently small, this term is bounded by

$$C e^{s_0^2} \sum_{i=0}^{\infty} \int_{\{\mathbf{t} \in (-r_0, r_0)^n : 2^{-i-1} \leq |S^*(\mathbf{t})| < 2^{-i}\}} 2^{-is_0} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \tag{3.5}$$

Using (1.5) (and recalling that $S^*(\mathbf{t})$ is even in each variable) we see that this is bounded by $C' e^{s_0^2} \sum_{i=0}^{\infty} 2^{-is_0} \times i^{d_0} 2^{-ia_0}$. Since we are assuming that $s_0 > -g \geq -a_0$, we have that $s_0 + a_0 > 0$ and this sum converges with a bound depending on s_0. Thus the first term in (3.4) satisfies the needed bounds.

Moving on to bounding a given term of the sum on the right of (3.4), let k_0 be such that t_i is one of the coordinate functions in $\mathbf{t}_{k_0}$. Then integrating the term in the the t variables not part of $\mathbf{t}_{k_0}$, we get a bound of

$$C \int_{(-r_0, r_0)^{l_{k_0}}} |t_i|^{s_0} |\mathbf{t}_{k_0}|^{-\alpha_{k_0}} \, d\mathbf{t}_{k_0} \tag{3.6}$$

Performing the integral in the $\mathbf{t}_{k_0}$ variables other than t_i by breaking into $|\mathbf{t}_{k_0}| < 2|t_i|$ and $|\mathbf{t}_{k_0}| > 2|t_i|$ portions, we obtain

$$C' \int_{(-r_0, r_0)^{l_{k_0}}} |t_i|^{s_0} \max(|t_i|^{l_{k_0} - \alpha_{k_0} - 1}, 1) \, dt_i \tag{3.7}$$

(If $l_{k_0} - \alpha_{k_0} - 1 = 0$ we get an additional $\log |t_i|$ factor that does not affect our argument.) So we have two cases, depending on whether or not $l_{k_0} - \alpha_{k_0} - 1 < 0$. If this does hold, since $s_0 > -g \geq -(l_{k_0} - \alpha_{k_0})$, the integral in (3.7) converges. On the other hand if $l_{k_0} - \alpha_{k_0} - 1 \geq 0$, since $s_0 > -\frac{1}{\max(o(S),2)} \geq -\frac{1}{2}$, the integral once again converges. This concludes the proof of Lemma 3.1.

Lemma 3.2. *Suppose $s_1 > \max(0, \frac{1}{\max(o(S),2)} - g)$. For z on the line $Re(z) = s_1$ there exists a constant C_{s_1} such that $||T_z f||_{L_\beta^2} \le C_{s_1} ||f||_{L^2}$ where $\beta = \frac{1}{\max(o(S),2)}$.*

Proof. Note that $T_z f$ is of the form $f * \nu_z$, so to prove Lemma 3.2 one must that $|\hat{\nu}_z(\lambda)| \le C_{s_1}(1 + |\lambda|)^{-\beta}$. Since the integrand in (3.8) below is integrable with an L^1 norm bounded by a function of s_1, it suffices to prove the estimate $|\hat{\nu}_z(\lambda)| \le C_{s_1} |\lambda|^{-\beta}$ for say $|\lambda| > 2$.

Explicitly, we have

$$\hat{\nu}_z(\lambda) = e^{z^2} \int_{\mathbb{R}^n} e^{i\lambda_1 t_1 + i\lambda_2 t_2 + \dots + i\lambda_n t_n + i\lambda_{n+1} S(\mathbf{t})} (\min(|t_1|, \dots, |t_n|, |S^*(\mathbf{t})|))^z K(\mathbf{t}) \, d\mathbf{t} \qquad (3.8)$$

We write $\hat{\nu}_z(\lambda) = \sum_j I_j$, where for $j = (j_1, \dots, j_n)$, I_j denotes the portion of the integral (3.8) over the dyadic rectangle $R_j = \{\mathbf{t} \in \mathbb{R}^n : 2^{-j_i-1} \le |t_i| < 2^{-j_i} \text{ for each } i\}$. By Lemma 2.2, we can further write I_j as the sum of boundedly many integrals over rectangles on which (2.6) holds for the function $S(\mathbf{t})$. Each such rectangle can in turn be written as the sum of boundedly many subsets on which the minimum in $\min(|t_1|, \dots, |t_n|, |S^*(\mathbf{t})|)$ is achieved by a specific $|t_1|, \dots, |t_n|$, or $|S^*(\mathbf{t})|$. We label the subsets corresponding to a given I_j as R_{jk} and we correspondingly write (3.8) as the sum of I_{jk}, where

$$I_{jk} = e^{z^2} \int_{R_{jk}} e^{i\lambda_1 t_1 + i\lambda_2 t_2 + \dots + i\lambda_n t_n + i\lambda_{n+1} S(\mathbf{t})} (\min(|t_1|, \dots, |t_n|, |S^*(\mathbf{t})|))^z K(\mathbf{t}) \, d\mathbf{t} \qquad (3.9)$$

Regardless of what λ is, we may select a λ_i such that $|\lambda_i| > \frac{1}{n+1}|\lambda|$. We will have two separate arguments, depending on whether $i < n + 1$ or $i = n + 1$. We first suppose that $i < n + 1$.

Case 1. $|\lambda_i| > \frac{1}{n+1}|\lambda|$ for some $i \le n$.

Let $P(\mathbf{t})$ denote the phase function in (3.9). Note that $\partial_{t_i} P(\mathbf{t}) = \lambda_i + \lambda_{n+1}\partial_{t_i} S(\mathbf{t})$. Since $S(\mathbf{t})$ has a zero of order at least two at the origin by (1.3), we can assume that we are in a sufficiently small neighborhood of the origin such that

$$|\partial_{t_i} P(\mathbf{t})| > \frac{1}{2n+2}|\lambda| \qquad (3.10)$$

We focus our attention on the t_i integration in (3.9) for the other variables fixed. There are boundedly many intervals of integration, and on each we integrate by parts, integrating $e^{\lambda_1 t_1 + \lambda_2 t_2 + \dots + \lambda_n t_n + \lambda_{n+1} S(\mathbf{t})}$ and differentiating the rest. Due to (3.10), we gain a factor of $C\frac{1}{|\lambda|}$ in the integration by parts, but due to (1.2b) and the corresponding derivative estimates on $|S^*(\mathbf{t})|^z$ we also get a factor of $C|z|\frac{1}{|t_i|}$ from the differentiation. As a result, substituting in (1.2a) we have

$$|I_{jk}| \le C|ze^{z^2}| \int_{R_j} \frac{1}{|\lambda t_i|} |(\min(|t_1|, \dots, |t_n|, |S^*(\mathbf{t})|))^z| \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.11a)$$

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(The endpoint terms in the integration by parts will satisfy the same bounds.) Since $|ze^{z^2}|$ is bounded on any vertical line in the complex plane, the above is bounded by

$$= C_{s_1} \int_{R_j} \frac{1}{|\lambda t_i|} (\min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|))^{s_1} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.11b)$$

Using that $s_1 > 0$, we may take $|t_i|$ in the minimum and therefore (3.11b) is bounded by

$$C'_{s_1} \int_{R_j} \frac{1}{|\lambda t_i|} |t_i|^{s_1} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.12)$$

By simply taking absolute values and integrating in (3.9), we also have a bound of

$$|I_{jk}| \leq C''_{s_1} \int_{R_j} |t_i|^{s_1} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.13)$$

Combining (3.12) and (3.13) gives

$$|I_{jk}| \leq C'''_{s_1} \int_{R_j} \min\left(1, \frac{1}{|\lambda t_i|}\right) |t_i|^{s_1} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.14)$$

Adding (3.14) over all j and k gives for some $r_0 > 0$ that

$$|\hat{\nu}_z(\lambda)| \leq C'''_{s_1} \int_{(-r_0, r_0)^n} \min\left(1, \frac{1}{|\lambda t_i|}\right) |t_i|^{s_1} \prod_{k=1}^{m} |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.15)$$

Integrating (3.15) in the t variables other than t_i, similarly to (3.7) we get

$$|\hat{\nu}_z(\lambda)| \leq C''''_{s_1} \int_0^{r_0} \min\left(1, \frac{1}{|\lambda| t_i}\right) t_i^{s_1} \max(t_i^{l_i - \alpha_i - 1}, 1) \, dt_i \qquad (3.16)$$

Once again if $l_i - \alpha_i - 1$ happens to equal zero we get an additional $|\ln t_i|$ that does not affect our arguments. It is natural to break up the integral of (3.16) into $t_i < \frac{1}{|\lambda|}$ and $t_i > \frac{1}{|\lambda|}$ portions. The first part is given by

$$\int_0^{\frac{1}{|\lambda|}} t_i^{s_1} \max(t_i^{l_i - \alpha_i - 1}, 1) \, dt_i \qquad (3.17)$$

Since $s_1 > 0$, if one is taking 1 in the maximum one obtains a bound of $C|\lambda|^{-1}$ which is far better than we need. So assume that $l_i - \alpha_i - 1 < 0$ and we are taking $t^{l_i - \alpha_i - 1}$ in the maximum. Since this lemma assumes that $s_1 > \frac{1}{\max(o(S), 2)} - g$ and by definition of g we have $g \leq l_i - \alpha_i$, we have that $s_1 + l_i - \alpha_i > \frac{1}{\max(o(S), 2)} - g + l_i - \alpha_i > \frac{1}{\max(o(S), 2)}$. Then integrating (3.17) gives a bound of $C|\lambda|^{-(s_1 + l_i - \alpha_i)} \leq C|\lambda|^{-\frac{1}{\max(o(S), 2)} - \epsilon_{s_1}}$ for some $\epsilon_{s_1} > 0$, giving the bounds needed in this lemma.

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Moving on to the $t_i > \frac{1}{|\lambda|}$ portion of (3.16), note that the magnitude of the part of the integral over the dyadic pieces $t_i \sim 2^{-j}$ will increase or decrease exponentially in j, with the integral over the outermost dyadic piece being of order $|\lambda|^{-1}$, which is better than the estimate needed. Given that the integral over the innermost dyadic piece has the same bound of $C|\lambda|^{-\frac{1}{\max(o(S),2)}-\epsilon_{s_1}}$ as above, the sum of these magnitudes will again have a bound of $C|\lambda|^{-\frac{1}{\max(o(S),2)}-\epsilon_{s_1}}$. This completes the argument for the case where $|\lambda_i| > \frac{1}{n+1}|\lambda|$ for some $i \leq n$.

Case 2. $|\lambda_{n+1}| > \frac{1}{n+1}|\lambda|$.

We will make use of the Van der Corput lemma (see p. 334 of [S]).

Van der Corput lemma. *Suppose $P(x)$ is a real-valued C^k function on the interval $[a,b]$ with $|P^{(k)}(x)| > M$ on $[a,b]$ for some $M > 0$. Let $\psi(x)$ be a complex-valued C^1 function on $[a,b]$. If $k \geq 2$ there is a constant c_k depending only on k such that*

$$\left| \int_a^b e^{iP(x)}\psi(x)\,dx \right| \leq c_k M^{-\frac{1}{k}}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,dx \right) \tag{3.18}$$

If $k = 1$, the same is true if we add the conditions that $P(x)$ is C^2 and that $P'(x)$ is monotonic on $[a,b]$.

We examine the function $S(\mathbf{t})$ on the set R_{jk} of (3.9). Recall that R_{jk} is contained in the dyadic rectangle $R_j = \{\mathbf{t} \in \mathbb{R}^n : 2^{-j_i-1} \leq |t_i| < 2^{-j_i}$ for each $i\}$. Let $S_1(\mathbf{t}) = S(2^{-j_1}t_1, ..., 2^{-j_n}t_n)$ and $S_1^*(\mathbf{t}) = S^*(2^{-j_1}t_1, ..., 2^{-j_n}t_n)$. Then by Lemma 2.2 there is a constant $\eta > 0$, a direction v and an a satisfying $2 \leq a \leq \max(o(S), 2)$ such that on $R_{jk}^* = \{(t_1, ..., t_n) : (2^{-j_1}t_1, ..., 2^{-j_n}t_n) \in R_{jk}\}$ for any fixed $\mathbf{t}_0$ in R_j one has the estimate

$$|\partial_v^a S_1(\mathbf{t})| > \eta|S_1^*(\mathbf{t}_0)| \tag{3.19}$$

Note that in the dilated coordinates, if $K_1(\mathbf{t})$ denotes $K(2^{-j_1}t_1, ..., 2^{-j_n}t_n)$, then equation (3.9) becomes

$$I_{jk} = 2^{-(j_1+...+j_n)}e^{z^2}\int_{R_{jk}^*} e^{i2^{-j_1}\lambda_1 t_1 + i2^{-j_2}\lambda_2 t_2 + ... + i2^{-j_n}\lambda_n t_n + i\lambda_{n+1}S_1(\mathbf{t})}$$

$$\times (\min(2^{-j_1}|t_1|, ..., 2^{-j_n}|t_n|, |S_1^*(\mathbf{t})|))^z K_1(\mathbf{t})\,d\mathbf{t} \tag{3.20}$$

We apply the Van der Corput lemma for ath derivatives in the v direction in (3.20), and then integrate the result in the $n - 1$ orthogonal directions. Note that due to the way the R_j were divided into R_{jk}, there may be boundedly many intervals of integration in the v direction, in which case we apply the Van der Corput lemma on each interval and add the result.

In the situation at hand the quantity denoted by $M^{-\frac{1}{k}}$ in (3.18) is given by $|\lambda_{n+1}S_1^*(\mathbf{t}_0)|^{-\frac{1}{a}}$. Since $|\lambda_{n+1}| > \frac{1}{n+1}|\lambda|$ in our argument, this in turn is bounded by $C|\lambda S_1^*(\mathbf{t}_0)|^{-\frac{1}{a}}$. In

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view of the $|\psi'(x)|$ in the integral (3.18), we examine the effect of a derivative in the v direction on the various factors in (3.20). Since $R_{jk} \subset [\frac{1}{2}, 1]^n$, a derivative landing on $(\min(2^{-j_1}|t_1|, ..., 2^{-j_n}|t_n|, |S_1^*(\mathbf{t})|))^z$ will introduce a factor of $C|z|$. In view of $(1.2a) - (1.2b)$ and the fact that $R_{jk} \subset [\frac{1}{2}, 1]^n$, a derivative landing on $K_1(\mathbf{t})$ introduces a factor of C when using $(1.2a) - (1.2b)$ in upper bounds. Thus applying the Van der Corput lemma in the v direction has the overall effect of introducing a factor bounded by $C|z||\lambda S_1^*(\mathbf{t}_0)|^{-\frac{1}{a_0}}$. Translating this back into the unscaled coordinates, we get

$$I_{jk} \le C|ze^{z^2}| \int_{R_j} |\lambda S^*(\mathbf{t}_0)|^{-\frac{1}{a}} \min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|)^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.21)$$

Note that $|ze^{z^2}|$ is uniformly bounded on a given line $\operatorname{Re} z = s_1$, and that $|S^*(\mathbf{t})|$ varies by a factor of at most C on any dyadic rectangle. Thus (3.21) implies

$$I_{jk} \le C_{s_1} \int_{R_j} |\lambda S^*(\mathbf{t})|^{-\frac{1}{a}} \min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|)|^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.22a)$$

Adding this over the boundedly many k for a given j, we get the same form for the estimate for I_j, the portion of the integral (3.8) coming from the rectangle R_j:

$$I_j \le C_{s_1} \int_{R_j} |\lambda S^*(\mathbf{t})|^{-\frac{1}{a}} \min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|)^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.22b)$$

By simply taking absolute values of the integrand and integrating, we also have

$$I_j \le C'_{s_1} \int_{R_j} \min(|t_1|, ..., |t_n|, |S^*(\mathbf{t})|)^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.23)$$

Combining (3.22b) and (3.23) and using $|S^*(\mathbf{t})|$ in the minimum (note that we are using that $s_1 \ge 0$ here), we get

$$I_j \le C''_{s_1} \int_{R_j} \min(1, |\lambda S^*(\mathbf{t})|^{-\frac{1}{a}}) |S^*(\mathbf{t})|^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.24)$$

Since $a \le \max(o(S), 2)$, this in turn is bounded by

$$I_j \le C''_{s_1} \int_{R_j} \min(1, |\lambda S^*(\mathbf{t})|^{-\frac{1}{\max(o(S),2)}}) |S^*(\mathbf{t})|^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.25)$$

Finally, adding (3.25) over all dyadic rectangles R_j gives for some $r_0 > 0$ that

$$|\hat{\nu}_z(\lambda)| \le C'''_{s_1} \int_{(-r_0, r_0)^n} \min(1, |\lambda S^*(\mathbf{t})|^{-\frac{1}{\max(o(S),2)}}) |S^*(\mathbf{t})|^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.26a)$$

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Since everything in the integral (3.26a) is even in all variables, we can replace $(-r_0, r_0)^n$ by $(0, r_0)^n$ and say

$$|\hat{\nu}_z(\lambda)| \leq C_{s_1}'''' \int_{(0,r_0)^n} \min(1, |\lambda S^*(\mathbf{t})|^{-\frac{1}{\max(o(S),2)}}) |S^*(\mathbf{t})|^{s_1} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} \qquad (3.26b)$$

In terms of the measure $d\mu = \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t}$, this can be rewritten as

$$|\hat{\nu}_z(\lambda)| \leq C_{s_1}'''' \int_{(0,r_0)^n} \min(1, |\lambda S^*(\mathbf{t})|^{-\frac{1}{\max(o(S),2)}}) |S^*(\mathbf{t})|^{s_1} \, d\mu \qquad (3.26c)$$

It is natural to break up (3.26c) into $|S^*(\mathbf{t})| < |\lambda|^{-1}$ and $|S^*(\mathbf{t})| > |\lambda|^{-1}$ portions. We obtain

$$|\hat{\nu}_z(\lambda)| \leq C_{s_1}'''' \int_{|S^*(\mathbf{t})| < |\lambda|^{-1}} |S^*(\mathbf{t})|^{s_1} \, d\mu$$

$$+ \; C_{s_1}'''' \int_{|S^*(\mathbf{t})| > |\lambda|^{-1}} |\lambda|^{-\frac{1}{\max(o(S),2)}} |S^*(\mathbf{t})|^{s_1 - \frac{1}{\max(o(S),2)}} \, d\mu \qquad (3.27)$$

If one writes the first integral in (3.27) as the sum in i of integrals over the points where $2^{-i-1}|\lambda|^{-1} < |S^*(\mathbf{t})| \leq 2^{-i}|\lambda|^{-1}$, and inserts (1.5) into each term, since $s_1 > 0$ one gets a geometric sum that decreases as i increases. Thus the overall sum can be bounded by a constant times the $i = 0$ term, which by (1.5) is bounded by $C|\lambda|^{-s_1 - a_0}(\ln|\lambda|)^{d_0}$. Since we are assuming $s_1 > \frac{1}{\max(o(S),2)} - g$ and since $g \leq a_0$, we have $s_1 + a_0 > \frac{1}{\max(o(S),2)}$ and as a result the bound $C|\lambda|^{-s_1 - a_0}(\ln|\lambda|)^{d_0}$ implies a bound of $C'|\lambda|^{-\frac{1}{\max(o(S),2)} - \epsilon_{s_1}}$ for some $\epsilon_{s_1} > 0$.

Similarly, one can write the second integral as the sum in i of integrals over the points $\mathbf{t}$ where $2^i|\lambda|^{-1} < |S^*(\mathbf{t})| \leq 2^{i+1}|\lambda|^{-1}$ and then insert (1.5). The resulting estimates change exponentially in i, with the $i = 0$ term being comparable to $C|\lambda|^{-s_1 - a_0}(\ln|\lambda|)^{d_0} < C'|\lambda|^{-\frac{1}{\max(o(S),2)} - \epsilon_{s_1}}$ like above, while the last term will be comparable to $C|\lambda|^{-\frac{1}{\max(o(S),2)}}$ As a result we have a bound of $C''|\lambda|^{-\frac{1}{\max(o(S),2)}}$ for the whole second integral in (3.27). Combining with the above bounds for the first integral, we see that we have an estimate of the form $|\hat{\nu}_z(\lambda)| \leq C_{s_1}'''''|\lambda|^{-\frac{1}{\max(o(S),2)}}$, as is needed for Lemma 3.2. This concludes the argument for the case where $|\lambda_{n+1}| > \frac{1}{n+1}|\lambda|$ and therefore the proof of Lemma 3.2.

The end of the proof of part 1 of Theorem 1.1.

By Lemma 3.2, for $Re(z) = s_1 > \max(\frac{1}{\max(o(S),2)} - g, 0)$ we have $\|T_z f\|_{L^2_\beta} \leq C_{s_1}\|f\|_{L^2}$ with $\beta = \frac{1}{\max(o(S),2)}$. Furthermore, by Lemma 3.1, if $2 < p < \infty$ and $Re(z) = s_0 > \max(-g, -\frac{1}{\max(o(S),2)})$, then we have an estimate $\|T_z f\|_{L^p} \leq C_{s_0}\|f\|_{L^p}$.

Note that $0 = \alpha \max(-g, -\frac{1}{\max(o(S),2)}) + (1 - \alpha)(\max(\frac{1}{\max(o(S),2)} - g, 0))$, where $\alpha = \max(0, 1 - g\max(2, o(S)))$. Thus if $1 > \alpha' > \alpha$, one can write $0 = \alpha' s_0 + (1 - \alpha') s_1$, where $0 > s_0 > \max(-g, -\frac{1}{\max(o(S),2)})$ and $s_1 > \max(\frac{1}{\max(o(S),2)} - g, 0)$. Hence by complex interpolation

$T = T_0$ is bounded from L^q to L^q_β, where $\frac{1}{q} = \alpha'\frac{1}{p} + (1-\alpha')\frac{1}{2}$ and $\beta = \alpha'0 + (1-\alpha')\frac{1}{\max(o(S),2)}$. Explicitly, we have $q = \frac{1}{\frac{1}{2}+\alpha'(\frac{1}{p}-\frac{1}{2})}$ and $\beta = \frac{1-\alpha'}{\max(o(S),2)}$.

Using interpolation again, we have that T is bounded from L^r to L^r_γ for $(\frac{1}{r}, \gamma)$ in the closed triangle with vertices $(0,0), (1,0)$, and $(\frac{1}{q}, \beta)$. Taking the union of these triangles as α' approaches α and p approaches ∞, we get that T is bounded from L^r to L^r_γ for $(\frac{1}{r}, \gamma)$ in the open triangle with vertices $(0,0), (1,0)$, and $(\frac{1}{q'}, \beta')$, where $q' = \frac{1}{\frac{1}{2}-\frac{1}{2}\alpha} = \frac{2}{\min(1, g\max(o(S),2))} = \max(2, \frac{2}{g\max(o(S),2)})$ and where $\beta' = \frac{1-\alpha}{\max(o(S),2)} = \min(\frac{1}{\max(o(S),2)}, g)$.

In the case where $g \geq \frac{1}{\max(o(S),2)}$, the union of these triangles is the open triangle with vertices $(0,0), (1,0)$, and $(\frac{1}{2}, \frac{1}{\max(o(S),2)})$, which is the boundedness region stipulated by part 1 of Theorem 1.1 in this case. If $g < \frac{1}{\max(o(S),2)}$, the union of these triangles is the open triangle with vertices $(0,0), (1,0)$, and $(\frac{g\max(o(S),2)}{2}, g)$. So T is bounded from L^r to L^r_γ for $(\frac{1}{r}, \gamma)$ in this region. By duality, it is also bounded from L^r to L^r_γ for $(\frac{1}{r}, \gamma)$ such that $(1 - \frac{1}{r}, \gamma)$ is in this region, giving the triangle with vertices $(0,0), (1,0)$, and $(1 - \frac{g\max(o(S),2)}{2}, g)$. Thus T is bounded from L^r to L^r_γ for $(\frac{1}{r}, \gamma)$ in the open trapezoidal region with vertices $(0,0), (1,0)$, $(\frac{g\max(o(S),2)}{2}, g)$, and $(1 - \frac{g\max(o(S),2)}{2}, g)$. This is the region stipulated by Theorem 1.1 in the case where $g < \frac{1}{\max(o(S),2)}$. This completes the proof of part 1 of Theorem 1.1.

4   Proof of part 2 of Theorem 1.1.

Assume the hypotheses of part 2 of Theorem 1.1 hold. Namely, assume that $g < 1$, $K(\mathbf{t})$ is nonnegative, and there exists a positive constant C_0 and a neighborhood N_0 of the origin such that $K(\mathbf{t}) > C_0 \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k}$ on $\{\mathbf{t} \in N_0 : t_i \neq 0 \text{ for all } i\}$. Suppose that T is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^p_\beta(\mathbb{R}^{n+1})$ for some $1 < p < \infty$. Then by duality, T is bounded from $L^q(\mathbb{R}^{n+1})$ to $L^q_\beta(\mathbb{R}^{n+1})$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Since either $p \leq 2 \leq q$ or $q \leq 2 \leq p$, interpolation shows that T is bounded from $L^2(\mathbb{R}^{n+1})$ to $L^2_\beta(\mathbb{R}^{n+1})$. As a result, if ν is the measure such that $Tf = f * \nu$, we have an estimate $|\hat{\nu}(\lambda)| \leq C(1+|\lambda|)^{-\beta}$. Explicitly, we have

$$\left| \int_{\mathbb{R}^n} e^{i\lambda_1 t_1 + i\lambda_2 t_2 + \ldots + i\lambda_n t_n + i\lambda_{n+1} S(\mathbf{t})} K(\mathbf{t}) \, d\mathbf{t} \right| \leq C(1+|\lambda|)^{-\beta} \tag{4.1}$$

We will first show that $\min(\beta, 1) \leq a_0$, and then we will show that $\min(\beta, 1) \leq l_i - \alpha_i$ for each i. Since $g = \min(a_0, l_1 - \alpha_1, \ldots, l_m - \alpha_m)$, this leads to $\min(\beta, 1) \leq g$. Since we are assuming $g < 1$, we obtain that $\beta \leq g$ as desired.

Since (4.1) holds in all directions, it holds in the $(0, \ldots, 0, \lambda_{n+1})$ direction, so (4.1) implies that for all λ_{n+1} we have

$$\left| \int_{\mathbb{R}^n} e^{i\lambda_{n+1} S(\mathbf{t})} K(\mathbf{t}) \, d\mathbf{t} \right| \leq C(1+|\lambda_{n+1}|)^{-\beta} \tag{4.2}$$

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Denote the integral on the left of (4.2) by $U(\lambda_{n+1})$. Let $B(x)$ be a bump function on $\mathbb{R}$ whose Fourier transform is nonnegative, compactly supported, and equal to 1 on a neighborhood of the origin, and let ϵ be a small positive number. If $0 < \beta' < \min(\beta, 1)$, then (4.2) implies that for some constant A independent of ϵ one has

$$\int_{\mathbb{R}} |U(\lambda_{n+1})\lambda_{n+1}^{\beta'-1} B(\epsilon\lambda_{n+1})| \, d\lambda_{n+1} < A \tag{4.3}$$

As a result we have

$$\left| \int_{\mathbb{R}^{n+1}} e^{i\lambda_{n+1}S(\mathbf{t})} K(\mathbf{t})|\lambda_{n+1}|^{\beta'-1} B(\epsilon\lambda_{n+1}) \, d\lambda_{n+1} \, d\mathbf{t} \right| < A \tag{4.4}$$

We do the integral in λ_{n+1} in (4.4). Letting $b_\epsilon(y)$ be the convolution of $|y|^{-\beta'}$ with $\frac{1}{\epsilon}\hat{B}(\frac{y}{\epsilon})$, for a constant A' independent of ϵ we get

$$\left| \int_{\mathbb{R}^n} b_\epsilon(-S(\mathbf{t})) \, K(\mathbf{t}) \, d\mathbf{t} \right| < A' \tag{4.5}$$

Note that both $b_\epsilon(-S(\mathbf{t}))$ and $K(\mathbf{t})$ are nonnegative here. Thus we may remove the absolute value and let $\epsilon \to 0$ to obtain

$$\int_{\mathbb{R}^n} |S(\mathbf{t})|^{-\beta'} K(\mathbf{t}) < \infty \tag{4.6}$$

Since $K(\mathbf{t})$ is bounded below by $C_0 \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k}$ on a neighborhood N_0 of the origin, we therefore have

$$\int_{N_0} |S(\mathbf{t})|^{-\beta'} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} < \infty \tag{4.7}$$

In other words, $|S(\mathbf{t})|^{-\beta'}$ is in $L^1(N_0)$ with respect to the measure μ. Hence it is in weak L^1, and we have the existence of a constant C such that

$$\mu(\{\mathbf{t} \in N_0 : |S(\mathbf{t})|^{-\beta'} > \epsilon\}) \leq C\frac{1}{\epsilon} \tag{4.8}$$

Replacing ϵ by $\epsilon^{-\beta'}$, we get

$$\mu(\{\mathbf{t} \in N_0 : |S(\mathbf{t})| < \epsilon\}) \leq C\epsilon^{\beta'} \tag{4.9}$$

By Lemma 2.1 of [G2] there is a constant C' such that $|S(\mathbf{t})| \leq C'S^*(\mathbf{t})$, so we also have

$$\mu(\{\mathbf{t} \in N_0 : S^*(\mathbf{t}) < \epsilon\}) \leq C''\epsilon^{\beta'} \tag{4.10}$$

In view of the definition of a_0, we have $\beta' \leq a_0$. Since this holds for each β' satisfying $0 < \beta' < \min(\beta, 1)$, we conclude that $\min(\beta, 1) \leq a_0$.

Showing that $\min(\beta, 1) \leq l_i - \alpha_i$ for each i is quite similar. This time we use (4.1) in the $(0, ..., 0, \lambda_i, 0, ..., 0)$ direction, and the steps from (4.2) to (4.7) lead to

$$\int_{N_0} |t_i|^{-\beta'} \prod_{k=1}^m |\mathbf{t}_k|^{-\alpha_k} \, d\mathbf{t} < \infty \tag{4.11}$$

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Suppose k_0 is such that t_i is a component of $\mathbf{t}_{k_0}$. Then integrating (4.11) in the remaining variables first leads to the following holding for some $\delta > 0$.

$$\int_{(0,\delta)^{l_i}} |t_i|^{-\beta'} |\mathbf{t}_{k_0}|^{-\alpha_{k_0}} \, d\mathbf{t}_{k_0} < \infty \tag{4.12}$$

As a result, $\beta' + \alpha_{k_0} < l_{k_0}$. Since this is true for all $0 < \beta' < \min(\beta, 1)$, we conclude that $\min(\beta, 1) \leq l_{k_0} - \alpha_{k_0}$. Since t_i was arbitrary, this holds for all k_0. This completes the proof of part 2 of Theorem 1.1.

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