Simply Nondegenerate Multilinear Oscillatory Integral Operators with Smooth Phase

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1) Introduction

As in [CLTaTh], in this paper we consider operators of the form

$$T_\lambda(f_1, ..., f_n) = \int_{R^m} e^{i\lambda S(x)} \prod_{j=1}^{n} f_j(\pi_j(x))\eta(x)\, dx \tag{1.1}$$

Here $S(x)$ is a smooth real-valued phase function, λ is a parameter, and $\eta(x)$ is a C^1 cutoff function supported in the unit ball centered at the origin. π_j denotes projection from $\mathbf{R}^m$ to some subspace V_j of dimension strictly less than m, and each f_j is an L^∞ function. A natural question to ask is for which $\{V_j\}_{j=1}^n$ and which smooth phase functions $S(x)$ do we get an estimate of the form

$$|T_\lambda(f_1, ..., f_n)| \leq C(1 + |\lambda|)^{-\epsilon} \prod_{j=1}^{n} ||f_j||_\infty \tag{1.2}$$

Here $\epsilon > 0$ is to be independent of $\eta(x)$. In [CLTaTh] this question was considered in considerable depth, and attention was focused on the case where $S(x)$ is a polynomial. Here we will extend several of their results to the general smooth case. When the π_j are all projections onto 1 dimensional spaces and one considers L^2 norms instead of L^∞ norms, much additional work has been done, such as [PS] [PSSt] [Se].

Note that if $S(x) = \sum_j p_j \circ \pi_j(x)$, then there can be no positive ϵ satisfying (1.2). For one can let $f_j = e^{-i\lambda p_j}$ and then (1.1) is equal to $\int \eta(x)$, a quantity independent of λ. In [CLTaTh], such a polynomial $S(x)$ is referred to as *degenerate relative to* $\{V_j\}_{j=1}^n$, and a polynomial $S(x)$ that has no realization of the form $S(x) = \sum_j p_j \circ \pi_j(x)$, p_j polynomials, is referred to as *nondegenerate relative to* $\{V_j\}_{j=1}^n$.

One way one can ensure that a polynomial is nondegenerate relative to $\{V_j\}_{j=1}^n$ is for there to exist a differential operator L of the form $L = \prod_{j=1}^n (w_j \circ \nabla)$, with each w_j a unit vector in $V_j^\perp$, such that $L(S)$ is not the zero function. For if one applies L to a polynomial of the form $S(x) = \sum_j p_j \circ \pi_j(x)$, each factor knocks out one of the terms and one is left with zero. [CLTaTh] refers to a polynomial S for which there exists such an L as a *simply nondegenerate* polynomial. Hence simply nondegenerate polynomials are all nondegenerate. The converse does not hold in general, as explained in [CLTaTh]. However for several important classes of $\{V_j\}_{j=1}^n$, simple nondegeneracy is equivalent to nondegeneracy. In [CLTaTh], the existence of an ϵ satsifying (1.2) is shown for simply nondegenerate polynomials $S(x)$ (among others), and therefore for all nondegenerate polynomials when simply nondgenerate and nondegenerate are equivalent. Their estimates are uniform in the sense that for a given family $\{V_j\}_{j=1}^n$, ϵ depends only on the degree of $S(x)$ and $\max_{|x|\leq 1} |L(S(x))|$, while C depends on these quantities as well as the C^1 norm of $\eta(x)$.

The purpose of this paper is to extend the simple nondegeneracy results of [CLTaTh] to the case of general C^∞ phase. Since the ϵ in the proofs in [CLTaTh] depend on the degree of $S(x)$, the results there do not carry over immediately.

Definition: Suppose $S(x)$ is a smooth function defined in a neighborhood of a point a. $S(x)$ is called degenerate at a if there are smooth functions $s_j : V_j \rightarrow \mathbf{R}$ for which $S(x) - \sum_{j=1}^n s_j(\pi_j(x))$ has a zero of infinite order at a. We say $S(x)$ is nondegenerate at a if there do not exist such functions.

Definition: Suppose $S(x)$ is a smooth function defined in a neighborhood of a point a. $S(x)$ is called simply nondegenerate at a if there exists a differential operator L of the form $L = \prod_{j=1}^n (w_j \circ \nabla)$, with each w_j a unit vector in $V_j^\perp$, such that $L(S(x))$ does not vanish to infinite order at a.

As in the polynomial case, in the smooth case it is immediate that if $S(x)$ is simply nondegenerate at a then it is nondegenerate at a.

Our main theorem gives uniform decay for simply nondegenerate $S(x)$ in the smooth case:

Theorem 1.1: Suppose $S(x)$ is simply nondegenerate at 0. Let L be as above and α be a multiindex such that $\partial^\alpha (L \circ S)(0) = c \neq 0$. Then there is an open U containing the origin, U depending on c, m, n, $|\alpha|$, and the $C^{|\alpha|+n+1}$ norm of S, such that if $\eta(x)$ is supported in U, for $\epsilon = \frac{2^{-n+1}}{|\alpha|+n}$ and any $\delta > 0$ we have

$$|T_\lambda(f_1, ..., f_n)| \leq C(1 + |\lambda|)^{-\epsilon+\delta} \prod_{j=1}^n ||f_j||_\infty \tag{1.3}$$

Here C depends on c, m, n, $|\alpha|$, $||\eta||_{C^1}$, and the C^l norm of $S(x)$ for some l depending on δ. For fixed values of the other parameters, C is $O(||\eta||_{C^1})$. In the case where S is a polynomial, we can take $\delta = 0$.

We will see in Lemma 1.4 below that if $\{V_j\}_{j=1}^n$ is such that simple nondegeneracy is equivalent to nondegeneracy for polynomial $S(x)$, then simple nondegeneracy is also equivalent to nondegeneracy for smooth $S(x)$ at any point. As a result, by Proposition 3.1 and Lemma 3.5 of [CLTaTh], Theorem 1.1 immediately implies the following smooth analogues of Theorem 2.2 and Theorem 2.4 of [CLTaTh]:

Theorem 1.2: Suppose each V_j has codimension 1. Then (1.2) holds for some $\epsilon > 0$ for all η with sufficiently small support if and only if $S(x)$ is nondegenerate at the origin.

Theorem 1.3: Suppose $\{V_j : 1 \leq j \leq n\}$ are each of dimension $k < n$. Suppose the orthocomplements $V_j^{\perp}$ together span a space of dimension at most $(m-k)n$, and $(m-k)n \leq m$. Then (1.2) holds for some $\epsilon > 0$ for all η with sufficiently small support if and only if $S(x)$ is nondegenerate at the origin.

In view of Theorem 1.1, Theorems 1.2 and 1.3 are uniform in the sense that ϵ depends only on n and the order of vanishing of $L \circ S$ at the origin.

Lemma 1.4: Suppose $\{V_j\}_{j=1}^{n}$ are such that simple nondegeneracy is equivalent to nondegeneracy for any polynomial $S(x)$. Then for smooth $S(x)$, simple nondegeneracy is equivalent to nondegeneracy at any point.

Proof: Since simple nondegeneracy readily implies nondegeneracy, we assume that $S(x)$ is not simply nondegenerate at a point a, and we will show that $S(x)$ is degenerate at a. Clearly it suffices to assume that $a = 0$. Let $\sum_{\alpha} S_{\alpha} x^{\alpha}$ denote the (possibly divergent) Taylor expansion of $S(x)$ about the origin. For each positive integer i, we define

$$R_i(x) = \sum_{|\alpha|=i} S_{\alpha} x^{\alpha} \tag{1.4}$$

Suppose $w_j \in V_j^{\perp}$ for $1 \leq j \leq n$. Analogous to (1.4) we can write

$$\prod_{j=1}^{n} (w_j \circ \nabla) S = \sum_{i} \prod_{j=1}^{n} (w_j \circ \nabla) R_i(x) \tag{1.5}$$

Note that $\prod_{j=1}^{n} (w_j \circ \nabla) R_i(x)$ is the sum of the terms of (1.5) of degree $i - n$. Since $S(x)$ is not simply nondegenerate at the origin, for any such w_j the right hand side of (1.5) must have no nonvanishing terms. In other words, for all i and all choices of w_j we have

$$\prod_{j=1}^{n} (w_j \circ \nabla) R_i(x) = 0 \tag{1.6}$$

Since we are assuming simple nondegeneracy is equivalent to nondegeneracy for polynomials, this means we can write

$$R_i(x) = \sum_{j=1}^{n} p_{ij} \circ \pi_j(x) \tag{1.7}$$

Here p_{ij} are polynomials. Since the terms of $R_i(x)$ are all of degree i, the terms of the righthand sum of all other degrees must add to zero. Hence we may replace each p_{ij} by the sum of its monomials of degree i if necessary, and assume each p_{ij} is homogeneous of degree i.

Next, for a given j we let r_j be a rotation such that the image of $r_j \circ \pi_j$ is $\{(y,z) \in \mathbf{R}^{k_j} \times \mathbf{R}^{m-k_j} : z = 0\}$. Since $p_{ij} \circ \pi_j = p_{ij} \circ r_j^{-1} \circ (r_j \circ \pi_j)$ and $p_{ij} \circ \pi_j$ is homogeneous of degree i, $p_{ij} \circ r_j^{-1}$ is a homogeneous polynomial in the y variables of degree i. Thus the monomials of $p_{ij} \circ r_j^{-1}(y)$ for different i's do not overlap. As a result, it makes sense to write $\sum_i p_{ij} \circ r_j^{-1}(y)$ as a (possibly nonconvergent) power series in the y variables.

By a famous theorem of Borel (see p.16 of [H]), we can let $q_j(y)$ be a C^∞ function whose Taylor series about the origin is given by $\sum_i p_{ij} \circ r_j^{-1}(y)$. So for a fixed i_0, $q_j(y) - \sum_{i<i_0} p_{ij} \circ r_j^{-1}(y)$ has a zero of order i_0 at the origin. Similarly, $q_j \circ r_j \circ \pi_j(x) - \sum_{i<i_0} p_{ij} \circ \pi_j(x)$ has a zero of order i_0 at the origin. Adding this over all j, we get that $\sum_j q_j \circ r_j \circ \pi_j(x) - \sum_{i<i_0} R_i(x)$ has a zero of order i_0 at the origin. By (1.4), this means that $\sum_j q_j \circ r_j \circ \pi_j(x) - S(x)$ has a zero of infinite order at the origin. Since each $q_j \circ r_j$ is C^∞, this means that $S(x)$ is degenerate to infinite order at the origin, and we are done.

2) Proof of Theorem 1.1

Case 1: n = 1, $S(x)$ a polynomial:

Let g denote the degree of $S(x)$ and let k_1 denote the dimension of V_1. Rotate coordinates such that $V_1 = \{x = (y,z) \in R^{k_1} \times R^{m-k_1} : z = 0\}$, and such that the vector w_1 in the definition of simply nondegenerate is in the z_1 direction. Our arguments will resemble the proof of the Van der Corput lemma in the z_1 direction. Namely, for some $a > 0$ to be determined we divide the domain of (1.1) into 2 parts. The first part is where $|\partial_{z_1} S(x)| < |\lambda|^{-a}$, and the second where $|\partial_{z_1} S(x)| \geq |\lambda|^{-a}$. On the first part, we take absolute values and integrate, obtaining a term of absolute value at most

$$m\{x \in supp(\eta) : |\partial_{z_1} S(x)| < |\lambda|^{-a}\}||f_1||_\infty ||\eta||_\infty$$

Here m denotes Lebesgue measure. This in turn is bounded by

$$m\{x \in supp(\eta) : |\partial_{z_1} S(x)| < |\lambda|^{-a}\}\ ||f_1||_\infty ||\eta||_{C^1} \tag{2.1}$$

On the second part we do an integration by parts in z_1 on the oscillatory factor. We get

$$\frac{1}{i\lambda} \int_{|S_{z_1}(y,z)| \geq |\lambda|^{-a}} e^{i\lambda S(y,z)} [\frac{S_{z_1 z_1}(y,z)}{S_{z_1}(y,z)^2} \eta(y,z) - \eta_{z_1}(y,z)] f_1(y)\, dy\, dz \tag{2.2}$$

The expression (2.2) is of absolute value at most

$$\frac{1}{|\lambda|} ||f_1||_\infty ||\eta||_\infty (\int_{|(y,z)|<1,\, |S_{z_1}(y,z)| \geq |\lambda|^{-a}} |\frac{S_{z_1 z_1}(y,z)}{S_{z_1}(y,z)^2}|\, dy\, dz) + \frac{1}{|\lambda|} ||f_1||_\infty ||\eta||_{C^1} \tag{2.3}$$

For fixed y and $(z_2, ..., z_n)$, the domain of integration of (2.3) can be written as the union of at most $g-1$ intervals on which S_{z_1} is monotonic as a function of z_1. Consequently, on each of these intervals, one may perform the integration of $|\frac{S_{z_1 z_1}(y,z)}{S_{z_1}(y,z)^2}|$ as in the proof of the Van der Corput lemma, obtaining the difference of $\frac{1}{S_{z_1}(y,z)}$ at the endpoints, so that (2.3) is at most

$$(2g-2)|\lambda|^{a-1}||f_1||_\infty||\eta||_\infty + \frac{1}{|\lambda|}||f_1||_\infty||\eta||_{C^1}$$
$$\leq (2g-1)|\lambda|^{a-1}||f_1||_\infty||\eta||_{C^1} \tag{2.4}$$

Combining (2.1) and (2.4), in the current situation the oscillatory integral (1.1) is bounded in absolute value by:

$$[m\{x \in supp(\eta) : |\partial_{z_1} S(x)| < |\lambda|^{-a}\} + (2g-1)|\lambda|^{a-1}]\ ||f_1||_\infty||\eta||_{C^1} \tag{2.5}$$

By the simple nondegeneracy of S, there is some multiindex α for which $\partial^\alpha \partial_{z_1} S(0) = c \neq 0$. As a result, by [C1] for example, if the support of η is sufficiently small, one has

$$m\{x \in supp(\eta) : |\partial_{z_1} S(x)| < |\lambda|^{-a}\} < C|\lambda|^{-\frac{a}{|\alpha|}} \tag{2.6}$$

Here C depends on c, $|\alpha|$, m, and the $C^{|\alpha|+2}$ norm of S. As a result, we can select $a = \frac{|\alpha|}{|\alpha|+1}$, and (2.5) is at most

$$(C + 2g - 1)|\lambda|^{-\frac{1}{1+|\alpha|}}||f_1||_\infty||\eta||_{C^1} \tag{2.7}$$

This gives us the estimate required by Theorem 1.1 and we are done when $n = 1$ and $S(x)$ is a polynomial.

Case 2: $n > 1$, $S(x)$ **a polynomial**.

To prove Theorem 1.1 in this case we will need the following procedure. Given a smooth function $Q(x)$ defined in a neighborhood of the origin, we successively define functions $Q^1(y,z)$, $Q^2(y,z,\zeta^2)$, ..., $Q^n(y,z,\zeta^2,...,\zeta^n)$. Let $Q^1(y,z)$ be $Q(x)$ in rotated coordinates such that V_1 is $\{(y,z) \in R^{k_1} \times R^{m-k_1} : z = 0\}$ and such that w_1 is the z_1 direction. Once Q^{j-1} is defined, one defines Q^j as follows. Consider coordinates such that $V_j = \{(y,z) \in R^{k_j} \times R^{m-k_j} : z = 0\}$ with w_j being the z_1 direction. Letting $Q_\zeta(y,z) = Q(y, z+\zeta) - Q(y,z)$ in these coordinates, we define $Q^j(y,z,\zeta^2,...,\zeta^j)$ to be $(Q_{\zeta^j})^{j-1}(y,z,\zeta^2,...,\zeta^{j-1})$. I claim that inductively we have

$$\frac{\partial^{j-1} Q^j}{\partial_{\zeta_1^j} ....\partial_{\zeta_1^2}}(y,z,0) = \prod_{i=2}^{j}(w_i \circ \nabla)Q(y,z) \tag{2.8}$$

For if we know the $j-1$ case (the case $j = 1$ is a tautology) we have the following:

$$\frac{\partial^{j-2} Q^j}{\partial_{\zeta_1^{j-1}} ....\partial_{\zeta_1^2}}(y,z,0,..,0,\zeta^j) = \frac{\partial^{j-2}(Q_{\zeta^j})^{j-1}}{\partial_{\zeta_1^{j-1}} ....\partial_{\zeta_1^2}}(y,z,0) = \prod_{i=2}^{j-1}(w_i \circ \nabla)Q_{\zeta^j}(y,z)$$

$$= \prod_{i=2}^{j-1}((w_i \circ \nabla)Q)(y, z+\zeta^j) - \prod_{i=2}^{j-1}((w_i \circ \nabla)Q)(y,z)$$

Then taking directional derivatives in the ζ_1^j direction gives (2.8). We will be most interested in (2.8) when $Q = (w_1 \circ \nabla)S$. In fact, Theorem 1.1 in the polynomial case will follow readily from the following lemma:

Lemma 2.1: Suppose T_λ is of the form (1.1) with $S(x)$ a polynomial. Let δ be such that $|x| < \delta$ for all $x \in supp(\eta)$. Let $s(x)$ denote $(w_1 \circ \nabla)S(x)$, and let $m\{s^n(y,z,\zeta^2,...\zeta^n) < b\}$ denote $m\{(y,z,\zeta^2,...\zeta^n) \in [-2\delta, 2\delta]^{m+\sum_{j>1}(m-k_j)} : s^n(y,z,\zeta^2,...\zeta^n) < b\}$. For any $a > 0$ we have $|T_\lambda(f_1,...,f_n)| <$

$$[m\{(s^n(y,z,\zeta^2,...\zeta^n) < |\lambda|^{-a}\} + (2g-1)|\lambda|^{a-1}]^{2^{1-n}} ||\eta||_{C^1} \prod_{j=1}^{n} ||f_j||_\infty \tag{2.9}$$

Proof: The case $n = 1$ follows from (2.5), so we can assume the lemma for $n-1$ and prove it for n. We work in the general setup used in the inductive step in Theorem 2.3 of [CLTaTh], originating in [CaCW]. Like above and in [CLTaTh], we rotate to coordinates such that $V_n = \{(y,z) \in R^{k_n} \times R^{m-k_n} : z = 0\}$. As above, we assume the direction w_n in the definition of simply nondegenerate is the z_1 direction. We now have

$$T_\lambda(f_1,...,f_n) = \int f_n(y)(\int e^{i\lambda S(y,z)} \prod_{j=1}^{n-1} f_j(\pi_j(y,z))\eta(y,z)\,dz)\,dy \tag{2.10}$$

This equals $< U_\lambda(f_1,...,f_{n-1}), \bar{f}_n >$ for an appropriate operator U_λ. Hence we have

$$|T_\lambda(f_1,...,f_n)| \leq ||f_n||_2 ||U_\lambda(f_1,...,f_{n-1})||_2 \leq ||f_n||_\infty ||U_\lambda(f_1,...,f_{n-1})||_2 \tag{2.11}$$

As in [CLTaTh], we write $||U_\lambda(f_1,...,f_{n-1})||_2^2$ as

$$\int_{\mathbf{R}^{m-k_n}} [\int_{\mathbf{R}^m} e^{i\lambda(S(y,z)-S(y,z+\zeta))} \prod_{j<n} f_j(\pi_j(y,z))\bar{f}_j(\pi_j(y,z+\zeta))\eta(y,z)\bar{\eta}(y,z+\zeta)dz\,dy]\,d\zeta \tag{2.12}$$

For fixed ζ, the integral inside the brackets is of the form (1.1), with phase $S(y,z)-S(y,z+\zeta)$ and acting on the $n-1$ functions $f_j(\pi_j(y,z))\bar{f}_j(\pi_j(y,z+\zeta))$ for $j < n$. Thus one can apply the inductive hypothesis to the bracketed integral and say it is bounded in absolute value by

$$[m\{(s_\zeta)^{n-1}(y,z,\zeta^2,...,\zeta^{n-1}) < |\lambda|^{-a}\} + (2g-1)|\lambda|^{a-1}]^{2^{2-n}} ||\eta||_{C^1}^2 \prod_{j<n} ||f_j||_\infty^2 \tag{2.13}$$

Here $s_\zeta(y,z) = s(y,z+\zeta) - s(y,z) = (w_1 \circ \nabla)(S(y,z+\zeta) - S(y,z))$. We integrate (2.13) in ζ, using Jensen's inequality to push the exponent 2^{2-n} to the outside. Given that $(s_\zeta)^{n-1}(y,z,\zeta^2,...\zeta^{j-1}) = s^n(y,z,\zeta^2,...\zeta^{j-1},\zeta)$, we get that (2.12) is bounded by

$$[m\{s^n(y,z,\zeta^2,...,\zeta^n) < |\lambda|^{-a}\} + (2g-1)|\lambda|^{a-1}]^{2^{2-n}} ||\eta||_{C^1}^2 \prod_{j<n} ||f_j||_\infty^2 \tag{2.14}$$

Putting this upper bound for $||U_\lambda(f_1,...,f_{n-1})||_2^2$ into (2.11) gives the lemma and we are done.

We now may prove Theorem 1.1 in the polynomial case. Applying (2.8) to $s(x)$ for $j = n$, the simple nondenegeracy hypothesis implies that for some α we have

$$\partial^\alpha\Big(\frac{\partial^{n-1}s^n}{\partial_{\zeta_1^n}....\partial_{\zeta_1^2}}\Big)(0,0,0) = c \neq 0$$

Thus s^n has a zero of order $|\alpha|+n-1$ at the origin. Using [C] again for example, assuming the support of η is sufficiently small, depending on c, m, n, $|\alpha|$ and the $C^{|\alpha|+n+1}$ norm of S, we have that

$$m\{s^n(y,z,\zeta^1,...,\zeta^n) < |\lambda|^{-a}\} < C|\lambda|^{-\frac{a}{|\alpha|+n-1}} \tag{2.15}$$

Here C is a function of m, $|\alpha|$, n, and c. We choose $a = \frac{|\alpha|+n-1}{|\alpha|+n}$ in (2.15). Then (2.9) gives

$$|T_\lambda(f_1,...,f_n)| < C'|\lambda|^{-\frac{2^{1-n}}{|\alpha|+n}}||\eta||_{C^1}\prod_{j=1}^n ||f_j||_\infty \tag{2.16}$$

Here C' is a function m, $|\alpha|$, n, c, and g. This completes the proof of Theorem 1.1 in the polynomial case.

Case 3: General smooth $S(x)$

Because the ϵ obtained in the polynomial case of Theorem 1.1 did not depend on the degree of $S(x)$, we will be able to extend to general smooth $S(x)$ by dividing the domain into cubes of radius $|\lambda|^{-e}$ for small e, and then approximating $S(x)$ by polynomials of sufficiently high degree on each of these cubes. To be precise, let l be some positive integer, and write the cutoff function $\eta(x)$ as $\sum_{i=1}^I \eta_i(x)$, where each $\eta_i(x)$ is supported on a set of diameter $< |\lambda|^{-\frac{1}{l}}$ and where $I < C_0|\lambda|^{\frac{m}{l}}$. We may also assume $||\eta_i||_{C^1} < C_0|\lambda|^{\frac{1}{l}}||\eta||_{C^1}$. Here C_0 is a uniform constant. We correspondingly write $T_\lambda = \sum_{i=1}^I T_\lambda^i$, where

$$T_\lambda^i(f_1,...,f_n) = \int_{R^m} e^{i\lambda S(x)}\prod_{j=1}^n f_j(\pi_j(x))\eta_i(x)\,dx \tag{2.17}$$

We break each term (2.17) into two parts. For some fixed i, let x_0 be a point in the support of η_i. Let $\bar{S}(x)$ be the sum of the first $2l$ terms of the Taylor expansion of $S(x)$ taken about $x = x_0$. We write $T_\lambda^i = T_\lambda^{i,1} + T_\lambda^{i,2}$, where

$$T_\lambda^{i,1}(f_1,...,f_n) = \int_{R^m} e^{i\lambda\bar{S}(x)}\prod_{j=1}^n f_j(\pi_j(x))\eta_i(x)\,dx \tag{2.18a}$$

$$T_\lambda^{i,2}(f_1,...,f_n) = \int_{R^m} [e^{i\lambda S(x)} - e^{i\lambda\bar{S}(x)}]\prod_{j=1}^n f_j(\pi_j(x))\eta_i(x)\,dx \tag{2.18b}$$

We can apply the polynomial case treated above to (2.18a), and get that

$$|T_\lambda^{i,1}(f_1, ..., f_n)| \leq C|\lambda|^{\frac{1}{l}}|\lambda|^{-\epsilon}||\eta||_{C^1}\prod_{j=1}^{n}||f_j||_\infty \tag{2.19}$$

Here $\epsilon = \frac{2^{1-n}}{|\alpha|+n}$, and C is a constant depending on c, l, m, n, and $|\alpha|$. For (2.18b), we use Taylor's theorem on the difference of exponentials, obtaining

$$|e^{i\lambda S(x)} - e^{i\lambda \bar{S}(x)}| \leq C'(|\lambda|^{-\frac{1}{l}})^{2l} = C'|\lambda|^{-2}$$

Here C' depends on the C^{2l+1} norm of S. Hence we have

$$|T_\lambda^{i,2}(f_1, ..., f_n)| \leq C'|\lambda|^{-2+\frac{1}{l}}||\eta||_{C^1}\prod_{j=1}^{n}||f_j||_\infty \tag{2.20}$$

Adding up (2.19) and (2.20) over all i, we obtain

$$|T_\lambda(f_1, ..., f_n)| \leq C''(|\lambda|^{\frac{m+1}{l}}|\lambda|^{-\epsilon} + |\lambda|^{-2+\frac{m+1}{l}})||\eta||_{C^1}\prod_{j=1}^{n}||f_j||_\infty \tag{2.21}$$

Here C'' depends on c, l, m, n, $|\alpha|$, and the C^{2l+1} norm of S. Given $\delta > 0$, by picking l large enough, one can make (2.21) bounded by $2C''|\lambda|^{-\epsilon+\delta}||\eta||_{C^1}\prod_{j=1}^{n}||f_j||_\infty$ This completes the proof of Theorem 1.1.

3) References

[C] M. Christ, *Hilbert transforms along curves. I. Nilpotent groups*, Annals of Mathematics (2) **122** (1985), no.3, 575-596.

[CaCW] A. Carbery, M. Christ, J. Wright, *Multidimensional van der Corput and sublevel set estimates.*, J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015.

[CLTaTh] M. Christ, X. Li, T. Tao, C. Thiele, *On multilinear oscillatory integrals, nonsingular and singular*, Duke Math. J. **130** (2005), no. 2, 321–351.

[H] L. Hormander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, 2nd ed. Springer-Verlag, Berlin, (1990). xii+440 pp.

[PS] D. H. Phong, E. M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Mathematica **179** (1997), 107-152.

[PSSt] D. H. Phong, E. M. Stein, J. Sturm, *Multilinear level set operators, oscillatory integral operators, and Newton diagrams*, Math. Annalen, **319** (2001), 573-596.

[Se] A. Seeger, *Radon transforms and finite type conditions*, Journal of the American Mathematical Society **11** (1998) no.4, 869-897.

[S] E. Stein, *Harmonic analysis; real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematics Series Vol. 43, Princeton University Press, Princeton, NJ, 1993.

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