Oscillatory Integral Decay, Sublevel Set Growth, and the Newton Polyhedron

Michael Greenblatt
greenbla@uic.edu

February 9, 2009

1. Introduction

In this paper we consider two types of integrals. Suppose $S(x)$ is a real-analytic function defined in a neighborhood of the origin in $\mathbf{R}^n$. The first type of integral being considered are sublevel set integrals of the form

$$I_{S,\phi}(\epsilon) = \int_{\{x: 0 < S(x) < \epsilon\}} \phi(x)\, dx \tag{1.1a}$$

$$I_{|S|,\phi}(\epsilon) = \int_{\{x: |S(x)| < \epsilon\}} \phi(x)\, dx \tag{1.1b}$$

Here $\phi(x)$ is a smooth nonnegative real-valued function supported within the domain of definition of $S(x)$ satisfying $\phi(0) > 0$. Such integrals have been considered for example in [PSSt] and [Va], and are closely related to Gelfand-Leray functions. We are interested in the behavior of $I_{S,\phi}(\epsilon)$ or $I_{|S|,\phi}(\epsilon) = I_{S,\phi}(\epsilon) + I_{-S,\phi}(\epsilon)$ as $\epsilon \to 0$.

The second type of integral under consideration are oscillatory integrals

$$J_{S,\phi}(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda S(x)} \phi(x)\, dx \tag{1.2}$$

Again $\phi(x)$ is a smooth real-valued function supported within the domain of definition of $S(x)$, but we make no assumption of nonnegativity on $\phi(x)$. Here we are interested in the behavior of $J_{S,\phi}(\lambda)$ as $|\lambda| \to \infty$. Since $\phi(x)$ is real, it suffices to consider the behavior of $J_{S,\phi}(\lambda)$ as $\lambda \to +\infty$.

In this paper, extending the methods of [G1] we will prove theorems generalizing a well-known theorem of Varchenko (Theorem 1.1 below) concerning oscillatory integrals $J_{S,\phi}$. They will be derived from analogous results proven here for the sublevel integrals $I_{|S|,\phi}$. Varchenko's theorem requires a certain nondegeneracy condition on the faces of the Newton polyhedron on S. In this paper, we will show in Theorems 1.2 and 1.3 that the

This research was supported in part by NSF grant DMS-0654073

estimates he obtained also hold for a significant class of $S(x)$ for which this nondegeneracy condition does not hold. Thus in problems where one wants to switch coordinates to a coordinate system where Varchenko's estimates are valid, one has greater flexibility by using the results of this paper. It should be pointed out that the methods of [G1] were influenced by those of [V] and therefore [V] can be viewed as an antecedent to this paper.

We will also exhibit some weaker estimates for more general situations, including some where the estimates of Theorem 1.1 in fact do not hold. We will see that our conditions on $S(x)$ in Theorem 1.3 for Varchenko's estimates to hold are optimal in some situations (Theorem 1.4). In two dimensions (Theorem 1.5), we will give a characterization of the $S(x)$ for which the Newton polygon determines sharp estimates in the fashion of Theorem 1.1; this too will hold for both the sublevel and oscillatory integrals. This may be viewed as a generalization of [G3], at least for real-analytic phase.

Integrals of the form $(1.1a) - (1.1b)$ and (1.2) come up frequently in analysis. For example, oscillatory integrals of the form (1.2) arise in PDE's, mathematical physics, and in harmonic analysis applications such as finding the decay of Fourier transforms of surface-supported measures and associated problems concerning the restriction and Kakeya problems. We refer to [AGV] chapter 6 and [S] chapter 8 for more information on such issues. The stability of oscillatory integrals of this kind under perturbations of the phase function $S(x)$ is related to a number of issues in complex geometry and has been studied for example in [PSSt] and [V]. Also, operator versions of these oscillatory integrals have been extensively analyzed, for example in [G4] [G5] [GrSe] [R] [PS] [Se]. Furthermore, as will be seen, our theorems concerning $I_{|S|,\phi}$ directly imply corresponding results for how the measure of $\{x \in U : 0 < |S(x)| < \epsilon\}$ goes to zero as $\epsilon \rightarrow 0$. Here U is a sufficiently small open set containing the origin. These come up for example in the analysis of Radon transforms such as in [C2] or [G5].

If $S(0) \neq 0$ and ϕ is supported on a sufficiently small neighborhood of the origin, then $I_{S,\phi}(\epsilon) = 0$ for small enough ϵ and thus is not interesting to analyze. In studying (1.2), one can always reduce to the case where $S(0) = 0$ by factoring out a $e^{i\lambda S(0)}$. Hence it does no harm to assume that $S(0) = 0$ in the analysis of $J_{S,\phi}$ either. Furthermore, if $\nabla S(0) \neq 0$, one easily has that $I_{S,\phi}(\epsilon) \sim \epsilon$ as $\epsilon \rightarrow 0$ for ϕ supported near the origin. Also, by integrating by parts repeatedly in the $\nabla S(0)$ direction, one also has $|J_{S,\phi}| < C_N \lambda^{-N}$ as $\lambda \rightarrow +\infty$ if the support of ϕ is sufficiently small. Therefore the interesting situation for both $I_{S,\phi}$ and $J_{S,\phi}$ is when $\nabla S(0) = 0$. Hence in this paper we will always assume that

$$S(0) = 0 \qquad \nabla S(0) = 0 \tag{1.3}$$

By Hironaka's resolution of singularities one has asymptotic expansions for both $I_{S,\phi}$ and $J_{S,\phi}$ if ϕ is supported in a sufficiently small neighborhood of the origin (see [G2] for elementary proofs). Namely, if $S(0) = 0$ and ϕ is supported in a sufficiently small neighborhood of the origin one can asymptotically write

$$I_{S,\phi}(\epsilon) \sim \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} c_{ij}(\phi) \ln(\epsilon)^i \epsilon^{r_j} \tag{1.4a}$$

$$J_{S,\phi}(\lambda) \sim \sum_{j=0}^{\infty}\sum_{i=0}^{n-1} d_{ij}(\phi)\ln(\lambda)^i\lambda^{-s_j} \tag{1.4b}$$

Here $\{r_j\}$ and $\{s_j\}$ are increasing arithmetic progressions of positive rational numbers independent of ϕ deriving from the resolution of singularities of S. Using resolution of singularities one can show that the smallest r_j for which some $c_{ij}(\phi)$ is nonzero will not depend on what ϕ is, and similarly the largest i for which $c_{ij}(\phi)$ is nonzero for this j also is independent of ϕ. (This uses the nonnegativity assumption on ϕ and that $\phi(0) > 0$). Hence as $\epsilon \rightarrow 0$, $I_{S,\phi}(\epsilon)$ will always be of the same order of magnitude. Inspired by terminology from the text [AGV], we refer to the value of r_j in this case as the *growth index* of S at the origin, and the corresponding value of i is referred to as the *multiplicity* of this index. We define the growth index of $|S|$ to be the minimum of the growth indices of S and $-S$, with its multiplicity that of S or $-S$. The multiplicity taken to be the maximum of the multiplicities of this growth index for S and $-S$ if they both have the same growth index. Note that the above considerations imply that if U is a sufficiently small neighborhood of the origin, then the measure of $\{x \in U : 0 < S(x) < \epsilon\} \sim |\ln\epsilon|^i\epsilon^{r_j}$ as $\epsilon \rightarrow 0$, where r_j is the growth index and i is the multiplicity of that index. As a result, knowing the growth index and its multiplicity gives the correct order of magnitude for such sublevel set volumes as $\epsilon \rightarrow 0$.

In the case of $J_{S,\phi}$, one does not necessarily have that the smallest r_j for which a $d_{ij}(\phi)$ is nonzero is the same for all ϕ (which is no longer even assumed to be nonnegative), so the above definition of index does not make sense. Instead, similar to [AGV] we define the *oscillation index* of S at the origin to be the minimal s_j for which for any sufficiently small neighborhood U of the origin, $d_{ij}(\phi)$ is nonzero for some ϕ supported in U. The multiplicity of this index s_j is defined to be the maximal i such that for any sufficiently small neighborhood U of the origin there is a ϕ supported on U such that $d_{ij}(\phi)$ is nonzero for this minimal s_j.

In general, the growth or oscillation index and their multiplicities are determined by the zero set of S in a complicated way. However, there are a number of situations when they can be determined from the Taylor series of $S(x)$ at the origin in a nice geometric way, a fact discovered by Varchenko in [V]. Heuristically speaking, these situations correspond to when the zero of $S(x)$ at the origin is stronger than any zero of $S(x)$ outside the coordinate hyperplanes $\{x_i = 0\}$. To indicate how the index and its multiplicity are determined in these situations, we first define some terminology.

Definition 1.1. Let $S(x) = \sum_\alpha s_\alpha x^\alpha$ denote the Taylor expansion of $S(x)$ at the origin. For any α for which $s_\alpha \neq 0$, let Q_α be the octant $\{x \in \mathbf{R}^n : x_i \geq \alpha_i \text{ for all } i\}$. Then the *Newton polyhedron* $N(S)$ of $S(x)$ is defined to be the convex hull of all Q_α.

In general, a Newton polyhedron can contain faces of various dimensions in various configurations. These faces can be either compact or unbounded. In this paper as well as in [V], an important role is played by the following functions, defined for compact faces of the Newton polyhedron. A vertex is always considered to be a compact face of dimension zero.

Definition 1.2. Suppose F is a compact face of the $N(S)$. Then if $S(x) = \sum_\alpha s_\alpha x^\alpha$ denotes the Taylor expansion of S like above, define $S_F(x) = \sum_{\alpha \in F} s_\alpha x^\alpha$

Also useful is the following terminology.

Definition 1.3. Assume $S(x)$ is not identically zero. Then the *Newton distance* of $S(x)$ is defined to be $\inf\{t : (t, t, ..., t, t) \in N(S)\}$.

The above-mentioned characterization in [V] of the oscillation index S at 0 and its multiplicity is as follows.

Theorem 1.1. (Varchenko) Suppose for each compact face F of $N(S)$, the function $\nabla S_F(x)$ is nonvanishing on $(\mathbf{R} - \{0\})^n$. Further suppose that the Newton distance of S is equal to some $d > 1$. Then the oscillation index of S at 0 is given by $\frac{1}{d}$. If the face of $N(S)$ (compact or not) that intersects the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ in its interior has dimension k, then the multiplicity of this index is given by $n - k - 1$.

For the purposes of Theorem 1.1, if the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ intersects $N(S)$ at a vertex, then one takes $k = 0$.

In this paper, we generalize Theorem 1.1 to a large class of functions where the $S_F(x)$ are not required to have nonvanishing gradient, and prove analogues for the sublevel set integrals. We also prove weaker substitutes for more degenerate situations including some when the conclusions of Theorem 1.1 do not necessarily hold. The methods of this paper are closely tied to the methods of [G1]. In turn, [G1] has antecedents in the earlier two-dimensional algorithms [G4]-[G5], and also [PS] and [V]. There has furthermore been much important work in sublevel set estimates and associated stability problems in the complex-analytic setting, such as in [DKo] [PSt1] [PSt2]. In [PSt1] and [PSt2], the method of algebraic estimates is used for this purpose; in [PSt2] resolution of singularities algorithms of Bierstone and Milman such as [BM] are also used. The complex methods tend to be rather different from the real ones since the results obtainable in the complex case are quite a bit stronger than those obtainable in the real situation.

In the theorems below, $S(x)$ is a real-analytic function, not identically zero, defined in a neighborhood of the origin and satisfying (1.3). $d > 0$ denotes the Newton distance of $S(x)$. $C(S)$ denotes the face (compact or not) of $N(S)$ intersecting the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ in its interior, and k denotes the dimension of $C(S)$. If the line intersects $N(S)$ at a vertex, we let $C(S)$ be this vertex and take $k = 0$.

Theorem 1.2.

a) As $\epsilon \rightarrow 0$, one has

$$I_{|S|,\phi}(\epsilon) > C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$$

b) If for each compact face F of $N(S)$ any zero of $S_F(x)$ in $(\mathbf{R}-\{0\})^n$ has order at most d, then as $\epsilon \rightarrow 0$ one has

$$I_{|S|,\phi}(\epsilon) < C'|\ln\epsilon|^{n-k}\epsilon^{\frac{1}{d}}$$

In this situation, as long as there is no compact face F of $N(S)$ with $F \subset C(S)$ such that $S_F(x)$ has a zero of order d somewhere in $(\mathbf{R}-\{0\})^n$, then one has the stronger estimate (compare with part **a**))

$$I_{|S|,\phi}(\epsilon) < C'|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$$

c) If the maximum order of any zero of any $S_F(x)$ (F compact) on $(\mathbf{R}-\{0\})^n$ is $d' > d$, then as $\epsilon \rightarrow 0$ one can at least say that

$$I_{|S|,\phi}(\epsilon) < C'\epsilon^{\frac{1}{d'}}$$

We next come to our three-dimensional result. One can get somewhat stronger results in three dimensions using a theorem of Karpushkin in [K] concerning the stability of growth indices under deformations of the phase in $n-1=2$ dimensions. A version of this theorem that sufficies for our purposes is as follows.

Theorem (Karpushkin) Suppose $f(x_1, x_2, t_1, ..., t_m)$ is a real-analytic function on a neighborhood of the origin in $\mathbf{R}^{m+2}$ and $f(x_1, x_2, 0, ..., 0)$ has growth index c at the origin as a function of x_1 and x_2. Then for any $\mu > 0$, there is a constant A_μ and a neighborhood $U_1^\mu \times U_2^\mu$ of the origin in the (x,t) variables such that for $(t_1, ..., t_m) \in U_2^\mu$ one has

$$|\{(x_1, x_2) \in U_1^\mu : |f(x_1, x_2, t_1, ..., t_m)| < \epsilon\}| \leq A_\mu \epsilon^{c-\mu}$$

To state our three-dimensional theorem, we need to consider the growth index of a polynomial $S_F(x)$ at a point $a \neq 0$. By this we mean the growth index of $S_F(x+a)$ at $x=0$. When $S_F(a) \neq 0$, we define this growth index to infinity, and when $S_F(a)=0$ but $\nabla S_F(a) \neq 0$, we take the growth index to be 1.

Theorem 1.3. Suppose $n=3$. Then the following hold.

a) As $\epsilon \rightarrow 0$, one has

$$I_{|S|,\phi}(\epsilon) > C|\ln\epsilon|^{2-k}\epsilon^{\frac{1}{d}}$$

b) Suppose the growth index of every $|S_F|$ (F compact) at any point in $(\mathbf{R}-\{0\})^3$ is at least $\frac{1}{d}$. Then as $\epsilon \rightarrow 0$ one has

$$I_{|S|,\phi}(\epsilon) < C'|\ln\epsilon|^2\epsilon^{\frac{1}{d}}$$

If the growth index of every $|S_F|$ (F compact) on $(\mathbf{R}-\{0\})^3$ is actually greater than $\frac{1}{d}$ at each point in $(\mathbf{R}-\{0\})^3$, then as $\epsilon \rightarrow 0$ one has the stronger (compare with **a)**)

$$I_{|S|,\phi}(\epsilon) < C'|\ln\epsilon|^{2-k}\epsilon^{\frac{1}{d}}$$

c) Let a denote the infimum over all compact faces F of $N(S)$ and all $x \in (R-\{0\})^3$ of the growth index of $|S_F|$ at x. If $a < \frac{1}{d}$, then as $\epsilon \rightarrow 0$ one has

$$I_{|S|,\phi}(\epsilon) < C'|\ln\epsilon|^2\epsilon^a$$

We next have the following result, which may be viewed as a sort of converse to the type of result given in Theorem 1.3, at least for the face $C(S)$. It holds in all dimensions.

Theorem 1.4. Suppose $C(S)$ is a compact face of $N(S)$.

a) Suppose there is some $x \in (R-\{0\})^n$ such that the growth index of $|S_{C(S)}|$ at x is $a < \frac{1}{d}$. Then for some $a' < \frac{1}{d}$, as $\epsilon \rightarrow 0$ one has

$$I_{|S|,\phi}(\epsilon) > C\epsilon^{a'}$$

b) Suppose there is a $x \in (R-\{0\})^n$ such that $|S_{C(S)}|$ has a growth index of $\frac{1}{d}$ at x, with multiplicity q. Then as $\epsilon \rightarrow 0$ one has

$$I_{|S|,\phi}(\epsilon) > C|\ln\epsilon|^{n-k+q}\epsilon^{\frac{1}{d}}$$

In [V] it is shown that for any real-analytic phase in two dimensions, there are necessarily "adapted coordinates" in which the reciprocal of the Newton distance gives the correct oscillation index. These results were generalized to smooth phase in [IM]. There are many situations where the hypotheses of Theorems 1.2b do not hold, but where they do hold after a coordinate change; take $S(x,y) = (x-y)^n$ in two-dimensions for example. A natural question to ask is in which situations is there a coordinate change after which one is in the setting of Theorem 1.2b) or 1.3b). The two-dimensional proofs of [V] and [IM] use facts arising from two-dimensional resolution of singularities such as Puiseux's theorem. Thus it would be reasonable to believe that proving analogues of such theorems in higher dimensions would use higher-dimensional resolution of singularities methods (and may be correspondingly more involved).

In the other extreme, if one works in two dimensions and fixes a coordinate system, one has the following theorem, analogous to the results of [G3]. It will be a rather direct consequence of Theorems 1.2 and 1.4.

Theorem 1.5. Suppose $n = 2$. Then the following hold.

a) The growth index of $|S|$ at the origin is given by $\frac{1}{d}$ if and only if $C(S)$ is not a compact edge of $N(S)$ such that $S_{C(S)}$ has a zero on $(\mathbf{R}-\{0\})^2$ of order greater than d. If $C(S)$ is such a compact 1-dimensional face, then the growth index is less than $\frac{1}{d}$.

b) When the growth index of $|S|$ at the origin is $\frac{1}{d}$, then the multiplicity of this index is equal to $1-k$, unless $S_{C(S)}$ has a zero on $(\mathbf{R}-\{0\})^2$ of order d, in which case it is equal to 1.

By well-known methods relating sublevel integrals to oscillatory integrals, the above results about the $I_{|S|,\phi}$ have direct implications for the $J_{S,\phi}$. Namely we have

Theorem 1.6.

a) Suppose ϕ is nonnegative with $\phi(0) > 0$.

If $d > 1$, or if $S(x)$ is either everywhere nonnegative or everywhere nonpositive in some neighborhood of the origin, then all statements and estimates analogous to those of Theorems 1.2-1.5 hold for $J_{S,\phi}$ in place of $I_{|S|,\phi}$. If one is not in these situations, as long as the growth index of $|S|$ is not an odd integer, then Theorems 1.2 and 1.3 hold for $J_{S,\phi}$ in place of $I_{|S|,\phi}$. In particular, they hold under any of the hypotheses of Theorem 1.2b) or 1.3b) if d is not the reciprocal of an odd integer.

b) For general smooth $\phi(x)$ and any d, $J_{S,\phi}$ decays as fast or faster than the decay rates corresponding to any upper bound given by Theorems 1.2, 1.3 or 1.5 for $I_{|S|,\phi}$.

Stability of Integrals.

Karpushkin's theorem above can be described as a stability theorem for level set measures of two-dimensional integrals; he proved analogues for oscillatory integrals as well. The analogues of these results in three or more dimensions do not hold, as exemplified by the following result contained in [V].

Theorem [V]. Let $S_t(x,y,z) = (x^4 + tx^2 + y^2 + z^2)^2 + x^p + y^p + z^p$, where $p \geq 9$. Then

a) If $t > 0$, the oscillatory index of S_t is $\frac{3}{4}$.
b) The oscillatory index of S_0 is $\frac{5}{8}$.
c) If $t < 0$, the oscillatory index of S_t is given by $\frac{1}{2} + \gamma(p)$, where $\gamma(p) \rightarrow 0$ as $p \rightarrow \infty$.

The next two theorems are simple examples of this phenomenon that follow from Theorem 1.2; in particular we avoid using the full Zariski three-dimensional resolution of singularities needed in [V] to prove the above result.

Theorem 1.7. Let $U_t(x,y,z) = x^4 + tx^2 + y^2 + z^2$.

a) If $t > 0$, the growth index of U_t at the origin is $\frac{3}{2}$.
b) The growth index of U_0 at the origin is $\frac{5}{4}$.
c) If $t < 0$, the growth index U_t at the origin is 1.

Theorem 1.7 will quickly lead to the following oscillatory integral analogue.

Theorem 1.8. Let $V_t(x,y,z) = (x^4 + tx^2 + y^2 + z^2)^2$

a) If $t > 0$, the oscillatory index of V_t is $\frac{3}{4}$.
b) The oscillatory index of V_0 is $\frac{5}{8}$.
c) If $t < 0$, the oscillatory index of V_t is $\frac{1}{2}$.

Proofs of Theorem 1.7 and 1.8.

If $t \geq 0$, then for each compact face F of $N(U_t)$ the corresponding polynomial $(U_t)_F(x,y,z)$ has no zeroes on $(\mathbf{R} - \{0\})^3$. Hence the growth index of U_t in these situations is given by Theorem 1.2a)-b). Computing the Newton distances, one sees that the growth index at the origin is equal to $\frac{3}{2}$ if $t > 0$ and equal to $\frac{5}{4}$ when $t = 0$. This gives parts a) and b) of Theorem 1.7. Now assume $t < 0$. Since each polynomial $(U_t)_F(x,y,z)$ has zeroes of order at most 1 on $(\mathbf{R} - \{0\})^3$, Theorem 1.2c) implies that the growth index of U_t is at least 1. To show it is exactly 1, we do a variable change, writing $(x,y,z) = (x, xy', xz')$. In the new coordinates $U_t(x,y,z)$ becomes the function $W_t(x,y',z') = x^2(x^2 + t + (y')^2 + (z')^2)$. This has zeroes on the sphere $x^2 + (y')^2 + (z')^2 = -t$, and $\nabla W_t(x,y',z')$ is nonzero at any such zero with $0 < |x| < \sqrt{-\frac{t}{2}}$. Going back into the (x,y,z) coordinates, this means $U_t(x,y,z)$ has zeroes arbitrarily close to the origin at which $\nabla U_t(x,y,z) \neq 0$. In a small neighborhood of each such zero, the measure of $\{(x,y,z) : U_t(x,y,z) < \epsilon\}$ is bounded below by $C\epsilon$. Hence the growth index of U_t is at most 1. We conclude that the growth index of U_t is exactly 1, giving part c) of Theorem 1.7 and completing the proof of that theorem.

We move to Theorem 1.8. The growth index at the origin of V_t is half that of U_t. So if $t > 0$, the growth index is $\frac{3}{4}$, if $t = 0$ it is $\frac{5}{8}$, and if $t < 0$ it is $\frac{1}{2}$. We will see in the last paragraph of section 5 that if the growth index is less than 1 then the oscillatory index and the growth index are the same (this also follows pretty directly from Ch 7 of [AGV]). The desired properties immediately follow and we are done.

2. Geometric constructions from the Newton polyhedron

In this section we do a number of geometric constructions which will used in later sections in proving the various estimates of this paper. As indicated above, they are based on the resolution of singularities methods of [G1]. However, we do not need a full-fledged resolution of singularities algorithm for the purposes of this paper.

Heuristically speaking, what we will do is as follows. Suppose $S(x)$ is a real-analytic function defined on a neighborhood of the origin. We will take a small neighborhood of the origin, and divide it (modulo sets of measure zero) into open slivers W_{ij} whose closures each contains the origin. Each W_{ij} corresponds to one vertex or compact face F_{ij} of $N(S)$ of dimension i in the sense that on W_{ij}, the monomials x^v for v a vertex of $N(S)$

on F_{ij} dominate the monomials x^v for $v \in N(S)$ not on F_{ij}. Lemmas 2.0 and 2.1 make this notion precise.

Next, each W_{ij} will be further subdivided, modulo sets of measure zero, into open slivers W_{ijp} to each of which there will be assigned an invertible map $\beta_{ijp} : Z_{ijp} \rightarrow W_{ijp}$. Each component function of each β_{ijp} is plus or minus a monomial in $x_1^{\frac{1}{N}}...x_n^{\frac{1}{N}}$ for some integer N, and for $i > 0$ each domain Z_{ijp} satisfies inclusions of the form

$$(0, \eta')^{n-i} \times D_{ij} \subset Z_{ijp} \subset (0,1)^{n-i} \times D_{ij} \tag{2.0}$$

Here D_{ij} is a bounded open set whose closure is contained in $\{(x_{n-i+1}, ..., x_n) : x_k > 0$ for all $k\}$. Furthermore, the map β_{ijp} is such that $S_{F_{ij}} \circ \beta_{ijp}(x)$ can be expressed as $m(x_1, ..., x_{n-i})T(x_{n-i+1}, ..., x_n)$, where $m(x_1, ..., x_{n-i})$ is a monomial in the first $n - i$ variables. As a result, a condition that the zeroes of $S_{F_{ij}}$ on $(\mathbf{R} - \{0\})^n$ are of order less than d implies that the same condition holds for $T(x_{n-i+1}, ..., x_n)$. Similarly, the various other conditions stipulated on $S_{F_{ij}}(x)$ in the different lemmas imply that the same condition holds for $T(x_{n-i+1}, ..., x_n)$. In addition, since the $x_{n-i+1}, ..., x_n$ variables are bounded above on Z_{ijp} (by the boundedness of the D_{ij}), the function $S_{F_{ij}} \circ \beta_{ijp}(x)$ is bounded above by $Cm(x_1, ..., x_{n-i})$. Analogously, using that the points in D_{ij} have coordinates bounded below away from zero, a local nonvanishing pth derivative condition on $S_{F_{ij}}(x)$ will imply the corresponding pth derivative of $T(x_{n-i+1}, ..., x_n)$ is bounded below on some open set, so that this derivative of $S_{F_{ij}} \circ \beta_{ijp}(x)$ is bounded below by $C'm(x_1, ..., x_{n-i})$ in some neighborhood. These facts are proven via the constructions of Theorem 2.2 and Lemmas 2.3 and 2.4.

Because the terms x^v for $v \in F_{ij}$ dominate on W_{ijp}, the difference $|S_{F_{ij}} \circ \beta_{ijp}(x) - S \circ \beta_{ijp}(x)|$ is bounded above by $\epsilon\, m(x_1, ..., x_{n-i})$ for a small ϵ (Lemma 2.1). So one has $|S \circ \beta_{ijp}(x)| < C''m(x_1, ...x_{n-i})$. One can also do the constructions are done so that any given derivative of $S \circ \beta_{ijp}(x)$ is a also a small perturbation of the corresponding derivative of $S_{F_{ij}} \circ \beta_{ijp}(x)$; if $S_{F_{ij}} \circ \beta_{ijp}(x)$ satisfies a nonvanishing pth derivative condition on a small open set, then so does $S \circ \beta_{ijp}(x)$ which is therefore bounded below by $C'''m(x_1, ..., x_{n-i})$. This enables one to use van der Corput type lemmas in the $x_{n-i+1}....x_n$ variables to prove various desired estimates. The most convenient such van der Corput lemma for our purposes is that of [C1], which says that if $f(x)$ is a C^{k+1} function on an interval I whose kth derivative is bounded below by η, then one has

$$|\{x \in I : |f(x)| < \delta\}| < C_k(\frac{\delta}{\eta})^{\frac{1}{k}} \tag{2.1}$$

Note that the properties being used here are quite a bit weaker than those of a full resolution of singularities theorem since we only have an upper bound for the blown-up function and a lower bound for its derivative in terms of a monomial (the blown-up function can even have a complicated zero set), but this suffices for our purposes.

For the three dimensional result, instead of getting uniform estimates from a Van der Corput-type lemma, one considers the growth index directly. One uses Karpushkin's theorem to show that locally the growth index of $S \circ \beta_{ijp}(x)$ is the same as that of $S_{F_{ij}} \circ \beta_{ijp}(x)$, which in turn is the same as that of $T(x_3)$ or $T(x_2, x_3)$ (for $i = 1$ and 2 respectively.) One always has has to be careful that perturbing $S_{F_{ij}} \circ \beta_{ijp}(x)$ into $S \circ \beta_{ijp}(x)$ can be done in such a way that Karpushkin's result applies.

To enable us to use van der Corput lemmas most effectively, one should have a good idea of what the monomials $m(x_1, ..., x_{n-i})$ are. Fortunately, Lemmas 2.5 and 2.6 give us a way of doing this. Namely, if one redefines $\beta_{ijp}(x)$ such that for $q \leq n-i$ one replaces each x_q by $x_q^{l_q}$, where the l_q are chosen such that the determinant of $\beta_{ijp}(x)$ is constant, then each variable x_q for $q \leq n-i$ appears to at most the dth power in $m(x_1, ..., x_{n-i})$, where as usual d is the Newton distance. Furthermore, the dth power appears at in most $n-k$ variables, where k is the dimension of the central face $C(S)$, and it appears $n-k$ times if and only if $F_{ij} \subset C(S)$. These things are proven in Lemma 2.6. One then proves the estimates of Theorems 1.2b-c by first using the appropriate Van der Corput-type lemma in a direction in the $x_{n-i+1}, ..., x_n$ variables, then taking absolute values and integrating in the remaining $x_{n-i+1}, ..., x_n$ variables, and then integrating the resulting function of the first $n-i$ variables. As one might guess, one needs to take a lot of care in carrying out this strategy.

It should be pointed out that in the above description, we always assumed $i > 0$. But there are also W_{0jp}; fortunately these are easy to deal with since the functions $T(x_{n-i+1}, ..., x_n)$ are replaced by a constant.

To give a concrete and easy-to-understand example of the above considerations, in three dimensions consider the function $S(x, y, z) = x^2 + y^2 - z^2$. Then the Newton distance of S is $\frac{2}{3}$, and the functions $S_e(x, y, z)$ either have no zeroes on $(\mathbf{R} - \{0\})^3$, or have zeroes of order 1 on $(\mathbf{R} - \{0\})^3$. In the above language, this says that the exponents appearing in each monomial $m(x_1)$ or $m(x_1, x_2)$ are at most $\frac{2}{3}$, while the functions $T(x_2, x_3)$ or $T(x_3)$ can have zeroes of order as high as 1. We focus our attention on the situation where F_{ij} is the main 2-dimensional face; the 1-dimensional faces where $S_e(x, y, z)$ has a zero will behave similarly to the following. Then $i = 2$, and $T(x_2, x_3)$ has zeroes of order 1. By first using the Van der Corput lemma (2.1) in an appropriate direction in the x_2x_3 variables, then integrating in the orthogonal x_2x_3 direction, and lastly integrating in x_1, using (2.0) one gets that for some positive δ and δ', $|\{(x_1, x_2, x_3) : |S \circ \beta_{ijp}(x_1, x_2, x_3)| < \epsilon\}|$ is comparable to $\int_0^\delta \max(\delta', \frac{\epsilon}{x_1^{\frac{2}{3}}})\, dx_1 \sim \epsilon$. These are the weaker bounds of Theorem 1.2c). It is worth pointing out that the oscillation is index here is the value $\frac{3}{2}$ given by the Newton polyhedron since the phase has nonvanishing Hessian. This is an example where one gets a smaller growth index (which is in fact 1 in this example) than oscillation index; by Theorem 1.6 for this to happen d must be less than 1.

Next, suppose that instead of $S(x, y, z) = x^2 + y^2 - z^2$, one chooses $S(x, y, z) = x^4 + y^4 - z^4$. Then the Newton distance doubles to $\frac{4}{3}$, yet the maximum order of any zero of any $S_e(x, y, z)$ is still 1. Since $1 < \frac{4}{3}$, the stronger results of Theorem 1.2b) apply (It doesn't help in this particular situation to use Theorem 1.3). Instead of ending out with an integration of $\int_0^\delta \max(\delta', \frac{\epsilon}{x_1^{\frac{2}{3}}})\, dx_1$, one ends out with an integration of $\int_0^\delta \max(\delta', \frac{\epsilon}{x_1^{\frac{4}{3}}})\, dx_1$. Since the exponent in the denominator is now greater than 1, the result is now comparable to $\epsilon^{\frac{3}{4}}$. Simply put, the zero of $S(x,y,z)$ at the origin now dominates the zeroes of $S(x,y,z)$ away from the origin on W_{ijp}, so the Newton polyhedron now determines the growth index. On the other hand, in the previous example the reverse was true, so that the zeroes of $T(x_2,x_3)$ and its analogues from the other W_{ijp} force the growth index to be smaller. Theorem 1.2c) says that, like in this example, that the growth index is bounded below by the reciprocal of the maximal order of a zero of the functions $T(x_2,x_3)$ or $T(x_3)$. In general, when the Newton polyhedron determines the growth index, the powers of at least one variable appearing in the integration for one Z_{ijp} will have exponent at least 1, while when the zeroes are too strong for that, all powers of all variables will be less than one. So these two examples, however simple, are fairly indicative.

The lower bounds of Theorems 1.2a) and 1.3a) are not affected by the behavior of the zeroes of the various $S_{F_{ij}}(x_1,...,x_n)$ since the zeroes can only cause one to obtain worse estimates than those given by the Newton polyhedron. Thus in proving the lower bounds one can just restrict attention to some small subregion of D_{ij} away from the zeroes of the associated $T(x_{n-i+1},...,x_n)$. On this region $S \circ \beta_{ijp}(x) \sim m(x_1,...,x_{n-i})$ and the lower bounds determined by the Newton polyhedron are readily proven.

We now begin proving our various lemmas.

Lemma 2.0. (Lemma 3.2 of [G1]) Let $v(S)$ denote the set of vertices of $N(S)$. There are $A_1, A_2 > 1$ such that if $C_0, ..., C_n$ are constants with $C_0 > A_1$ and $C_{i+1} > C_i^{A_2}$ for all i, then one can define the W_{ij} so that

a) Let $i < n$. If the following two statements hold, then $x \in W_{ij}$.

1) If $v \in v(S) \cap F_{ij}$ and $v' \in v(S) \cap (F_{ij})^c$ we have $x^{v'} < C_n^{-1} x^v$.
2) For all $v, w \in v(S) \cap F_{ij}$ we have $C_i^{-1} x^w < x^v < C_i x^w$.

b) There is a $\delta > 0$ depending on $N(S)$, and not on A_1 or A_2, such that if $x \in W_{ij}$, then the following two statements hold.

1) If $v \in v(S) \cap F_{ij}$ and $v' \in v(S) \cap (F_{ij})^c$ we have $x^{v'} < C_{i+1}^{-\delta} x^v$.
2) For all $v, w \in v(S) \cap F_{ij}$ we have $C_i^{-1} x^w < x^v < C_i x^w$.

Informally, this gives a way of saying that the vertices of F_{ij} dominate the Taylor series of S when $x \in W_{ij}$. Another way of making this precise is the following lemma.

Lemma 2.1. Suppose $x \in W_{ij}$. Let $V \in v(S)$ be such that $x^V \geq x^v$ for all $v \in v(S)$; if there is more than one such vertex let V be any of them. Then if A_1 is sufficiently large and η is sufficiently small, for any positive d one has the following estimate:

$$\sum_{\alpha \notin F_{ij}} |s_\alpha||\alpha|^d x^\alpha < K(C_{i+1})^{-\delta''} x^V$$

Here K is a constant depending on d as well as the function $S(x)$, and $\delta'' > 0$ is a constant depending on the Newton polyhedron of S.

Proof. There are several one-dimensional faces of $N(S)$ that contain V, and there are vectors $w_1, ...w_N$ so that a given edge is given by $V + tw_l$ for a set of nonnegative t. If any component of a vector w_l is negative, the corresponding edge will terminate at a vertex which we denote by v_l. Rescaling w_l if necessary, we can assume that $v_l = V + w_l$. If all components of a w_l are nonnegative, then the edge is an infinite ray. (It is not hard to show that w_l is in fact some unit coordinate vector $\mathbf{e}_m$). In this situation we define $v_l = V + w_l$. Consequently, for all l we have

$$x^{v_l} = x^V x^{w_l} \tag{2.2}$$

I claim that, shrinking η if necessary, we may assume that for all l such that $v_l \notin F_{ij}$ we have

$$x^{v_l} < (C_{i+1})^{-\delta} x^V \tag{2.3}$$

This is true if v_l is a vertex of $N(S)$ by Lemma 2.0 above. It is true if v_l is not a vertex since $\frac{x^{v_l}}{x^V} = x^{w_l}$, which can be made less than $(C_{i+1})^{-\delta}$ by shrinking η appropriately since w_l has only nonnegative components. So we can assume (2.3) holds. Next, note that since $N(S)$ is a convex polyhedron we have

$$N(S) \subset \{V + \sum_{l=1}^{N} t_l w_l : t_l \geq 0\} \tag{2.4}$$

For a positive integer k, define B_k to be the set of points α with integer coordinates that are in $N(S)$ but not on F_{ij} such that α can be written as $V + \sum_l t_l w_l, t_l \geq 0$ with $k - 1 < \sum_{v_l \notin F_{ij}} |t_l| \leq k$. Let E be a separating hyperplane for $N(S)$ such that $E \cap N(S) = F_{ij}$. Since F_{ij} is bounded, we may let a be a vector normal to F_{ij} such that each component of a is positive. For each w_l not parallel to F_{ij}, the vector w_l points "inward"; that is, $a \cdot w_l > 0$. Consequently, for a constant C depending only on $N(S)$, the points in B_k are contained in the points of $(\mathbf{R}^+)^n$ between E and its translate $E + Cka$. In particular each coordinate of a point in B_k is bounded by Ck and there at most Ck^n of them. Next, writing a given $\alpha \in B_k$ as $V + \sum_l t_l w_l$ with $k - 1 < \sum_{v_l \notin F_{ij}} |t_l| < k$, we have

$$x^\alpha = x^V \prod_l (x^{w_l})^{t_l} = x^V \prod_{v_l \in F_{ij}} (x^{w_l})^{t_l} \prod_{v_l \notin F_{ij}} (x^{w_l})^{t_l} \leq x^V \prod_{v_l \notin F_{ij}} (x^{w_l})^{t_l} \tag{2.5}$$

The last inequality follows from (2.2) and the maximality of x^V. Using (2.3) and the definition of B_k we have

$$x^V \prod_{v_l \notin F_{ij}} (x^{w_l})^{t_l} < x^V (C_{i+1})^{-\delta \sum_{v_l \notin F_{ij}} t_l} < (C_{i+1})^{-\delta(k-1)} x^V \tag{2.6}$$

When $k = 1$, one has an inequality

$$x^\alpha < x^V (C_{i+1})^{-\delta \sum_{v_l \notin F_{ij}} t_l} < (C_{i+1})^{-\delta'} x^V \tag{2.7}$$

Here δ' is the minimum of the finitely many positive numbers $\delta \sum_{v_l \notin F_{ij}} t_l$ that can appear in the right hand side of (2.7). Since S is real analytic, the coefficients s_α satisfy $|s_\alpha| < CM^{|\alpha|}$ for some M. Since the components of any α in any B_k are at most Ck, we have

$$|s_\alpha| < C' M^k \tag{2.8}$$

Since there are most Ck^n points with integer coordinates in any B_k, inserting (2.8) in (2.6) or (2.7) and adding gives the following for $k > 1$.

$$\sum_{\alpha \in B_k} |s_\alpha||\alpha|^d x^\alpha < C' k^{d+n} M^k C_{i+1}^{-\delta(k-1)} x^V$$

$$= C' M^2 k^{d+n} C_{i+1}^{-\delta} (M C_{i+1}^{-\delta})^{(k-2)} x^V \tag{2.9a}$$

If $k = 1$ we have

$$\sum_{\alpha \in B_1} |s_\alpha||\alpha|^d x^\alpha < C' M C_{i+1}^{-\delta'} x^V \tag{2.9b}$$

Adding this over all k, as long as $A_1^\delta > 2M$ so that each $MC_{i+1}^{-\delta} < \frac{1}{2}$, we get

$$\sum_{\alpha \notin F_{ij}} |s_\alpha||\alpha|^d x^\alpha < C'' C_{i+1}^{-\delta''} x^V \tag{2.10}$$

Here $\delta'' = \min(\delta, \delta')$. This gives the lemma and we are done.

Corollary. There is a constant C such that on a sufficiently small neighborhood of the origin $|S(x)| \leq C \sum_{v \in v(S)} x^v$.

Proof. It suffices to prove the corollary on a given W_{ij}. We have

$$|S(x)| \leq \sum_\alpha |s_\alpha| x^\alpha = \sum_{\alpha \in F_{ij}} |s_\alpha| x^\alpha + \sum_{\alpha \notin F_{ij}} |s_\alpha| x^\alpha < C_0 x^V + K(C_{i+1})^{-\delta'} x^V$$

$$= (C_0 + K(C_{i+1})^{-\delta'}) x^V \leq (C_0 + K(C_{i+1})^{-\delta'}) \sum_{v \in v(S)} x^v$$

The corollary follows.

For the purposes of this paper, we need to do a further subdivision of a given W_{ij} into finitely many pieces W_{ijp}. The relevant properties of the W_{ijp} are encapsulated by the following theorem.

Theorem 2.2. If A_1 and A_2 are sufficiently large, each W_{ij} can be, modulo a set of measure zero, written as the union of finitely many open nonempty sets W_{ijp} to each of which is associated a bijective map $\beta_{ijp} : Z_{ijp} \rightarrow W_{ijp}$ depending on $N(S)$ and (i,j,p), but not the particular subdivision being done, such that each component of $\beta_{ijp}(z)$ is a monomial in $(z_1^{\frac{1}{N}}, ..., z_n^{\frac{1}{N}})$ for some N, and such that for some $\mu' > 0$ that is allowed to depend on the particular subdivision we have

a) When $i = 0$, $(0, \mu')^n \subset Z_{ijp} \subset (0,1)^n$.

b) When $i > 0$, there are sets $D_{ij} \subset (C_i^{-e}, C_i^{e})^i$ for some $e > 0$ depending on $N(S)$ such that $(0,\mu')^{n-i} \times D_{ij} \subset Z_{ijp} \subset (0,1)^{n-i} \times D_{ij}$

c) When $i > 0$, write $z \in \mathbf{R}^n$ as (σ, t) where $\sigma \in \mathbf{R}^{n-i}$ and $t \in \mathbf{R}^i$. For any $v \in N(S)$, denote by $\sigma^{v'}t^{v''}$ the function in z coordinates that x^v transforms into under the x to z coordinate change. When $i = 0$, write $z = \sigma$ and for $v \in N(S)$ denote by $\sigma^{v'}$ the the function x^v transforms into. Then for any $v_1, v_2 \in F_{ij}$ we have $v_1' = v_2'$, while if $v_1 \in F_{ij}$ and v_2 is in $N(S)$ but not in F_{ij}, then $(v_2')_k \geq (v_1')_k$ for all k with at least one component strictly greater.

The proof of Theorem 2.2 is very similar to the arguments of section 4 of [G1]. However, there are enough differences that we prove it separately here. We will do it through some constructions resembling Lemmas 4.1-4.3 of [G1], after which we will prove Theorem 2.2.

For each i and j let f_{ij} be any vertex on on F_{ij}. Since the face F_{ij} is of dimension i, we may let $\{P_l\}_{l=1}^{n-i}$ be separating hyperplanes for $N(S)$ such that $F_{ij} = \cap_{l=1}^{n-i} P_l$. We write these hyperplanes as

$$P_l = \{x : a^l \cdot x = c^l\}$$

We can assume the a^l have rational coefficients. The hyperplanes satisfy

$$N(S) \subset \cap_{l=1}^{n-i}\{x : a^l \cdot x \geq c^l\} \tag{2.11}$$

Since $\cap_{m=1}^{n}\{x : x_m \geq f_{ijm}\} \subset N(S)$, we also have

$$\cap_{m=1}^{n}\{x : x_m \geq f_{ijm}\} \subset \cap_{l=1}^{n-i}\{x : a^l \cdot x \geq c^l\} \tag{2.12}$$

Since $a^l \cdot f_{ij} = c^l$ for all l, if we shift x in (2.11) by $-f_{ij}$ we get

$$\cap_{m=1}^{n}\{x : x_m \geq 0\} \subset \cap_{l=1}^{n-i}\{x : a^l \cdot x \geq 0\} \tag{2.13}$$

In the case where $i > 0$, we would like to extend the hyperplanes $a^l \cdot x = 0$ to a collection of n independent hyperplanes such that

$$\cap_{m=1}^{n}\{x : x_m \geq 0\} \subset \cap_{l=1}^{n}\{x : a^l \cdot x \geq 0\} \tag{2.14}$$

(Note that (2.14) is (2.13) when $i = 0$.) We do this by defining a^l for $i < l < n$ to be unit coordinate vectors such that $a^1, ..., a^n$ are linearly independent. Once we do this, we have

$$\cap_{m=1}^{n}\{x : x_m \geq 0\} \subset \cap_{l=n-i+1}^{n}\{x : a^l \cdot x \geq 0\} \tag{2.15}$$

Combining with (2.13) gives (2.14).

Since the $a^l \cdot x \geq 0$ are n independent hyperplanes intersecting at the origin, any $n-1$ of the hyperplanes intersect along a line through the origin. Write the directions of these lines as b_l, chosen so that the b_l have rational components and $a_l \cdot b_l > 0$. The b_l span $\mathbf{R}^n$, so we may write the mth unit coordinate vector $\mathbf{e}_m$ in the form

$$\mathbf{e}_m = \sum_{l=1}^{n} d_{lm} b_l \tag{2.16}$$

Lemma 2.3. The coefficients d_{lm} are all nonnegative rational numbers.

Proof. By definition of b_l, we have

$$\cap_{l=1}^{n} \{x : a^l \cdot x \geq 0\} = \{s : s = \sum_{p=1}^{n} s_p b_p \text{ with } s_p \geq 0\} \tag{2.17}$$

Since each $\mathbf{e}_m$ is in $\cap_{m=1}^{n} \{x : x_m \geq 0\} \subset \cap_{l=1}^{n} \{x : a^l \cdot x \geq 0\}$, (2.17) says that each d_{lm} is nonnegative. Elementary linear algebra gives a formula for the d_{lm} which shows that they are rational. This completes the proof.

We now do a coordinate change on each W_{ij} for $i > 0$. Denoting the original coordinates of a point x by $(x_1, ..., x_n)$, we let the new coordinates be denoted by $(y_1, ..., y_n)$, where

$$y_m = \prod_{l=1}^{n} x_l^{d_{lm}} \tag{2.18}$$

Observe that a monomial x^α becomes $y^{L(\alpha)}$ in the new coordinates, where L is the linear map such that $L(b_l) = \mathbf{e}_l$ for all l. If $\bar{f}_{ij} = (\bar{f}_{ij1}, ..., \bar{f}_{ijn})$ denotes $L(f_{ij})$, then each $\bar{f}_{ijk} \geq 0$ since each d_{lm} is nonnegative. Furthermore, L takes each hyperplane P_l to $\{y : y_l = \bar{f}_{ijl}\}$. Notice that each point p of F_{ij} is on P_l for $l \leq n-i$. This means that the lth component of $L(p)$ is equal to $\bar{f}_{ijl}$ for $l \leq n-i$. So if v and v' are vertices of $N(S)$ on F_{ij}, the first $n-i$ components of $L(v-v')$ are zero. Hence $y^{L(v-v')}$ is a function of the last i y-variables only. Write $y = (s,t)$, where s is the first $n-i$ variables and t is the last i variables. Similarly, write $L = (L_1, L_2)$, where L_1 is the first $n-i$ components and L_2 is the last i components. Recall from Lemma 2.0 that for any such v and v', any $x \in W_{ij}$ satisfies the inequalities

$$C_i^{-1} < x^{v-v'} < C_i \tag{2.19a}$$

In terms of the t variables this translates as

$$C_i^{-1} < t^{L_2(v-v')} < C_i \tag{2.19b}$$

Write $\log(t) = (\log(t_1), \log(t_2), ..., \log(t_n))$. Equation (2.19$b$) becomes

$$-\log(C_i) < \log(t) \cdot L_2(v-v') < \log(C_i) \tag{2.20}$$

Since the set of all possible $L_2(v-v')$ for v and v' vertices of S on F_{ij} spans an i-dimensional space, and since $\log(t)$ is an i-dimensional vector, there must be a constant d depending on the function S such that for each l we have

$$-d\log(C_i) < \log(t_l) < d\log(C_i) \tag{2.21a}$$

Equation (4.11a) is equivalent to

$$C_i^{-e} < t_l < C_i^e \tag{2.21b}$$

In particular, the variables t_l are bounded away from 0. Next, continuing to focus on the $i > 0$ case, we examine how the x to (s,t) coordinate change affects W_{ij} in the first $n-i$ variables. It turns out that the relevant inequalities are those provided by Lemma 2.0. This lemma says that if $x \in W_{ij}$, w is in the vertex set $v(S)$ of $N(S)$ and on the face F_{ij}, and $w' \in v(S)$ but $w' \notin F_{ij}$, then we have

$$x^{w'-w} < (C_{i+1})^{-\delta}$$

Writing in y coordinates, this becomes

$$y^{L(w'-w)} < (C_{i+1})^{-\delta} \tag{2.22a}$$

We would like to encapsulate the condition that $x \in (0,\eta)^n$ through an equation analogous to (2.22a). Shrinking η if necessary, we can assume that for each m, $x_m = x^{\mathbf{e}_m} < (C_{i+1})^{-\delta}$, and we express this in y coordinates as

$$y^{L(\mathbf{e}_m)} < (C_{i+1})^{-\delta} \tag{2.22b}$$

Writing $L = (L_1, L_2)$ and $y = (s,t)$ like before, equations (2.22) become

$$s^{L_1(w'-w)} < (C_{i+1})^{-\delta} t^{L_2(w-w')} \tag{2.23a}$$

$$s^{L_1(\mathbf{e}_m)} < (C_{i+1})^{-\delta} t^{L_2(-\mathbf{e}_m)} \tag{2.23b}$$

Equation (4.11b) says that each component of t is between C_i^{-e} and C_i^e. So there is a constant d' depending only $N(S)$ such that in (2.23) one has

$$C_i^{-e'} < t^{L_2(w-w')} < C_i^{e'} \tag{2.24a}$$

$$C_i^{-e'} < t^{L_2(-\mathbf{e}_m)} < C_i^{e'} \tag{2.24b}$$

So as long as A_2 from the beginning of section 3 is sufficiently large, equations (2.23) give

$$s^{L_1(w'-w)} < 1 \tag{2.25a}$$

$$s^{L_1(\mathbf{e}_m)} < 1 \tag{2.25b}$$

Summarizing, if $x \in W_{ij}$, then the corresponding (s,t) in y coordinates satisfy $(2.19b)$ and $(2.25a) - (2.25b)$. We now use in a similar fashion the other inequalities of Lemma 2.0. Namely, $x \in (0,\eta)^n$ is in W_{ij} if $(2.19a)$ holds and x satisfies the following for all $w \in v(S) \cap F_{ij}$, $w' \in v(S) \cap (F_{ij})^c$

$$x^{w'} < C_n^{-1} x^w \tag{2.26a}$$

Analogous to above, we incorporate the condition $x \in (0,\eta)^n$ by stipulating that $\eta < (C_n)^{-1}$ and write

$$x^{\mathbf{e}_m} < C_n^{-1} \tag{2.26b}$$

Analogous to (2.23), these can be written as

$$s^{L_1(w'-w)} < (C_n)^{-1} t^{L_2(w-w')} \tag{2.27a}$$

$$s^{L_1(\mathbf{e}_m)} < (C_n)^{-1} t^{L_2(-\mathbf{e}_m)} \tag{2.27b}$$

Again using (2.24), there is some μ such that equations (2.27) hold whenever for all $w' - w$ and all $\mathbf{e}_m$ we have

$$s^{L_1(w'-w)} < \mu \tag{2.28a}$$

$$s^{L_1(\mathbf{e}_m)} < \mu \tag{2.28b}$$

Hence if a point (s,t) is such that s satisfies $(2.28a) - (2.28b)$ and t satisfies $(2.19a)$, then the corresponding x is in W_{ij}. Putting (2.25) and (2.28) together, let Y_{ij} denote the set W_{ij} in the y coordinates. Let u_1, u_2,... be an enumeration of the set of all $L_1(w'-w)$ for vertices $w \in F_{ij}$ and $w' \notin F_{ij}$, as well as the distinct $L_1(\mathbf{e}_m)$. We define the sets E_1 and E_2 by

$$E_1 = \{s : 0 < s^{u_l} < \mu \text{ for all } l\} \times D_{ij} \tag{2.29a}$$

$$E_2 = \{s : 0 < s^{u_l} < 1 \text{ for all } l\} \times D_{ij} \tag{2.29b}$$

Then by (2.25) and (2.28) we have

$$E_1 \subset Y_{ij} \subset E_2 \tag{2.29c}$$

It is worth pointing out that none of the u_l are zero: If some $\bar{w}_l - \bar{w}_0$ were zero this would imply that they came from a $w \in F_{ij}$ and a $w' \notin F_{ij}$ such that $w' - w$ is a function of only the t-variables. This would mean that $w' - w$ is tangent to F_{ij}, which can never happen when $w \in F_{ij}$ and $w' \notin F_{ij}$. If some $L_1(\mathbf{e}_m)$ were zero, that would imply $\mathbf{e}_m$ is a function of the t variables only, meaning that $\mathbf{e}_m$ is tangent to F_{ij}. Since F_{ij} is a bounded face, this cannot happen either.

Equations $(2.29a) - (2.29c)$ are for $i > 0$, and there are analogous equations when $i = 0$. Fortunately, these require less effort to deduce; a coordinate change is not required. There is a single vertex v on a given F_{0j}. Lemma 2.0 tells us that if μ is sufficiently small, if we define

$$F_1 = \{x \in (0,\eta)^n : x^{v'} < \mu x^v \text{ for all } v' \in v(S) - \{v\}\}$$

$$F_2 = \{x \in (0,\eta)^n : x^{v'} < x^v \text{ for all } v' \in v(S) - \{v\}\}$$

Then we have $F_1 \subset W_{0j} \subset F_2$. To combine this with the $i > 0$ case, we rename the x variables s and define $Y_{0j} = W_{0j}$. Let $\{u_l\}_{l>0}$ be an enumeration of the $v' - v$ for $v' \in v(S) - \{v\}$ as well as the unit coordinate vectors $\mathbf{e}_m$. When $i = 0$ define

$$E_1 = \{s : 0 < s^{u_l} < \mu \text{ for all } l > 0\}$$

$$E_2 = \{s : 0 < s^{u_l} < 1 \text{ for all } l > 0\} \tag{2.30}$$

Then, shrinking μ to less than η if necessary, like above we have $E_1 \subset Y_{0j} \subset E_2$.

In the remainder of this section, we consider the $i > 0$ and $i = 0$ cases together. We still have some work to do. Namely, we would like to replace the sets $\{s : 0 < s^{u_l} < \mu \text{ for all } l\}$ or $\{s : 0 < s^{u_l} < 1 \text{ for all } l\}$ by cubes. To this end, we will divide up Y_{ij} in the s variables into finitely many pieces. A coordinate change in the s variables will be performed on each piece taking it to a set which is a positive curved quadrant. This is done as follows. For $i > 0$ let E_1' and E_2' be defined by

$$E_1' = \{s : 0 < s^{u_l} < \mu \text{ for all } l > 0\}$$

$$E_2' = \{s : 0 < s^{u_l} < 1 \text{ for all } l > 0\}$$

When $i = 0$, let $E_1' = E_1$ and $E_2' = E_2$. Writing $S = (S_1, .., S_{n-i}) = (\log(s_1), .., \log(s_{n-i}))$, in the S coordinates E_2' becomes the set E_2^S given by

$$E_2^S = \{S : S \cdot u_l < 0 \text{ for all } l\}$$

The set of S satisfying (2.30) is the intersection of several hyperplanes passing through the origin. We subdivide E_2^S via the $n - i$ hyperplanes $S_m = 0$, resulting in (at most) 2^{n-i} pieces which we call $E_2^{S,1}$, $E_2^{S,2}$,... We focus our attention on the one for which all $S_m > 0$, which we assume is $E_2^{S,1}$. The intersection of $E_2^{S,1}$ with the hyperplane $\sum_m S_m = 1$ is a polyhedron, which we can triangulate into finitely simplices $\{Q_p\}$ whose vertices all have rational coordinates. By taking the convex hull of these Q_p's with the origin, one obtains a triangulation of $E_2^{S,1}$ into unbounded n-dimensional "simplices" which we denote by $\{R_p\}$. Each R_p has n unbounded faces of dimension $n - 1$ containing the origin. The equation for a given face can be written as $S \cdot q^{p,l} = 0$, where each $q^{p,l}$ has rational coordinates, so that

$$R_p = \{S : S \cdot q^{p,l} < 0 \text{ for all } 1 \leq l \leq n - i\} \tag{2.31}$$

Hence $\cup R_p = E_2^{S,1}$. The other $E_2^{S,m}$ can be similarly subdivided. We combine all simplices from all the $E_2^{S,m}$ into one list $\{R_p\}$. Note each R_p on the combined list satisfies (2.31). Furthermore, the R_p are disjoint and up to a set of measure zero $E_2^S = \cup_p R_p$. Converting back now into s coordinates, for $i > 0$ we define

$$Y_{ijp} = \{(s,t) \in Y_{ij} : \log(s) \in R_p\} = \{(s,t) \in Y_{ij} : 0 < s^{q^{p,l}} < 1 \text{ for all } 1 \leq l \leq n - i\} \tag{2.32a}$$

When $i = 0$ we let

$$Y_{0jp} = \{s \in Y_{0j} : \log(s) \in R_p\} = \{s \in Y_{0j} : 0 < s^{q^{p,l}} < 1 \text{ for all } 1 \leq l \leq n\} \tag{2.32b}$$

Then the Y_{ijp} are disjoint and up to a set of measure zero we have

$$\cup_p Y_{ijp} = Y_{ij} \subset E_2 \tag{2.33}$$

On each Y_{ijp} we shift from $y = (s, t)$ coordinates (or $y = s$ coordinates if $i = 0$) to $z = (\sigma, t)$ coordinates (or $z = \sigma$ coordinates if $i = 0$), where σ is defined by

$$\sigma_l = s^{q^{p,l}} \text{ for } l \leq n - i \tag{2.34}$$

In the new coordinates, Y_{ijp} becomes a set Z_{ijp} where

$$Z_{ijp} \subset (0, 1)^{n-i} \times D_{ij} \qquad (i > 0) \tag{2.35a}$$

$$Z_{ijp} \subset (0, 1)^n \qquad (i = 0) \tag{2.35b}$$

Let W_{ijp} denote the set Z_{ijp} in the original x coordinates. So the W_{ijp} are disjoint open sets and up to a set of measure zero $\cup_p W_{ijp} = W_{ij}$.

Lemma 2.4. If $i > 0$, write $z = (\sigma, t)$, where σ denotes the first $n - i$ components and t the last i components. For any vector w, we denote by (w', w'') the vector such that the monomial x^w transforms to $\sigma^{w'} t^{w''}$ in the z coordinates. In the case where $i = 0$, write $z = \sigma$ and say that x^w transforms into $\sigma^{w'}$.

a) If w is either a unit coordinate vector $\mathbf{e}_l$, or of the form $v' - v$ for v a vertex of S in F_{ij} and v' a vertex of S not in F_{ij}, then each component of w' is nonnegative, with at least one component positive.

b) If each component of w is nonnegative, then so is each component of w' and w''.

c) There exists some $\mu' > 0$ such that for all i, j, and p

$$(0, \mu')^{n-i} \times D_{ij} \subset Z_{ijp} \subset (0, 1)^{n-i} \times D_{ij} \qquad (i > 0) \tag{2.36a}$$

$$(0, \mu')^n \subset Z_{ijp} \subset (0, 1)^n \qquad (i = 0) \tag{2.36b}$$

In particular, when $i > 0$, for fixed t the cross-section of Z_{ijp} is a positive curved quadrant.

Proof. We assume that $i > 0$; the $i = 0$ case is done exactly the same way. If w is of one the forms of part a), then the monomial x^w in the x coordinates becomes a monomial of the form $s^{u_m} t^a$ in the y coordinates, where the u_m are as before. Since $Y_{ijp} \subset E_2$, where E_2 is as in (2.29) or (2.30), whenever each $s^{q^{p,l}} < 1$ for each l we have $s^{u_m} < 1$ for each m. Thus if we write $s^{u_m} = \prod_l (s^{q^{p,l}})^{\alpha_l}$, each α_l must be nonnegative; otherwise we could fix any $s^{q^{p,l}}$ for which α_l is nonnegative, and let the remaining $s^{q^{p,l}}$ go to zero, eventually forcing $s^{u_m} = \prod_l (s^{q^{p,l}})^{\alpha_l}$ to be greater than 1. This means that the α_l are nonnegative. If they were all zero, this would mean $u_m = 0$ which cannot happen by the discussion after (2.29c). So at least one α_l is positive. Since $s^{u_m}t^v$ transforms into $\sigma^{\alpha_l}t^v$ in the z coordinates, we have part a) of this lemma.

Next, we saw that any x_l transforms into some $s^{a_l}t^{b_l}$ in the y coordinates, where each component of a_l and b_l is nonnegative . When transforming from x to z coordinates, by part a) x_l transforms into some $\sigma^{a_l'}t^{b_l}$ with a_l' having nonnegative components. Hence part b) holds for the x_l. Therefore it holds for any x^w with each component of w nonnegative.

Moving to part c), the right-hand sides follow from (2.35). As for the left hand sides, from the expression $s^{u_m} = \prod_l (s^{q^{p,l}})^{\alpha_l}$ with nonnegative α_l, there is a $\mu' > 0$ such that each $s^{u_m} < \mu$ whenever $s^{q^{p,l}} < \mu'$ for all l. So if $s^{q^{p,l}} < \mu'$ for each l and $t \in D_{ij}$, then $(s,t) \in E_1$. By (2.29c), we conclude that whenever $s^{q^{p,l}} < \mu'$ for all l and if $t \in D_{ij}$, then $y = (s,t)$ is in Y_{ijp}. In the z coordinates this becomes the left hand inequality of (2.36a) for $i > 0$. When $i = 0$, the same argument holds; whenever $s^{q^{p,l}} < \mu'$ for each l then $s \in E_1$ and (2.36b) follows. Thus we are done with the proof of Lemma 2.4.

We can now give the proof of Theorem 2.2.

Proof of Theorem 2.2. Parts a) and b) follow from part c) of Lemma 2.4 except the statement that $D_{ij} \subset (C_i^{-e}, C_i^e)^i$ which is a consequence of (2.21b) and the fact that the y to z coordinate changes do not affect the t variables. Moving on to part c), the discussion prior to (2.19) showed that for v_1 and v_2 on F_{ij}, the x to y coordinate change takes $x^{v_2 - v_1}$ to a function of the t variables only. Since the coordinate change from y to z variables do not affect the t variables, the x to z coordinate change takes $x^{v_2-v_1}$ to a function of the t variables only as well, giving that $v_1' = v_2'$ as required.

Next, if v is a vertex of $N(S)$ on F_{ij} and v_l is a vertex of $N(S)$ not on F_{ij} or is of the form $v + \mathbf{e}_m$ for some m, then by Lemma 2.4a) $(v_l')_k \geq (v')_k$ for all k with at least one component strictly positive. Any $w \in N(S)$ satisfies $w - v = \sum c_l(v_l - v) + \sum d_m \mathbf{e}_m$ for some nonnegative c_l and d_m, where v_l are vertices of $N(S)$. As long as $w \notin F_{ij}$, there is either going to be some positive c_l for $v_l \notin F_{ij}$, or some positive d_m. Hence in this situation some $(w' - v')_k > 0$. This completes the proof of Theorem 2.2.

Suppose $x = f(z) = (z^{m_1}, ..., z^{m_n})$, where $m_i = (m_{i1}, ..., m_{in})$ are vectors such that $\det(m_{ij})$ is nonzero. Then a direct calculation reveals that the Jacobian determinant of $f(z)$ is given by

$$\det(m_{ij})(z^{\sum_i m_i - (1,1,...,1,1)}) \tag{2.37}$$

If in addition all the m_{ij} are nonnegative, we can find a $g(z)$ of the form $g(z) = (z_1^{k_1}, ... z_n^{k_n})$, $k_j > 0$, such that $f \circ g(z)$ has constant determinant. To see this, one uses the chain rule in conjunction with (2.37). One gets that the determinant of $f \circ g(z)$ is given by

$$(\prod_l k_l)\det(m_{ij})\prod_j z_j^{k_j \sum_i m_{ij} - 1}$$

Hence by setting $k_j = \frac{1}{\sum_i m_{ij}}$, one obtains that $f \circ g(z)$ has constant determinant. (The invertibility of (m_{ij}) insures that none of these sums are zero). Note that in Theorem 2.2, if one replaces $\beta_{ijp}(z)$ by such a $\beta_{ijp} \circ g(z)$, the conclusions of the theorem continue to hold. Hence in the rest of this paper, without losing generality we assume that for all i, j, and p, the Jacobian determinant of $\beta_{ijp}(z)$ is constant. One advantage of doing this is that integrals transform simply under $\beta_{ijp}(z)$ this way. Another is illustrated by the following lemma.

Lemma 2.5. Suppose $z = h(x) = (x^{b_1}, ..., x^{b_n})$, where $b_i = (b_{i1}, ..., b_{in})$ is such that the determinant of $B = (b_{ij})$ is nonzero and the Jacobian determinant of $h(x)$ is constant. Let β_i denote the hyperplane through the origin spanned by the vectors b_j for $j \neq i$. Then a monomial x^α transforms into the monomial $z^{\tilde{\alpha}}$ in z coordinates, where the ith component $\tilde{\alpha}_i$ is given by any component of the intersection of the hyperplane $\beta_i + \alpha$ with the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$

Proof. We use the notation $B_{j,v}$ to denote the matrix obtained by replacing the jth row of B by the vector v. The hyperplane $\beta_j + \alpha$ has equation $\det(B_{j,x}) = \det(B_{j,\alpha})$, so a component of the intersection of this plane with the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ is given by

$$\frac{\det(B_{j,\alpha})}{\det(B_{j,(1,1,...,1,1)})} \tag{2.38}$$

Next we examine how a monomial x^α transforms under the x to z coordinate change. To understand this, we work in logarithmic coordinates. Writing $X = (\log(x_1), ..., \log(x_n))$ and $Z = (\log(z_1), ..., \log(z_n))$, one has that $Z = BX$ or $X = B^{-1}Z$ where X and Z are viewed as n by 1 column matrices. The function $\log(x^\alpha)$ becomes $\alpha^T X = \alpha^T B^{-1} Z$. Thus in the z coordinates, x^α becomes $z^{\tilde{\alpha}}$, where $\tilde{\alpha} = (B^T)^{-1}\alpha$. By Cramer's rule, $(B^T)^{-1}\alpha = \frac{1}{\det(B)}(\det(B_{1,\alpha}), ..., \det(B_{n,\alpha}))$. Comparing with (2.38), to prove this lemma we must show that $\det(B_{j,(1,1,...,1,1)}) = \det(B)$ for all j.

To accomplish this, we use the fact that the Jacobian determinant of h is a constant function. By (2.37), this means we have $\sum_i b_i = (1, 1, ..., 1, 1)$. In matrix form, this can be written as

$$(1, 1, ..., 1, 1)B = (1, 1, ..., 1, 1) \tag{2.39}$$

Writing $\mathbf{1} = (1, 1, ..., 1, 1)$, taking transposes of (2.39) gives

$$B^T\mathbf{1} = \mathbf{1}$$

Equivalently,

$$\mathbf{1} = (B^T)^{-1}\mathbf{1}$$

By Cramer's rule this means that for all j we have

$$1 = \frac{\det(B_{j,(1,1,...,1,1)})}{\det(B)}$$

This is what we need to show and we are done.

Lemma 2.6 will interpret Lemma 2.5 in the setting of Theorem 2.2. To this end, let W_{ijp} be one of the open sets of Theorem 2.2 and $\beta_{ijp} : Z_{ijp} \rightarrow W_{ijp}$ be the associated map. We write $z = (\beta_{ijp})^{-1}(x) = (z^{b_1}, ..., z^{b_n})$. Here $b_i = (b_{i1}, ..., b_{in})$ where (b_{ij}) is an invertible matrix of rational numbers which can be negative.

Let v be a vertex of $N(S)$ on a face F_{ij}. Write $z = (\sigma, t)$, where σ are the first $n-i$ coordinates and t are the last i coordinates. The monomial x^v transforms into some $\sigma^{v'} t^{v''}$ in the z coordinates in accordance with Lemma 2.5. The $t^{v''}$ factor is of little interest; by Theorem 2.2 the t coordinates are bounded above and below away from zero and thus so is $t^{v''}$. The vector v' on the other hand is very important for the purposes of this paper, and Lemma 2.6 gives the relevant properties:

Lemma 2.6. Let $v' = (v'_1, ..., v'_{n-i})$ be as above. Let d be the Newton distance of S, and let $C(S)$ be the face of $N(S)$ (possibly unbounded) such that the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ intersects $C(S)$ in its interior. Let k be the dimension of $C(S)$, where $k = 0$ if the line intersects $N(S)$ at a vertex. Then the following hold.

a) Each v'_m satisfies $0 \leq v'_m \leq d$.

b) At most $n - k$ of the v'_m are equal to d.

c) If $n - k$ of the v'_m are equal to d, then the face F_{ij} is a subset of $C(S)$.

d) If $F_{ij} = C(S)$, then all $n - k$ of the v'_m are equal to d.

Proof. Let $1 \leq m \leq n - i$. As in the proof of Lemma 2.5, a monomial z^α becomes $x^{B^T\alpha}$ in the x coordinates. The image of the hyperplane $\{\alpha : \alpha_m = 0\}$ under B^T is the span of the b_l for $l \neq m$, denoted by β_m in Lemma 2.5. Hence the image of $\{\alpha : \alpha_m = v'_m\}$ is the hyperplane $\beta_m + v$. Suppose $w \in N(S)$. Let x^w transform into $\sigma^{w'} t^{w''}$ in the z coordinates. By Theorem 2.2c), $\sigma^{w'}$ has at least as high a power of σ_m appearing as does $\sigma^{v'}$. In other words $w'_m \geq v'_m$. Translating back into the x coordinates, any such w must be on a single side of the hyperplane $\beta_m + v$; we conclude that $\beta_m + v$ is a separating hyperplane for $N(S)$. Furthermore, by Theorem 2.2 c), for $v, w \in F_{ij}$ one has that $w'_m = v'_m$. Translating this into the x coordinates, we have that this separating hyperplane in fact contains the face F_{ij}.

Since $\beta_m + v$ is a separating hyperplane for $N(S)$, it cannot intersect the line $\{(t, t, ..., t, t) : t \in \mathbf{R}\}$ in the interior of $N(S)$. Thus the intersection point is some $(t, t, ..., t, t)$ with $t \leq d$, with $t = d$ only if $(d, d, ..., d, d) \in \beta_m + v$. Hence by Lemma 2.5, $v'_m \leq d$. By Lemma 2.4 b) we also have $v'_m \geq 0$, so we conclude that $0 \leq v'_m \leq d$ for all $1 \leq m \leq n - i$, giving a).

We now analyze how many of the v'_m can actually be equal to d. Let p denote the number of v'_m that are equal to d. By the above discussion, if m satisfies $v'_m = d$, then $\beta_m + v$ must contain F_{ij} as well as the point $(d, d, ..., d, d)$. Since any separating hyperplane for $N(S)$ containing $(d, d, ..., d, d)$ must also contain all of $C(S)$, we have that such a $\beta_m + v$ in fact contains $span(C(S), F_{ij})$. Hence the intersection of all p of these $\beta_m + v$ contains $span(C(S), F_{ij})$. We conclude that

$$n - p = dim(\cap_{\{m: v'_m = d\}} (\beta_m + v)) \geq dim(span(C(S), F_{ij}) \geq dim(C(S)) = k \qquad (2.40)$$

We conclude that $p \leq n - k$, giving b). Furthermore, if $p = n - k$, all the inequalities in (2.40) must be equalities. In particular, $dim(span(C(S), F_{ij}) = dim(C(S))$. The only way this can happen is if $F_{ij} \subset C(S)$, giving c). Lastly, suppose $F_{ij} = C(S)$. Then since each hyperplane $\beta_m + v$ for $1 \leq m \leq n - i = n - k$ contains F_{ij}, each such hyperplane also contains $(d, d, ..., d, d)$. Hence by Lemma 2.5, each $v'_m = d$ and we have part d). This concludes the proof.

3. Proofs of lower bounds of Theorems 1.2 - 1.4

We start with this elementary lemma, which we will make repeated use of.

Lemma 3.1. Suppose $m_1, ..., m_n$ are nonnegative numbers not all zero. Let $M = \max_i m_i$, and let l denote the number of m_i equal to M. Then if $|E|$ denotes Lebesgue measure, we have the following for all $0 < \delta < 1$, where C and C' are constants depending on the m_i.

a)

$$C|\ln\delta|^{l-1}\delta^{\frac{1}{M}} < |\{x \in (0,1)^n : x_1^{m_1}...x_n^{m_n} < \delta\}| < C'|\ln\delta|^{l-1}\delta^{\frac{1}{M}} \qquad (3.1)$$

b) If $M < 1$, then

$$C\delta < \int_{\{x\in(0,1)^n : \frac{\delta}{x_1^{m_1}...x_n^{m_n}} < 1\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\, dx < C'\delta$$

c) If $M = 1$, then

$$C|\ln\delta|^l\delta < \int_{\{x\in(0,1)^n : \frac{\delta}{x_1^{m_1}...x_n^{m_n}} < 1\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\, dx < C'|\ln\delta|^l\delta$$

d) If $M > 1$, then

$$\int_{\{x\in(0,1)^n : \frac{\delta}{x_1^{m_1}...x_n^{m_n}} < 1\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\, dx < C'|\{x \in (0,1)^n : x_1^{m_1}...x_n^{m_n} < \delta\}| \qquad (3.2)$$

Proof. We first deal with parts b) and c). Note that when each $m_i \leq 1$, we have

$$(\delta^{\frac{1}{n}}, 1)^n \subset \{x \in (0,1)^n : \frac{\delta}{x_1^{m_1}...x_n^{m_n}} < 1\} \subset (\delta, 1)^n$$

Thus

$$\int_{\{x\in(0,1)^n:(\delta^{\frac{1}{n}},1)^n\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\,dx < \int_{\{x\in(0,1)^n:\frac{\delta}{x_1^{m_1}...x_n^{m_n}}<1\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\,dx$$

$$< \int_{(\delta,1)^n} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\,dx \tag{3.3}$$

One can integrate the left and right hand sides of (3.3) directly and get parts b) and c). Moving on to a), we proceed by induction on n. When $n = 1$ it is immediate, so assume $n > 1$ and the result is known for $n-1$. Without losing generality, we may assume that $m_n = M$. We regard $|\{x \in (0,1)^n : x_1^{m_1}...x_n^{m_n} < \delta\}|$ as the integral of the characteristic function of $\{x \in (0,1)^n : x_1^{m_1}...x_n^{m_n} < \delta\}$, integrating with respect to x_n first. We have

$$|\{x \in (0,1)^n : x_1^{m_1}...x_n^{m_n} < \delta\}| = \int_{(0,1)^{n-1}} \min(1, \frac{\delta^{1/M}}{x_1^{m_1/M}....x_{n-1}^{m_{n-1}/M}})\,dx \tag{3.4}$$

Break (3.4) into 2 parts, depending on whether or not $x_1^{m_1}....x_{n-1}^{m_{n-1}} < \delta$. The portion where $x_1^{m_1}....x_{n-1}^{m_{n-1}} < \delta$ gives a contribution of $|\{x \in (0,1)^{n-1} : x_1^{m_1}....x^{m_{n-1}} < \delta\}|$, which by induction hypothesis will always be smaller by at least a factor of $C|\ln\delta|$ than the left and right hand sides of (3.1). As for the the portion where $x_1^{m_1}....x_{n-1}^{m_{n-1}} > \delta$, one obtains the integral

$$\int_{\{x\in(0,1)^{n-1}:\frac{\delta^{1/M}}{x_1^{m_1/M}...x_{n-1}^{m_{n-1}/M}}<1\}} (\frac{\delta^{1/M}}{x_1^{m_1/M}...x_{n-1}^{m_{n-1}/M}})\,dx \tag{3.5}$$

Since $M \geq m_i$ for all $i < n$, one can estimate (3.5) using parts b) or c) of this lemma. Since exactly $l-1$ of $m_1/M, ..., m_{n-1}/M$ are equal to 1, if $l > 1$ part c) says that (3.5) $\sim \delta^{1/M}|\ln\delta|^{l-1}$ as needed, while if $l = 1$ part b) says that (3.5) $\sim \delta^{1/M}$ as needed. This completes the proof of a).

Moving on to d), we again may assume that $m_n = M$ and perform the x_n integration first. We have

$$\int_{\{x\in(0,1)^n:\frac{\delta}{x_1^{m_1}...x_n^{m_n}}<1\}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}$$

$$= \int_{\{x\in(0,1)^{n-1}:\frac{\delta}{x_1^{m_1}...x_{n-1}^{m_{n-1}}}<1\}} (\int_{1>x_n>\frac{\delta^{1/M}}{x_1^{m_1/M}...x_{n-1}^{m_n/M}}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\,dx_n)dx_1...dx_{n-1}$$

Since $m_n > 1$, this is bounded by

$$C\int_{\{x\in(0,1)^{n-1}:\frac{\delta}{x_1^{m_1}...x_{n-1}^{m_{n-1}}}<1\}} (\int_{\frac{2\delta^{1/M}}{x_1^{m_1/M}...x_{n-1}^{m_{n-1}/M}}>x_n>\frac{\delta^{1/M}}{x_1^{m_1/M}...x_{n-1}^{m_{n-1}/M}}} \frac{\delta}{x_1^{m_1}...x_n^{m_n}}\,dx_n)$$

$$dx_1 \dots dx_{n-1} \tag{3.6}$$

The integrand is bounded above by a constant, so this is at most

$$C|\{x \in (0,1)^{n-1} \times (0,2) : x_1^{m_1} \dots x_n^{m_n} < 2^M \delta\}| \tag{3.7}$$

Rescaling in the x_n variable and using part a) gives us part d) and we are done.

We now start the proofs of the lower bounds of Theorems 1.2-1.4. Note that the lower bounds of Theorem 1.3 are contained in those of Theorem 1.2, so it suffices to prove the lower bounds of Theorem 1.2 to prove both.

Proof of Theorem 1.2a) Let $R(x) = \sum_{v \in v(S)} x^v$, where as earlier in this paper $v(S)$ denotes the set of vertices of $N(S)$. Note that $R(x)$ and $S(x)$ have the same Newton polyhedron. By the corollary to Lemma 2.1, there is a constant C such that $|S(x)| \leq C|R(x)|$ for all $x \in (0,\infty)^n$. Hence it suffices to show Theorem 1.2a) for $|R|$ in place of $|S|$.

Case 1) The face $C(S)$ is compact.

Let F_{kj} denote $C(S)$, and let W_{kjp} the corresponding sets from Theorem 2.2. We have $I_{|R|,\phi}(\epsilon) = \int_{|R|<\epsilon} \phi(x)\,dx$. Note that it suffices to show that each $\int_{\{x \in W_{kjp}:|R|<\epsilon\}} \phi(x)\,dx > C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$ for some constant C. The x to z coordinate change has constant Jacobian determinant by the discussion above Lemma 2.5, so if Φ denotes $\phi \circ \beta_{kjp}$, where β_{kjp} is as in Theorem 2.2 we have

$$\int_{\{x \in W_{kjp}:|R|<\epsilon\}} \phi(x)\,dx = c\int_{\{z \in Z_{kjp}:|R \circ \beta_{kjp}|<\epsilon\}} \Phi(z)\,dz$$

Since $\phi(0) > 0$, $\Phi(0) > 0$ as well, so for some $\delta, \xi > 0$ we have

$$\int_{\{z \in Z_{kjp}:|R \circ \beta_{kjp}|<\epsilon\}} \Phi(z)\,dz > \delta|\{z \in Z_{kjp} \cap (0,\xi)^n : |R \circ \beta_{kjp}(z)| < \epsilon\}|$$

By part a) of Theorem 2.2, we have $(0,\mu')^n \subset Z_{kjp}$ for some $\mu' > 0$. Hence for $\rho = \min(\mu', \xi)$ we have

$$\int_{\{x \in W_{kjp}:|R|<\epsilon\}} \phi(x)\,dx > C|\{z \in (0,\rho)^n : |R \circ \beta_{kjp}(z)| < \epsilon\}|$$

Writing $z = (\sigma, t)$ as in Theorem 2.2, each function x^v for $v \in v(S)$ transforms into some function $\sigma^{v'}t^{v''}$ in the z coordinates where the components of v' and v'' are all nonnegative. By part c) of Theorem 2.2, each component of v' is minimized for $v \in F_{kj} = C(S)$, and by part b), each t_i is bounded above and below away from zero. Hence if we fix some $V \in F_{kj}$, for $z \in Z_{kjp}$ we have

$$|R \circ \beta_{kjp}(z)| = |\sum_v \sigma^{v'}t^{v''}| < C\sigma^V \tag{3.8}$$

We conclude that

$$|\{z \in (0,\rho)^n : |R \circ \beta_{kjp}(z)| < \epsilon\}| > C''|\{z \in (0,\rho)^n : C|\sigma^V| < \epsilon\}| \tag{3.9}$$

By part d) of Theorem 2.6, each component of V is just equal to d. So by Lemma 3.1 a) (scaled), we have

$$|\{z \in (0,\rho)^n : C|\sigma^V| < \epsilon\}| > C''|\ln \epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$$

This gives the desired lower bounds and we are done in case 1.

Case 2) The face $C(S)$ is unbounded. Let $V = \sum_{l=1}^n a_l x_l = c$ denote a separating hyperplane for $N(S)$ such that $V \cap N(S) = C(S)$. Note that each a_l is nonnegative. Since $C(S)$ is unbounded, at least one $a_l = 0$. Without loss of generality, we may let $q < n$ such that $a_l > 0$ for $1 \leq l \leq q$ and $a_l = 0$ for $l > q$. Correspondingly write $x = (x', x'')$, where $x' \in \mathbf{R}^q$ and $x'' \in \mathbf{R}^{n-q}$. Let P denote the projection onto the first q coordinates. Define $\bar{R}(x') = \sum_{v \in P(v(S))}(x')^v$. From first principles one can verify that

$$P(N(R)) = N(\bar{R})$$

Using that V is a separating hyperplane for $N(R)$ it is also straightforward to verify that $P(V)$ is a separating hyperplane for $P(N(R)) = N(\bar{R})$ with $N(\bar{R}) \cap P(V) = P(C(S))$. But the equation for $P(V)$ is given by $\sum_{l=1}^q a_l x_l = c$ and each $a_l > 0$ for $l \leq q$. Thus $P(C(S))$ is a compact face of $N(\bar{R})$. Furthermore, since the directions $\mathbf{e}_l$ for $l > q$ are all parallel to $C(S)$ and $(d, d, ..., d, d)$(n times) is in the interior of $C(S)$, $(d, d, ..., d, d)$(p times) is in the interior of $P(C(S))$. For the same reasons, the codimension of $P(C(S))$ in $\mathbf{R}^q$ is the same as the codimension of $C(S)$ in $\mathbf{R}^n$, namely $n - k$. Hence we may apply Case 1 to $\bar{R}(x')$ and get the lower bounds of Theorem 1.2, for $\bar{R}(x')$ in place of $S(x)$.

For a given $v \in v(S)$, we write $v = (v', v'')$ where v' denotes the first q components and v'' the last $n - q$ components. We can write

$$I_{|R|,\phi}(\epsilon) = \int_{\mathbf{R}^{n-q}} \Big(\int_{\{x' \in \mathbf{R}^q : |R(x', x'')| < \epsilon\}} \phi(x', x'')\, dx' \Big)\, dx'' \tag{3.10}$$

Since $\phi(0) > 0$, there are $\delta, \xi > 0$ such that (3.10) is greater than

$$\delta \int_{(0,\xi)^{n-q}} |\{x' \in (0,\xi)^q : |R(x', x'')| < \epsilon\}|\, dx'' \tag{3.11}$$

For fixed x', one has $R(x', x'') = \sum_{v \in v(S)} (x')^{v'}(x'')^{v''}$, so consequently

$$|R(x', x'')| < C \sum_{v' \in P(v(S))} (x')^{v'} = C\bar{R}(x') \tag{3.12}$$

Hence by (3.10) and (3.11) we have

$$I_{|R|,\phi}(\epsilon) > \delta\xi^{n-q}|\{x' \in (0,\xi)^q : |\bar{R}(x')| < \frac{\epsilon}{C}\}| \tag{3.13}$$

As indicated above, case 1) of this lemma applies to $\bar{R}(x')$, which has the same values of d and k that $R(x)$ (and $S(x)$) do. Choosing an appropriate ϕ we get

$$|\{x' \in (0,\xi)^q : |\bar{R}(x')| < \frac{\epsilon}{C}\}| > C'|\ln \epsilon|^{n-k-1}\epsilon^{\frac{1}{d}} \tag{3.14}$$

Combining (3.13) and (3.14) gives the desired result and we are done.

To prepare for the proof of the lower bounds of Theorem 1.4, we consider the setting of Theorem 2.2, focusing on a specific i, j, and p. For now assume that $i > 0$. Note that β_{ijp} is defined on all of $[0,\infty)^n$, not just Z_{ijp}. Furthermore, by Lemma 2.4a), for any $t \in [0,\infty)^i$, $\beta_{ijp}(0,t) = 0$. Hence $S \circ \beta_{ijp}$ is defined on a neighborhood of $\{0\} \times [0,\infty)^i$ in $[0,\infty)^n$. Write $S(x) = \sum s_\alpha x^\alpha$ like before. By Theorem 2.2c), there is a single ω such that if $\alpha \in F_{ij}$, x^α transforms in the z coordinates into $\sigma^\omega t^{\alpha''}$ for some α'' that depends on α Hence $S_{F_{ij}}(x) = \sum_{\alpha \in F_{ij}} s_\alpha x^\alpha$ transforms into $\sigma^\omega P(t)$, where $P(t)$ is a polynomial in $t^{\frac{1}{N}}$ for some N. Any of our conditions on $S_{F_{ij}}(x)$ translates into a corresponding condition on $P(t)$. On Z_{ijp} we may write

$$S \circ \beta_{ijp}(z) = \sigma^\omega P(t) + \sum_{\alpha \notin F_{ij}} s_\alpha \sigma^{\alpha'} t^{\alpha''} \tag{3.15}$$

Equation (3.15) assumed that $i > 0$, but the $i = 0$ case can be incorporated by letting $t = 1$ and letting $P(1)$ be the appropriate coefficient. Using Theorem 2.2c) again, for a given α in the sum (3.15) each $\alpha_k' \geq \omega_k$ with at least one inequality strict. Since $\sum s_\alpha x^\alpha$ is a convergent Taylor series, we have $|s_\alpha| < CR^{|\alpha|}$ for some C and R. Because of Lemma 2.4b), $|\alpha|$ and $|\alpha'| + |\alpha''|$ are within a constant factor of one another. Hence we have an estimate $|s_\alpha| < C'(R')^{|\alpha'|+|\alpha''|}$ and therefore for some N the sum in (3.15) represents a Taylor series in $\sigma_k^{\frac{1}{N}}$ and $t_k^{\frac{1}{N}}$ convergent near the origin, not just on Z_{ijp}. Consequently, for some real-analytic functions $r_k(\sigma,t)$ of $\sigma_k^{\frac{1}{N}}$ and $t_k^{\frac{1}{N}}$ we can rewrite (3.15) as

$$S \circ \beta_{ijp}(z) = \sigma^\omega[P(t) + \sum_{k=1}^{n-i}(\sigma_k)^{\frac{1}{N}} r_k(\sigma,t)] \tag{3.16}$$

Equation (3.16) is valid near the origin. But it is also valid on a neighborhood of $\{0\} \times [0,\infty)^i$ in $[0,\infty)^n$. (If $i = 0$, we take $[0,\infty)^i$ to mean $\{1\}$). To see this, note that for any $\beta \leq N\omega$, we have $\partial_\sigma^\beta(S \circ \beta_{ijp}((\sigma_1^N, ..., \sigma_{n-i}^N, t) - \sigma^{N\omega}P(t))$ is zero on a set $\{0\} \times U$ where (3.16) is known to hold. Hence by real-analyticity it must be true on all of $\{0\} \times [0,\infty)^i$. This implies that (3.16) makes sense on a neighborhood of $\{0\} \times [0,\infty)^i$ in $[0,\infty)^n$.

We now proceed to the proof of Theorem 1.4. Assume $C(S)$ is compact face of codimension k, and there is some $x' \in (\mathbf{R} - \{0\})^n$ such that the growth index of $|S_{C(S)}|$ at x' is $a \leq \frac{1}{d}$ with multiplicity $q \geq 0$. Without loss of generality we may assume that $x' \in (0,\infty)^n$. Let $F_{kj} = C(S)$, and let W_{kjp} and Z_{kjp} be any of the sets of Theorem 2.2 corresponding to this face. In the z coordinates, $S_{C(S)}(x)$ becomes $\sigma^\omega P(t)$. Under this coordinate change, x' becomes some $z' = (\sigma', t')$ where $P(t)$ has growth index a at t' with multiplicity q. Since the coordinate change has constant determinant, if β_{kjp} as in Theorem 2.2 and Φ denotes $\phi \circ \beta_{kjp}$, then

$$I_{|S|,\phi}(\epsilon) = \int_{|S|<\epsilon} \phi(x)\, dx \geq \int_{\{x \in (\mathbf{R}^+)^n : |S(x)| < \epsilon\}} \phi(x)\, dx$$
$$= c \int_{\{z \in (\mathbf{R}^+)^n : |S \circ \beta_{kjp}(z) < \epsilon|} \Phi(z)\, dz \tag{3.17}$$

Since $\beta_{kjp}(0,t) = 0$ for all t by Lemma 2.4a), $\Phi(0,t) = \phi(0) > 0$ for all t. Thus we may let U be a neighborhood of $(0,t')$ in $(\mathbf{R}^+)^n$ such that $\Phi(\sigma, t) > \frac{\phi(0)}{2}$ on U. We then have

$$I_{|S|,\phi}(\epsilon) > \frac{\phi(0)}{2} |\{z \in U : |S \circ \beta_{kjp}(z)| < \epsilon\}| \tag{3.18}$$

Hence it suffices to find a lower bound for $|\{z \in U : |S \circ \beta_{kjp}(z)| < \epsilon\}|$ We will do this by finding a lower bound for

$$|\{z = (\sigma, t) \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k} \times U' : |S \circ \beta_{kjp}(z)| < \epsilon\}| \tag{3.19}$$

Here U' is a neighborhood of t', and μ is a sufficiently small positive number to be determined. We may assume ϵ is small enough that (3.16) holds on the set in (3.19). Using (3.16), we rewrite (3.19) as

$$|\{(\sigma, t) \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k} \times U' : |P(t) + \sum_{l=1}^{n-k} (\sigma_l)^{\frac{1}{N}} r_l(\sigma, t)| < \frac{\epsilon}{\sigma^\omega}\}| \tag{3.20}$$

By Lemma 2.6d), each $\omega_l = d$, so (3.20) is just

$$|\{(\sigma, t) \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k} \times U' : |P(t) + \sum_{l=1}^{n-k} (\sigma_l)^{\frac{1}{N}} r_l(\sigma, t)| < \frac{\epsilon}{\sigma_1^d \ldots \sigma_{n-k}^d}\}| \tag{3.21}$$

When $\sigma \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k}$, one has that $\frac{\epsilon}{\sigma_1^d \ldots \sigma_{n-k}^d} > \epsilon^{d\mu(n-k)}$. On the other hand, $\sum_{l=1}^{n-k} (\sigma_l)^{\frac{1}{N}} r_l(\sigma, t) < C\epsilon^{\frac{1}{dN(n-k)} + \frac{\mu}{dN}}$. Thus if μ were chosen appropriately small, then for small enough ϵ, if $\sigma \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k}$ one has

$$|\sum_{l=1}^{n-k} (\sigma_l)^{\frac{1}{N}} r_l(\sigma, t)| < \frac{1}{2} \frac{\epsilon}{\sigma_1^d \ldots \sigma_{n-k}^d} \tag{3.22}$$

Consequently, for such ϵ, (3.20) is bounded below by

$$|\{(\sigma, t) \in (\epsilon^{\frac{1}{d(n-k)}}, \epsilon^{\frac{1}{d(n-k)} + \mu})^{n-k} \times U' : |P(t)| < \frac{\epsilon}{2\sigma_1^d \ldots \sigma_{n-k}^d}\}|$$

$$= \int_{(\epsilon^{\frac{1}{d(n-k)}},\epsilon^{\frac{1}{d(n-k)}+\mu})^{n-k}} |\{t \in U' : P(t) < \frac{\epsilon}{2\sigma_1^d....\sigma_{n-k}^d}\}| \, d\sigma \tag{3.23}$$

By virtue of the facts that $t' \in U'$ and $P(t)$ has growth index a at t' with multiplicity q, the integrand in (3.23) is bounded below by $C(\ln|\frac{\epsilon}{\sigma_1^d....\sigma_{n-k}^d}|)^q(\frac{\epsilon}{\sigma_1^d....\sigma_{n-k}^d})^a$. Hence (3.23) is bounded below by

$$C\int_{(\epsilon^{\frac{1}{d(n-k)}},\epsilon^{\frac{1}{d(n-k)}+\mu})^{n-k}} (\ln|\frac{\epsilon}{\sigma_1^d....\sigma_{n-k}^d}|)^q(\frac{\epsilon}{\sigma_1^d....\sigma_{n-k}^d})^a \, d\sigma \tag{3.24}$$

Scaling each of the σ variables by $\epsilon^{\frac{1}{d(n-k)}}$, (3.24) becomes

$$C\epsilon^{\frac{1}{d}}\int_{(1,\epsilon^{-\mu})^{n-k}} (\ln(\sigma_1....\sigma_{n-k}))^q(\frac{1}{\sigma_1^{da}....\sigma_{n-k}^{da}}) \, d\sigma \tag{3.25}$$

We now evaluate (3.25) on a case by case basis. If $a = \frac{1}{d}$, one can do a term by term expansion of the logarithm in the integrand of

$$C\epsilon^{\frac{1}{d}}\int_{(1,\epsilon^{-\mu})^{n-k}} (\ln(\sigma_1) + ... + \ln(\sigma_{n-k}))^q(\frac{1}{\sigma_1...\sigma_{n-k}}) \, d\sigma \tag{3.26}$$

Integrating (3.26) term by term becomes immediate, and results in a lower bound of

$$C|\ln\epsilon|^{q+n-k}\epsilon^{\frac{1}{d}}$$

This is the lower bound of Theorem 1.4b). On the other hand if $a < \frac{1}{d}$, we may choose f with $a < f < \frac{1}{d}$, and we have

$$(\ln(\sigma_1....\sigma_{n-k}))^q(\frac{1}{\sigma_1^{da}....\sigma_{n-k}^{da}}) > C\frac{1}{\sigma_1^{df}....\sigma_{n-k}^{df}}$$

Hence it suffices to find lower bounds for

$$\epsilon^{\frac{1}{d}}\int_{(1,\epsilon^{-\mu})^{n-k}} \frac{1}{\sigma_1^{df}....\sigma_{n-k}^{df}} \, d\sigma \tag{3.27}$$

This is easily integrated directly to give a lower bound

$$C\epsilon^{\frac{1}{d}-(n-k)\mu(1-df)}$$

Setting $a' = \frac{1}{d} - (n-k)\mu(1-df)$ gives Theorem 1.4a) and we are done.

4. Proofs of upper bounds of Theorems 1.2 and 1.3. Recall that

$$I_{|S|,\phi}(\epsilon) = \int_{\{x:|S(x)|<\epsilon\}} \phi(x) \, dx$$

We will bound $\int_{\{x\in(\mathbf{R}^+)^n:|S(x)|<\epsilon\}}\phi(x)\,dx$ as the other octants are entirely analogous. We may assume that ϕ is supported in $(-\eta,\eta)^n$ where η is as in the constructions of section 2. Since ϕ is bounded, it suffices to bound a given

$$|\{x\in(0,\eta)^n:|S(x)|<\epsilon\}|=\sum_{ijp}|\{x\in W_{ijp}:|S(x)|<\epsilon\}|$$

Clearly it is enough to bound each term separately. Since for each i,j, and p the x to z coordinate change has constant Jacobian, it suffices to bound

$$|\{z\in Z_{ijp}:|S\circ\beta_{ijp}(z)|<\epsilon\}| \tag{4.0}$$

So our task is to bound (4.0) by the appropriate right hand side of Theorems 1.2 and 1.3. We now fix some i,j, and p. Let a denote the maximum order of any zero of $S_F(x)$ on $(\mathbf{R}-\{0\})^n$, for any compact face F of $N(S)$. In the notation of (3.15), this implies that the order of any zero of $P(t)$ on $(\mathbf{R}-\{0\})^i$ is at most a. By well known methods (see [S] Ch 8 sec 2.2), this means for any $t\in(\mathbf{R}-\{0\})^i$, there is some directional derivative ∂_w and some $0\leq a'\leq a$ such that $\partial_w^{a'}P$ is nonzero. (If $i=0$ we take $a'=0$). Note that by Theorem 2.2b) if $(\sigma,t)\in Z_{ijp}$ then $t\in(C_i^{-e},C_i^e)^n$. By continuity and compactness, we can let $\{E_l\}$ be a finite collection of cubes covering $[C_i^{-e},C_i^e]^n$, w_l be directions, a_l be nonnegative integers, and $\delta_0>0$ a constant such that on E_l

$$|\partial_{w_l}^{a_l}P(t)|>\delta_0 \tag{4.1}$$

We next examine the effect of taking such directional derivatives on the sum in (3.15). Using the fact that $|\alpha''|<C|\alpha|$ for some C, taking any t directional derivative of order at most a on this sum leads to a term bounded by

$$C\sum_{\alpha\notin F_{ij}}|s_\alpha||\alpha|^a\sigma^{\alpha'}t^{\alpha''}(\min_m t_m)^{-a} \tag{4.2}$$

We may assume that the E_l are small enough so that $t_m>\frac{1}{2}C_i^{-e}$ for each m on each E_l. Hence (4.2) is bounded by

$$C'C_i^{ae}\sum_{\alpha\notin F_{ij}}|s_\alpha||\alpha|^a\sigma^{\alpha'}t^{\alpha''} \tag{4.3}$$

By Lemma 2.1, if $(\sigma,t)\in Z_{ijp}$, then for some $V\in F_{ij}$ (4.3) is bounded by

$$C'C_i^{ae}C_{i+1}^{-\delta}x^V=C'C_i^{ae}C_{i+1}^{-\delta}\sigma^{V'}t^{V''}=C'C_i^{ae}C_{i+1}^{-\delta}\sigma^{\omega}t^{V''} \tag{4.4}$$

Here ω is as in (3.15). We can assume $|t_l|<2C_i^e$ for each l, so for some e' equation (4.4) is bounded by

$$C'C_i^{ae'}C_{i+1}^{-\delta}\sigma^{\omega} \tag{4.5}$$

We can assume C_{i+1} was chosen small enough so that $C'C_i^{ae'}C_{i+1}^{-\delta} < \frac{\delta_0}{2}$; shrinking C_{i+1} has no effect on any of the coordinate changes for the i-dimensional faces, or on the constant $C'C_i^{ae'}$ in (4.5). Hence we can assume that (4.5) is bounded by

$$\frac{\delta_0}{2}\sigma^\omega \tag{4.6}$$

Combining (4.1) and (4.6) in (3.15), we conclude that for $(\sigma, t) \in Z_{ijp}$ with $t \in E_l$ one has

$$|\partial_{w_l}^{a_l}(S \circ \beta_{ijp}(z))| > \frac{\delta_0}{2}\sigma^\omega \tag{4.7}$$

We now prove the appropriate bounds (4.0). Note that it suffices to bound each

$$|\{z = (\sigma, t) \in Z_{ijp} : t \in E_l, |S \circ \beta_{ijp}(z)| < \epsilon\}| \quad (i > 0) \tag{4.8a}$$

$$|\{z = \sigma \in Z_{0jp} : |S \circ \beta_{0jp}(z)| < \epsilon\}| \quad (i = 0) \tag{4.8b}$$

To do this, we separate into cases $a_l = 0$ and $a_l > 0$. For $a_l = 0$, by (4.7), equation (4.8a) or (4.8b) is at most

$$C|\{\sigma \in (0,1)^{n-i} : \sigma^w < \frac{2}{\delta_0}\epsilon\}| \tag{4.9}$$

By Lemma 2.6a), each component of ω is at most the Newton distance d, and the number of times d may appear in ω is at most the codimension $n-k$ of the face called $C(S)$. Hence by Theorem 3.1a), we have that (4.9) is at most

$$C'|\{\sigma \in (0,1)^{n-i} : \sigma^\omega < \epsilon\}| < C'|\ln \epsilon|^{n-k-1}\epsilon^{\frac{1}{d}} \tag{4.10}$$

This term is no greater than any of the right hand sides in Theorem 1.2, so we do not have to worry about it any further. We now move to the case when $a_l > 0$. Here we use Van der Corput's lemma in the w_l direction and then integrate the result. Since the Z_{ijp} are defined through monomial inequalities, their cross-sections in the w_l direction consist of boundedly many segments. Applying the van der Corput lemma (2.1) of [C1], we see that the w_l cross section of (4.8a) has measure at most $C(\frac{\epsilon}{\sigma^\omega})^{\frac{1}{a_l}} = \frac{\epsilon^{\frac{1}{a_l}}}{\sigma^{\omega/a_l}}$. Here ω/a_l denotes the vector where each component of ω is divided by a_l. It also of course has measure at most C since the t variables are bounded. Hence (4.8) is bounded by

$$C\int_{(0,1)^{n-i}} \min(1, \frac{\epsilon^{\frac{1}{a_l}}}{\sigma^{\omega/a_l}})\, d\sigma \tag{4.11}$$

It is natural to divide (4.9) depending on whether or not $\frac{\epsilon}{\sigma^\omega} < 1$. We get that (4.11) is bounded by

$$C|\{\sigma \in (0,1)^{n-i} : \sigma^\omega < \epsilon\}| + C\int_{\frac{\epsilon}{\sigma^\omega}<1} \frac{\epsilon^{\frac{1}{a_l}}}{\sigma^{\omega/a_l}}\, d\sigma$$

The left hand term is exactly (4.10) and satisfies the desired bounds in all cases. Since each a_l is at most the maximum order a of any zero of any $S_F(x)$, the second term of (4.10) is at most

$$C\int_{\frac{\epsilon}{\sigma^{\omega}}<1}\frac{\epsilon^{\frac{1}{a}}}{\sigma^{\omega/a}} = C\int_{\frac{\epsilon^{\frac{1}{a}}}{\sigma^{\omega/a}}<1}\frac{\epsilon^{\frac{1}{a}}}{\sigma^{\omega/a}}\,d\sigma \tag{4.12}$$

To analyze (4.12), we use the various parts of Lemma 3.1 to obtain the various upper bounds of Theorem 1.2. First suppose $a < d$. Then one or more components of ω/a may be greater than one. If this is in fact the case, Theorem 3.1d) says that (4.12) is bounded by the expression (4.10), which is the needed bound of the second statement of Theorem 1.2b). If all components of ω/a are at most 1, then by Theorem 3.1b) or c), (4.12) is at most $C|\ln\epsilon|^{n-i}\epsilon^{\frac{1}{a}}$. Since $a < d$, this is better than the bound $C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$ required by the second statement of Theorem 1.2b). This completes the proof of Theorem 1.2 for $a < d$.

If $a = d$, then each component of ω/a is at most 1, with at most $n-k$ equal to 1, so by Theorem 3.1c), (4.12) is at most $|\ln\epsilon|^{n-k}\epsilon^{\frac{1}{d}}$. This is the bound needed for the first statement of Theorem 1.2b). By Lemma 2.6c), the only way $n-k$ components of ω/a could be equal to 1 is for F_{ij} to be a subset of $C(S)$. If this is not the case, then Lemma 3.1c) says that (4.12) is at most $C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$. For a subface of $C(S)$ with zeroes of order at most $b < a = d$, then as in the $a < d$ case (4.11) is at most $C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$. Hence as long as $C(S)$ has no compact subface F such that $S_F(x)$ has a zero of order d, one gets the upper bound $C|\ln\epsilon|^{n-k-1}\epsilon^{\frac{1}{d}}$ of the second statement of Theorem 1.2b). Thus we have proven Theorem 1.2 for $a = d$.

If $a > d$, then each component of ω/a is less than 1, so by Theorem 3.1b) (4.12) is bounded by $C\epsilon^{\frac{1}{a}}$, the bound needed for Theorem 1.2 c) and we are done.

We now move on to the proof of the upper bounds in Theorem 1.3. As in the proof for Theorem 1.2, it suffices to prove upper bounds for

$$\int_{\{x\in(0,\eta)^n:|S(x)|<\epsilon\}}\phi(x)\,dx = \sum_{ijp}\int_{\{x\in W_{ijp}:|S(x)|<\epsilon\}}\phi(x)\,dx$$

$$= \sum_{ijp}c_{ijp}\int_{\{z\in Z_{ijp}:|S\circ\beta_{ijp}(z)|<\epsilon\}}\Phi_{ijp}(z)\,dz \tag{4.13}$$

Here $\Phi_{ijp}(z)$ denotes $\phi\circ\beta_{ijp}(z)$ and c_{ijp} is the (constant) Jacobian determinant of the x to z coordinate change. Clearly, it suffices to prove upper bounds for a given term of (4.13). The proof of Theorem 1.2 carries through when $i < 2$ since the nondegeneracy assumptions of Theorems 1.2 and 1.3 are the same for vertices and 1-dimensional edges and this is what was used in the analysis of the $i < 2$ terms. Hence the estimates of Theorem 1.2 hold for those terms, which imply the desired upper bounds in Theorem 1.3. So we assume that $i = 2$. Thus there are one σ variable and two t variables.

Let D_{2j} be as in Theorem 2.2. Fix $t' \in cl(D_{2j})$. We may let $U \times V$ be a neighborhood of $(0, t')$ in $[0,\infty)^3$ such that the expression $S \circ \beta_{ijp}(z) = \sigma^{\omega}[P(t) + \sigma^{\frac{1}{N}} r(\sigma, t)]$ of (3.16) is valid on $U \times V$. Let a denote the infimum over all compact faces F of $N(S)$ and all $x \in (R - \{0\})^3$ of the growth index of S_F at x. Since the x to z coordinate change transforms $S_{F_{2j}}(x)$ into $\sigma^{\omega} P(t)$, the infimum of the growth indices of $P(t)$ on $(R - \{0\})^2$ is at least a. In particular, if we denote the growth index of $P(t)$ at $t = t'$ by $a(t')$, we have

$$a(t') \geq a \tag{4.14}$$

In particular if $P(t') = 0$, then for a fixed $\mu > 0$ one has

$$a(t') > a - \mu \tag{4.15}$$

So in this situation, if V is sufficiently small, which we may assume, for any $\epsilon > 0$ we have

$$|\{t \in V : |P(t)| < \epsilon\}| < C\epsilon^{a(t') - \mu}$$

Furthermore, by a stability theorem of Karpushkin [K], if U is sufficiently small, which we may also assume, when each $\sigma_k \geq 0$ we have

$$|\{t \in V : |P(t) + \sigma^{\frac{1}{N}} r(\sigma, t)| < \epsilon\}| < C\epsilon^{a(t') - \mu} \tag{4.16}$$

(Technically Karpushkin's result applies to analytic functions of σ not $\sigma^{\frac{1}{N}}$, but a simple change of variables in σ gives us what we need). Using compactness, we may let $\{U_l \times V_l\}$ be a finite collection of $U \times V$ covering $\{0\} \times cl(D_{2j})$ such that for a given l either $P(t)$ doesn't vanish on $cl(V_l)$, or $P(t)$ has a zero on V_l with (4.16) holding for $\sigma \in U_l$. Since the continuous β_{2jp} takes $\{0\} \times [0,\infty)^2$ to the origin, and other points of $[0,\infty)^3$ to points other than the origin, if the support of ϕ is sufficiently small, which we may assume, then the support of $\Phi = \phi \circ \beta_{2jp}$ is contained in the neighborhood $\cup_l (U_l \times V_l)$ of $\{0\} \times cl(D_{2j})$. Hence to bound (4.13) it suffices to bound each

$$\int_{\{(\sigma,t) \in U_l \times V_l : |S \circ \beta_{2jp}(z)| < \epsilon\}} \Phi_{2jp}(z)\, dz$$

Since $\Phi_{2jp}(z)$ is bounded, this is at most

$$C|\{(\sigma, t) \in U_l \times V_l : |S \circ \beta_{2jp}(z)| < \epsilon\}| \tag{4.17}$$

For the $U_l \times V_l$ for which $P(t)$ doesn't vanish on $cl(V_l)$, one is in the setting of Theorem 1.2; namely (4.7) holds with $w_l = 0$ and the analysis there leading to (4.10) gives bounds as strong as all right-hand sides of Theorem 1.3. Hence we may restrict our attention to l for which $P(t)$ has a zero in V_l. In this case, (4.17) is at most

$$C|\{(\sigma, t) \in U_l \times V_l : |P(t) + \sigma^{\frac{1}{N}} r(\sigma, t)| < \frac{\epsilon}{\sigma^{\omega}}\}|$$

$$= \int_{U_l} |\{t \in V_l : |P(t) + \sigma^{\frac{1}{N}} r(\sigma,t)| < \frac{\epsilon}{\sigma^\omega}\}|\, d\sigma \tag{4.18}$$

Let a' be the minimum of all the $a(t')$ corresponding to the different $U_l \times V_l$. So in particular $a' \geq a$, where a is as in (4.14). By the above-mentioned stability result of Karpushkin, the integrand of (4.18) is at most $C\frac{\epsilon^{a'-\mu}}{\sigma^{\omega(a'-\mu)}}$. It is also uniformly bounded by the measure of V_l. Hence (4.18) is at most

$$C\int_{U_l} \min(1, \frac{\epsilon^{a'-\mu}}{\sigma^{\omega(a'-\mu)}})\, d\sigma \tag{4.19}$$

It is natural to break up the integral (4.19) into two parts, depending on whether or not $|\frac{\epsilon}{\sigma^\omega}|$ is less than or greater than 1. One gets that (4.19) is bounded by

$$C|\{\sigma \in U_l : \sigma^\omega < \epsilon\}| + C\int_{\{\sigma\in(0,1):\frac{\epsilon^{a'-\mu}}{\sigma^{\omega(a'-\mu)}}<1\}} \frac{\epsilon^{a'-\mu}}{\sigma^{\omega(a'-\mu)}}\, d\sigma \tag{4.20}$$

By Lemma 2.6, $\omega \leq d$. Thus the first term of (4.20) is bounded by $C\epsilon^{\frac{1}{d}}$. This is bounded by all the right hand sides of Theorem 1.3, so we need only consider the second term of (4.20).

Consider the situation where $a \leq \frac{1}{d}$. Then since $a' \geq a$, this second term of (4.20) is bounded by

$$C\int_{\{\sigma\in(0,1):\frac{\epsilon^{a-\mu}}{\sigma^{\omega(a-\mu)}}<1\}} \frac{\epsilon^{a-\mu}}{\sigma^{\omega(a-\mu)}}\, d\sigma \tag{4.21}$$

Since $\omega \leq d$, we have that $\omega(a-\mu) < 1$. Thus we can apply Lemma 3.1b) (or integrate directly) to obtain that the right term of (3.20) is at most $C\epsilon^{a-\mu}$. We conclude that the growth index of $|S|$ is at least $a-\mu$. Since this is true for all sufficiently small μ, we conclude that the growth index of $|S|$ is at least a. This gives us the first statement of Theorem 1.3b) as well as Theorem 1.3c), using that the multiplicity of this index is at most 2.

Next, we move to the setting of the second statement of Theorem 1.3b); that is, where the growth index of each $|S_F(x)|$ is greater than $\frac{1}{d}$ at every point in $(\mathbf{R}-\{0\})^n$. In this case a' is the minimum of finitely many numbers greater than $\frac{1}{d}$, and therefore $a' > \frac{1}{d}$. Assume μ is small enough that $a'-\mu > \frac{1}{d}$. In this case it is possible that $\omega(a'-\mu) > 1$ regardless of what μ is. If this happens, we use Lemma 3.1d), and obtain that the second term of (4.20) is bounded by a constant multiple of the first term, which as indicated above is bounded by all right-hand sides of Theorem 1.3. In the case that each $\omega(a'-\mu) \leq 1$, we apply Lemma 3.1b) or c) to obtain that the second term of (4.20) is at most $C|\ln\epsilon|\epsilon^{a'-\mu}$. Since $a'-\mu > \frac{1}{d}$, this is a better estimate than the right hand side of the first equation of Theorem 1.3b), and we are done.

5. Proofs of Theorems 1.5 and 1.6.

We start with the proof of Theorem 1.5, where we are working in two dimensions.

Lemma 5.1. If F is a 1-dimensional compact edge of $N(S)$ not intersecting the critical line $y = x$ in its interior, then $S_F(x)$ cannot have any zeroes on $(R - \{0\})^2$ of order greater than the Newton distance d.

Proof. Without loss of generality we assume F lies entirely on or below the line $y = x$. Denote by $cx^a y^b$ the term of $S_F(x, y)$ with highest power of y appearing. The line containing F is a separating line for $N(S)$, so it intersects $N(S)$ at some (d', d') for $d' \leq d$. it has negative slope, so $b \leq d' \leq d$. Since $\partial_y^b S_F(x, y) = cb!x^a$, we have a partial derivative of $S_F(x, y)$ of order at most d that doesn't vanish on $(R - \{0\})^2$. This completes the proof.

We now can prove Theorem 1.5. If the critical line doesn't intersect $N(S)$ in the interior of a compact edge, then by Lemma 5.1 we are in the setting of the second statement of Theorem 1.2b). So the growth index of S is $\frac{1}{d}$ and its multiplicity is $1 - k$. Hence the conculsions of Theorem 1.5 are satisfied.

Suppose now the critical line does intersect $N(S)$ in the interior of a compact edge F. If the associated $S_F(x)$ has zeroes of order less than d, then Lemma 5.1 implies we are once again in the setting of the second statement of Theorem 1.2b), and thus Theorem 1.5 is again satisfied. If $S_F(x)$ has a zero of order d but not greater, Theorem 1.4b) now says we have a growth index of $\frac{1}{d}$ but multiplicity 1. In other words, the final statement of Theorem 1.5b) is satisfied. If $S_F(x)$ has a zero of order greater than d, then by Theorem 1.4a) the growth index of S is less than $\frac{1}{d}$. Hence the last statement of Theorem 1.5a) is verified, and we are done.

We now turn to the proof of Theorem 1.6. As in equation $1.4a$ we write

$$I_{S,\phi}(\epsilon) \sim \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} c_{ij}(\phi) \ln(\epsilon)^i \epsilon^{r_j} \tag{5.1a}$$

Similarly, write

$$I_{-S,\phi}(\epsilon) \sim \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} C_{ij}(\phi) \ln(\epsilon)^i \epsilon^{R_j} \tag{5.1b}$$

We now no longer assume that ϕ has to be nonnegative. Recall that

$$J_{S,\phi}(\lambda) = \int_{\mathbf{R}^n} e^{i\lambda S(x)} \phi(x)\, dx \tag{5.2}$$

Doing the integration of (5.2) by first integrating over level sets $S = t$ and then with respect to t, one gets

$$\int_0^\infty \frac{dI_{S,\phi}(t)}{dt} e^{i\lambda t} \gamma(t)\, dt + \int_0^\infty \frac{dI_{-S,\phi}(t)}{dt} e^{-i\lambda t} \gamma(t)\, dt \tag{5.3}$$

Here $\gamma(t)$ is a bump function equal to 1 on the range of S. One can differentiate (5.1a) termwise, insert the result into (5.3), and then integrate termwise (we refer to [G2] for details). One obtains an expression

$$\sum_{j=0}^{\infty}\sum_{i=0}^{n-1} c'_{ij}(\phi)\int_0^{\infty} \ln(t)^i t^{r_j-1} e^{i\lambda t}\gamma(t)\,dt + \sum_{j=0}^{\infty}\sum_{i=0}^{n-1} C'_{ij}(\phi)\int_0^{\infty} \ln(t)^i t^{R_j-1} e^{-i\lambda t}\gamma(t)\,dt \quad (5.4)$$

It is well-known (see [F]) that for any $l > 0$, any real λ one has

$$\int_0^{\infty} e^{i\lambda t}\ln(t)^m t^{\alpha}\gamma(t)\,dt = \frac{\partial^m}{\partial\alpha^m}\frac{\Gamma(\alpha+1)}{(-i\lambda)^{\alpha+1}} + O(\lambda^{-l}) \quad (5.5)$$

The dominant term of (5.5) as $\lambda \to +\infty$ is given by $\frac{\Gamma(\alpha+1)\ln(\lambda)^m}{(-i\lambda)^{\alpha+1}}$ Next, note that the leading term of (5.1a) or (5.1b) will translate into the leading term of the asymptotic expansion for (5.2) unless their corresponding terms cancel out in (5.4). The leading terms of (5.1a) and (5.1b) will be at most the term corresponding to the growth index of $|S|$. If there is any cancellation in (5.4), then the result will be even faster decay for $J_{S,\phi}$. Hence the upper bounds of Theorem 1.2, 1.3, and 1.5 hold for $J_{S,\phi}$.

Suppose now $\phi(x)$ is a nonnegative function. It is not hard to check using (5.5) that the leading terms of the two series of (5.4) are given by $c_{ij}(\phi)r_j\frac{\Gamma(r_j)\ln(\lambda)^i}{(-i\lambda)^{r_j}}$ and $C_{i'j'}(\phi)R_j\frac{\Gamma(R_j)\ln(\lambda)^{i'}}{(i\lambda)^{R_{j'}}}$, where $c_{ij}\ln(t)^i t^{r_j}$ and $C_{i'j'}\ln(t)^{i'} t^{R'_j}$ are the leading terms of (5.1a) and (5.1b). They can only cancel out if $i = i'$ and $r_j = R_{j'}$. The numbers c_{ij} and $C_{ij'}$ are then both positive since the integrals they come from are of nonnegative functions. Hence for there to be cancellation, the ratio of $(-i\lambda)^{r_j}$ and $(i\lambda)^{r_j}$ must be a negative number. For this to happen, r_j must be an odd integer. We conclude that so long as the growth index of $|S|$ is not an odd integer, the oscillatory index of S is the same as this growth index. This implies that the results of Theorems 1.2-1.3 will hold for the oscillatory index. Furthermore, if $d > 1$ there will be no cancellation and therefore all of the statements analogous to Theorems 1.2-1.5 will hold for the oscillatory index. Similarly, if S does not take both positive and negative values in every neighborhood of the origin, then either (5.1a) or (5.1b) will be zero. Then there cannot be any cancellation; the growth index of S or $-S$ directly translates into the oscillatory index. Thus all of the statements analogous to Theorems 1.2-1.5 will hold for $J_{S,\phi}$. This completes the proof of Theorem 1.6.

6. References.

[AGV] V. Arnold, S Gusein-Zade, A Varchenko, *Singularities of differentiable maps Volume II*, Birkhauser, Basel, 1988.

[BM] E. Bierstone, P. Milman, *Resolution of singularities in Denjoy-Carleman classes.* Selecta Math. (N.S.) **10** (2004), no. 1, 1-28.

[C1] M. Christ, *Hilbert transforms along curves. I. Nilpotent groups*, Annals of Mathematics (2) **122** (1985), no.3, 575-596.

[C2] M. Christ, *Convolution, curvature, and combinatorics, a case study*, International Math. Research Notices **19** (1998) 1033-1048.

[DKo] J-P. Demailly, J. Kollar, *Semi-continuity of complex singularity exponents and Kahler-Einstein metrics on Fano orbifolds*, Ann. Sci. cole Norm. Sup. (4) **34** (2001), no. 4, 525-556.

[GrSe] A. Greenleaf, A. Seeger, *Oscillatory and Fourier integral operators with degenerate canonical relation*, Publicacions Matematiques special issue: Proceedings of the El Escorial Conference 2000 (2002), 93-141.

[F] M.V. Fedoryuk, *The saddle-point method*, Nauka, Moscow, 1977.

[IM] I. Ikromov, D. Müller, *On adapted coordinate systems*, to appear, Trans. Amer. Math. Soc.

[K] V. N. Karpushkin, *Uniform estimates for volumes*, Tr. Math. Inst. Steklova **221** (1998), 225-231.

[G1] M. Greenblatt, *A Coordinate-dependent local resolution of singularities and applications*, J. Funct. Anal. **255** (2008), no. 8, 1957-1994.

[G2] M. Greenblatt, *Resolution of singularities, asymptotic expansions of oscillatory integrals, and related Phenomena*, submitted.

[G3] M. Greenblatt, *Newton polygons and local integrability of negative powers of smooth functions in the plane*, Trans. Amer. Math. Soc. **358** (2006), no. 2, 657-670.

[G4] M. Greenblatt, *A direct resolution of singularities for functions of two variables with applications to analysis*, J. Anal. Math. **92** (2004), 233-257.

[G5] M. Greenblatt, *Sharp L^2 estimates for one-dimensional oscillatory integral operators with C^∞ phase.* Amer. J. Math. **127** (2005), no. 3, 659-695.

[PS] D. H. Phong, E. M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Mathematica **179** (1997), 107-152.

[PSSt] D. H. Phong, E. M. Stein, J. Sturm, *On the growth and stability of real-analytic functions*, Amer. J. Math. **121** (1999), no. 3, 519-554.

[PSt1] D. H. Phong, J. Sturm, *Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions*, Ann. of Math. (2) **152** (2000), no. 1, 277-329.

[PSt2] D. H. Phong, J. Sturm, *On the algebraic constructibility of varieties of integrable rational functions on C^n*, Math. Ann. **323** (2002), no. 3, 453-484.

[R] V. Rychkov, *Sharp L^2 bounds for oscillatory integral operators with C^∞ phases*, Math. Zeitschrift, **236** (2001) 461-489.

[Se] A. Seeger, *Radon transforms and finite type conditions*, Journal of the American Mathematical Society **11** (1998) no.4, 869-897.

[S] E. Stein, *Harmonic analysis; real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematics Series Vol. 43, Princeton University Press, Princeton, NJ, 1993.

[V] A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Functional Anal. Appl. **18** (1976), no. 3, 175-196.

[Va] V. Vassiliev, *The asymptotics of exponential integrals, Newton diagrams, and classification of minima*, Functional Analysis and its Applications **11** (1977) 163-172.

Department of Mathematics

244 Mathematics Building
University at Buffalo
Buffalo, NY 14260

Fields Institute
222 College Street
Toronto, Ontario M5T 3J1

email: greenbla@uic.edu