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Estimates for Fourier transforms of Surface Measures in $\mathbb{R}^3$ and PDE applications

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Abstract. An explicit local two-dimensional resolution of singularities theorem and arguments based on the Van der Corput lemma are used to give new estimates for the decay rate of the Fourier transform of a locally defined smooth hypersurface measure in $\mathbb{R}^3$, as well as to provide new proofs of some known estimates. These are then used to give L^q bounds on solutions to certain PDE problems in terms of the L^p norms of their initial data for various values of p and q.

1. Background and Theorem Statements

Let Q be a smooth two-dimensional surface in $\mathbb{R}^3$, and let a be a point on Q. We consider the Fourier transform of a small portion of the surface near a, localized using a smooth bump function supported near a. After a translation and rotation, without loss of generality we may take $a = (0, 0, 0)$ and assume that $(0, 0, 1)$ is normal to Q at the origin. In this situation, we are looking at the following, where $\phi(x, y)$ denotes a smooth real-valued bump function supported near the origin and where $S(x, y)$ denotes the function whose graph is given by Q.

$$T(\lambda_1, \lambda_2, \lambda_3) = \int_{\mathbb{R}^2} e^{i\lambda_1 S(x,y)+i\lambda_2 x+i\lambda_3 y}\, \phi(x, y)\, dx\, dy \tag{1.1}$$

Technically this is the Fourier transform of the surface measure at $(-\lambda_1, -\lambda_2, -\lambda_3)$, but to simplify notation we will consider $T(\lambda_1, \lambda_2, \lambda_3)$ as written here. Note that $S(0, 0) = 0$ and $\nabla S(0, 0) = (0, 0)$ due to our assumption that $(0, 0, 1)$ is normal to Q at the origin. When $S(x, y)$ is flat to infinite order, one gets very poor decay (if any) in λ_1 when $\lambda_2 = \lambda_3 = 0$ and there can be other pathologies, so we

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always assume that at least one partial $\partial_x^\alpha \partial_y^\beta S(0,0) \neq 0$. When $\lambda_2 = \lambda_3 = 0$, the function $U(\lambda_1) = T(\lambda_1, 0, 0)$ becomes a standard scalar oscillatory integral, and it is well-known (see [1] ch. 6) that when $S(x,y)$ is real-analytic, if ϕ is supported in a sufficiently small neighborhood of the origin then as $\lambda_1 \to \infty$ one has an asymptotic development of the form

$$U(\lambda_1) = c_{S,\phi}\lambda_1^{-\epsilon}(\ln \lambda_1)^m + o\big(\lambda_1^{-\epsilon}(\ln(\lambda_1))^m\big) \tag{1.2}$$

Here $m = 0$ or 1, and the pair (ϵ, m) is independent of ϕ and determined by the resolution of singularities of $S(x,y)$ at the origin. The constant $c_{S,\phi}$ will be nonzero whenever ϕ is nonnegative with $\phi(0,0) > 0$. When λ_1 is negative, then $U(\lambda_1)$ is just the complex conjugate of $U(|\lambda_1|)$. Thus an expansion of the form (1.2) in $|\lambda_1|$ also holds as $\lambda_1 \to -\infty$.

In the more general smooth case, in [8] it is shown there is always an (ϵ, m) with $\epsilon > 0$ and $m = 0$ or 1 such that for $|\lambda_1| > 2$ one has an upper bound

$$|U(\lambda_1)| \leq c_{S,\phi}|\lambda_1|^{-\epsilon}(\ln|\lambda_1|)^m \tag{1.3}$$

We stipulate that $|\lambda_1| > 2$ to avoid trivial cases where one has to change the formula due to the fact that $\ln(1) = 0$. This (ϵ, m) has the property that if ϕ is nonnegative with $\phi(0,0) > 0$, then (1.3) will not hold for (ϵ', m') with $\epsilon' > \epsilon$, or for $\epsilon' = \epsilon$ with $m' < m$. It was then shown in [13] that most of the time one even has a development of the form (1.2). It was shown in [18] for the real-analytic case and in [12] for the general smooth case that there are always certain "adapted" coordinate systems in which one can read off (ϵ, m) in terms of the Newton polygon of $S(x,y)$, and criteria can be given to determine if one is in such an adapted coordinate systems.

Note that in (1.1), if for a given $\delta > 0$ one has $|\lambda_2| + |\lambda_3| > \delta|\lambda_1|$ then if the support of ϕ is sufficiently small (depending on δ) the gradient of the phase in (1.1) is nonvanishing throughout the support of ϕ. Thus one can do repeated integrations by parts and for any N one can quickly get an estimate of the form $|T(\lambda)| < C_N|\lambda|^{-N}$. Thus one always assumes that $|\lambda_2| + |\lambda_3| \leq \delta|\lambda_1|$ for some small but fixed δ. In part because of this, in much of the work concerning the oscillatory integrals (1.1), people have viewed (1.1) as perturbations of $U(\lambda_1)$ and proven upper bounds of the form $|T(\lambda)| \leq C_{S,\phi}|\lambda_1|^{-\epsilon}(\ln|\lambda_1|)^m$, where ϵ and m are as in (1.2) or (1.3). In particular, in the real-analytic case, a theorem of Karpushkin [14]-[15] says that one always has upper bounds of this form. In the smooth situation, for the case where $\epsilon > \frac{1}{2}$ such upper bounds are a consequence of [6], and for the $\epsilon \leq \frac{1}{2}$ situation these upper bounds are proven in [11] [13]. One can obtain stronger results if one restricts to specific classes of functions, such as when the Hessian determinant is nonzero (where one has the strongest decay), the convex case considered in [3] [5], or the class of surfaces in [7]. Curvature has often played a prominent role in such theorems. Other oscillatory integrals related to surface measure Fourier transforms were analyzed in [10].

We now let $\mu = (\lambda_2, \lambda_3)$, so that λ may be written as (λ_1, μ). Our first theorem says that in the general real-analytic case, one has $|T(\lambda_1, \mu_1, \mu_2)| < C_{S,\phi}|\mu|^{-\frac{1}{2}}$. It goes beyond what follows from the perturbation results (Karpushkin's theorem) when $\epsilon \leq \frac{1}{2}$.

Theorem 1.1. Suppose $S(x,y)$ is real-analytic. There is a neighborhood V of the origin such that if ϕ is supported in V then for some constant C_S we have the following, where $|\mu|$ denotes the magnitude of the vector (μ_1, μ_2).

$$|T(\lambda_1, \mu_1, \mu_2)| < C_S |\mu|^{-\frac{1}{2}} \|\phi\|_{C^1(V)} \tag{1.4}$$

It can be shown that for many specific phases one gets a better exponent than $\frac{1}{2}$ in (1.4), but $\frac{1}{2}$ is the best exponent that holds for all phases, as can be seen when $S(x,y)$ is a function of x or y only. Typically one does not expect to get a better exponent than 1. This is because that is the decay rate for nondegenerate phases, so if one chooses ϕ supported in a small ball where ∇S and the Hessian determinant of S are nonvanishing one will get a decay rate $\sim |\mu|^{-1}$, which can be seen by examining the $|\lambda_1| \sim |\mu|$ range and letting $|\mu|, |\lambda_1| \to \infty$.

The next theorem will provide a new proof of the perturbation results for general smooth phase when $\epsilon \leq \frac{1}{3}$. In the terminology of Varchenko [18] and later papers, this corresponds to when the height of S is at least 3. Although such results are known in the real-analytic case by [14][15], and in the general smooth case by [11][13], we give a new proof here to illustrate that such theorems can also be proven with an appropriate resolution of singularities theorem, without reference to adapted coordinates and so on. While there are certainly commonalities between the proof of Theorem 1.2 and the arguments in [11][13], there are also noteworthy differences due to the use here of the resolution of singularities theorem of the next section and its consequences such as Lemma 2.2, as opposed to the type of subdivisions made in those papers.

Theorem 1.2. Suppose $S(x,y)$ is smooth and (ϵ, m) is as in (1.3). If $\epsilon \leq \frac{1}{3}$, then there is a neighborhood V of the origin such that if ϕ is supported in V then for $|\lambda_1| > 2$ one has

$$|T(\lambda_1, \mu_1, \mu_2)| \leq C_S |\lambda_1|^{-\epsilon} (\ln |\lambda_1|)^m \|\phi\|_{C^1(V)} \tag{1.5}$$

Again the $|\lambda_1| > 2$ condition is here to avoid concerning ourselves with trivial cases where one has to change the formula due to the fact that $\ln(1) = 0$.

PDE applications

We now assume $S(x_1, x_2)$ is real-analytic on some open ball B centered at the origin with $S(0,0) = 0$ and $\nabla S(0,0) = (0,0)$. Suppose $f(x)$ is a complex-valued function on $\mathbb{R}^2$ such that $\hat{f}(\xi)$ is L^1 and is supported in B. Let F denote the Fourier transform, and define $S(-i\partial)$ to be the operator such that $F(S(-i\partial)f)(\xi) = S(\xi)\hat{f}(\xi)$. When f is a function of (t, x_1, x_2) we interpret this to be this multiplier operator in the x_1 and x_2 variables, with t fixed. In Section 5, using Theorem 1.1 along with the general real-analytic version of Theorem 1.2 (i.e. Karpushkin's work) we will prove the following.

Theorem 1.3. Suppose (ϵ, m) is as in (1.3) for a real-analytic $S(x,y)$ and $\epsilon \leq \frac{1}{2}$. If B is sufficiently small, then the following holds. Let $1 \leq p < \infty$. For g such that $\hat{g} \in C_c^\infty(B)$, let $f(t, x_1, x_2)$ be the solution on $\mathbb{R}^3$ to the partial differential equation

$$\frac{\partial f}{\partial t}(t, x_1, x_2) = iS(-i\partial) f(t, x_1, x_2)$$

$$f(0, x_1, x_2) = g(x_1, x_2) \tag{1.6}$$

Then if $1 < q \leq \infty$ satisfies $\frac{1}{q} - \frac{1}{p} + \frac{3}{4} < 0$ there is a constant $C_{p,q,S}$ such that one has the estimate

$$\|f\|_q \leq C_{p,q,S}(|t|+2)^{4\epsilon(\frac{1}{q}-\frac{1}{p}+\frac{3}{4})}(\ln(|t|+2))^{-4m(\frac{1}{q}-\frac{1}{p}+\frac{3}{4})}\|g\|_p \tag{1.7}$$

The same is true if $\frac{1}{q} - \frac{1}{p} + \frac{3}{4} = 0$, as long as $p \neq 1$ and $q \neq \infty$. Here the L^p and L^q norms are in the x variables.

We have the condition $\epsilon \leq \frac{1}{2}$ in Theorem 1.3 since when $\epsilon > \frac{1}{2}$ one can get a stronger result by relatively rudimentary means. Consider now the case where $S(x_1, x_2) \geq 0$ in a neighborhood of the origin, and consider the oscillatory integral $R(\lambda_1, \mu_1, \mu_2)$ defined by

$$R(\lambda_1, \mu_1, \mu_2) = \int_{\mathbb{R}^2} e^{-\lambda_1 S(x_1,x_2) + i\mu_1 x_1 + i\mu_2 x_2}\, \phi(x_1, x_2)\, dx_1\, dx_2 \tag{1.8}$$

In other words, we replace the $i\lambda_1 S(x_1, x_2)$ in (1.1) by $-\lambda_1 S(x_1, x_2)$. In Lemma 5.1, we will see (by a much easier argument than those proving Theorems 1.1 and 1.2) that if the support of ϕ is sufficiently small then for $\lambda_1 > 2$ one has an estimate

$$|R(\lambda_1, \mu_1, \mu_2)| \leq C_{S,\phi} \min(\lambda_1^{-\epsilon}(\ln \lambda_1)^m, |\mu|^{-1}) \tag{1.9}$$

Then in analogy to Theorem 1.3, one has the following theorem.

Theorem 1.4. Assume S is real-analytic and nonnegative on a neighborhood of the origin. If B is sufficiently small, then the following holds. Let $1 \leq p < \infty$. For g such that $\hat{g} \in C_c^\infty(B)$, let $f(t, x_1, x_2)$ be the solution on $\mathbb{R}^3$ to the partial differential equation

$$\frac{\partial f}{\partial t}(t, x_1, x_2) = -S(-i\partial) f(t, x_1, x_2)$$

$$f(0, x_1, x_2) = g(x_1, x_2) \tag{1.10}$$

Then if $1 < q \leq \infty$ satisfies $\frac{1}{q} - \frac{1}{p} + \frac{1}{2} < 0$, there exists a constant $C_{p,q,S}$ such that for $t > 0$ one has estimate

$$\|f\|_q \leq C_{p,q,S}(t+2)^{2\epsilon(\frac{1}{q}-\frac{1}{p}+\frac{1}{2})}(\ln(t+2))^{-2m(\frac{1}{q}-\frac{1}{p}+\frac{1}{2})}\|g\|_p \tag{1.11}$$

The same is true if $\frac{1}{q} - \frac{1}{p} + \frac{1}{2} = 0$, as long as $p \neq 1$ and $q \neq \infty$. Here (ϵ, m) is as in (4.2) for $S(x, y)$, and the L^p and L^q norms are in the x variables.

Note that in Theorem 1.4, (ϵ, m) is as in (4.2) and not as in (1.3). By [8] these may differ only when $(\epsilon, m) = (1, 0)$ in (1.3); in this case the (ϵ, m) in (4.2) may be either $(1, 0)$ or $(1, 1)$.

Theorem 1.4 can be used to relatively quickly give the following. Here $[S(-i\partial)]^\delta$ refers to the operator with Fourier multiplier $(S(\xi_1, \xi_2))^\delta$ (we will only be considering it on ξ domains where $S(\xi_1, \xi_2)$ is nonnegative).

Theorem 1.5. Again assume S is real-analytic and nonnegative on a neighborhood of the origin, and again let (ϵ, m) be as in (4.2) for $S(x, y)$. If B is sufficiently small, then the following holds. Let $0 < \delta < \epsilon$. For g such that $\hat{g} \in C_c^\infty(B)$, let $f(x_1, x_2)$ solve the equation

$$[S(-i\partial)]^\delta f = g \tag{1.12}$$

Then if $p \in [1, \infty)$ and $q \in (1, \infty]$ such that $\frac{1}{q} - \frac{1}{p} + \frac{1}{2} + \frac{\delta}{2\epsilon} < 0$, one has an estimate of the form

$$\|f\|_q \leq C_{p,q,S} \|g\|_p \tag{1.13}$$

When $m = 0$, the same is true if $\frac{1}{q} - \frac{1}{p} + \frac{1}{2} + \frac{\delta}{2\epsilon} = 0$, so long as $p \neq 1$ and $q \neq \infty$.

The condition that $\delta < \epsilon$ is needed in Theorem 1.5 for the statement to make sense; if $\delta \geq \epsilon$ then $S(\xi_1, \xi_2)^{-\delta}$ is not integrable on a neighborhood of the origin and one cannot even automatically refer to the solution to (1.12).

Theorems 1.3-1.5 are not intended to give the best possible exponents, or in the case of Theorems 1.3-1.4, the best possible powers of $|t|$ and $\ln |t|$, for any particular $S(x, y)$. Rather, they are illustrations of how one may interpret in terms of PDE theorems the combination of Theorem 1.1 and 1.2 or their analogues for $R(\lambda_1, \mu_1, \mu_2)$, in such a way as to give a result for any given $S(x, y)$.

Because Theorem 1.3 is closely tied to the oscillatory integral estimates of Theorem 1.1-1.2 and Theorems 1.4-1.5 are closely tied to the analogous estimates of Lemma 5.1, previous results on such PDE problems have generally been those that follow from surface measure Fourier transform theorems such as those in [6] [7]. However, in the nondegenerate case (that is, when the Hessian of $S(x_1, x_2)$ is nonvanishing), it is relatively easy to bound $T(\lambda_1, \lambda_2, \lambda_3)$ and $R(\lambda_1, \mu_1, \mu_2)$ in a precise way because the phase functions are now nondegenerate. In this case, an analogue of Theorem 1.3 would be L^p to L^q asymptotic decay estimates for solutions to the homogeneous Schrödinger equation with initial conditions having Fourier transform in $C_c(\mathbb{R}^2)$, an analogue of Theorem 1.4 would be corresponding L^p to L^q asymptotic decay estimates for solutions to the homogeneous heat equation, and an analogue of Theorem 1.5 would be analogous estimates for fractional powers of the Laplacian. Theorems 1.3-1.5 can be then be viewed as generalizations to when $S(x_1, x_2)$ is degenerate.

2. The Resolution of Singularities Theorem

We next describe the resolution of singularities theorem that we need for this paper. There have been various resolution of singularities algorithms used in classical analysis problems in two dimensions, such as those of [16] [18] and the author's earlier work. For the purposes of this paper we will use a modification of the one used in [9], which was influenced by both [16] and [18].

Suppose $f(x,y)$ is any smooth function on a neighborhood of the origin such that $f(0,0)=0$ and such that the Taylor expansion of f at the origin has at least one nonvanishing term. After a linear change of coordinates if necessary we may assume that if k denotes the order of the zero of $f(x,y)$ at the origin then the Taylor expansion $\sum_{\alpha\beta} f_{\alpha\beta}x^\alpha y^\beta$ of f at the origin contains both a nonvanishing $f_{k0}x^k$ term and a nonvanishing $f_{0k}y^k$ term. We will now apply the resolution of singularities algorithm of Theorem 3.1 of [9] in the following fashion. We divide the xy plane into 8 triangles via the x and y axes as well as two lines through the origin, one of the form $y=mx$ for $m>0$ and one of the form $y=mx$ for $m<0$. For certain technical reasons, these two lines cannot be ones on which the function $f_0(x,y)=\sum_{\alpha+\beta=k} f_{\alpha\beta}x^\alpha y^\beta$ vanishes. After possible reflections about the x and/or y axes and/or the line $y=x$, modulo its boundary each of the triangles is of the form $B_a=\{(x,y)\in\mathbb{R}^2 : x>0,\ 0<y<ax\}$.

We now apply Theorem 3.1 of [9] to the (reflected) $f(x,y)$ on the portion of B_a contained in a sufficiently small neighborhood of the origin. Actually, we apply a slight variant. If in the first step of the proof of Theorem 3.1 of [9] one does a coordinate change of the form $(x,y)\to(x,y+cx+$ higher order terms), instead we just do a coordinate change $(x,y)\to(x,y+cx)$. This has some technical advantages; see the proof of Theorem 2.1 d) below. Other than this, we do exactly the algorithm of Theorem 3.1 of [9]. The following theorem is then a consequence of Theorem 3.1 of [9].

Theorem 2.1. Suppose $B_a=\{(x,y)\in\mathbb{R}^2 : x>0,\ 0<y<ax\}$ is as above. Abusing notation slightly, use the notation $f(x,y)$ to denote the reflected function $f(\pm x,\pm y)$ or $f(\pm y,\pm x)$ corresponding to B_a. Then there is a $b>0$ and a positive integer N such that if F_a denotes $\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq b,\ 0\leq y\leq ax\}$, then one can write $F_a=\cup_{i=1}^n cl(D_i)$, such that for to each i there is a $\psi_i(x)$ with $\psi_i(x^N)$ smooth and $\psi_i(0)=0$ such that after a coordinate change of the form $\eta_i(x,y)=(x,\pm y+\psi_i(x))$, the set D_i becomes a set D_i' on which the function $f\circ\eta_i(x,y)$ approximately becomes a monomial $d_ix^{\alpha_i}y^{\beta_i}$, α_i a nonnegative rational number and β_i a nonnegative integer as follows.

a) $D_i'=\{(x,y): 0<x<b,\ g_i(x)<y<G_i(x)\}$, where $g_i(x^N)$ and $G_i(x^N)$ are smooth. If we expand $G_i(x)=H_ix^{M_i}+...$, then $M_i\geq 1$ and $H_i>0$, and consists of a single term $H_ix^{M_i}$ when $\beta_i=0$. The function $g_i(x)$ is either identically zero or can be expanded as $h_ix^{m_i}+...$ where $h_i>0$ and $m_i>M_i$.

b) If $\beta_i=0$, then $g_i(x)$ is identically zero. Furthermore, the D_i' can be constructed such that for any predetermined $\delta > 0$ there is a $d_i \neq 0$ such that on D_i', for all $0 \leq l \leq \alpha_i$ one has

$$|\partial_x^l (f \circ \eta_i)(x,y) - d_i \alpha_i(\alpha_i - 1)...(\alpha_i - l + 1)x^{\alpha_i - l}| < \delta |d_i| x^{\alpha_i - l} \tag{2.1}$$

This δ can be chosen independent of all the exponents appearing in this theorem. Furthermore, if one Taylor expands $f \circ \eta_i(x,y)$ in powers of $x^{\frac{1}{N}}$ and y as $\sum_{\alpha,\beta} F_{\alpha,\beta} x^\alpha y^\beta$, then $\alpha_i \leq \alpha + M_i \beta$ for all (α, β) such that $F_{\alpha,\beta} \neq 0$, with equality holding for at least two (α, β), one of which is $(\alpha_i, 0)$ and another of which satisfies $\beta > 0$.

c) If $\beta_i > 0$, then one may write $f = f_1^i + f_2^i$ as follows. $f_2^i \circ \eta_i(x,y)$ has a zero of infinite order at $(0,0)$ and is identically zero if f is real-analytic. $f_1^i \circ \eta_i(x^N, y)$ is smooth and there exists a $d_i \neq 0$ such that for any predetermined $\delta > 0$ (which can be chosen independent of the exponents appearing in this theorem) the D_i' can be constructed such that on D_i', for any $0 \leq l \leq \alpha_i$ and any $0 \leq m \leq \beta_i$ one has

$$|\partial_x^l \partial_y^m (f_1^i \circ \eta_i)(x,y) - \alpha_i(\alpha_i - 1)....(\alpha_i - l + 1)\beta_i(\beta_i - 1)...(\beta_i - m + 1) d_i x^{\alpha_i - l} y^{\beta_i - m}|$$
$$\leq \delta |d_i| x^{\alpha_i - l} y^{\beta_i - m} \tag{2.2}$$

d) If $\beta_i = 0$ and we write $\psi_i(x) = k_i x^{r_i} + ...$, then either $\psi_i(x) = k_i x$ for some k_i, $\psi_i(x) = k_i x + l_i x^{s_i}$ with $s_i = M_i > 1$ and $l_i \neq 0$, or $\psi_i(x) = k_i x + l_i x^{s_i} +$ higher-order terms (if any), where $l_i \neq 0$ and $M_i > s_i > 1$.

Proof. Part a) is part of the statement of Theorem 3.1 of [9], other than the form of the upper edge of D_i' when $\beta_i = 0$, which is given in the proof itself. Part c) is also contained in the statement of Theorem 3.1 of [9].

As for part b), a weaker version was proved in [9] using equation (3.4) of that paper, and the stronger statement here also follows from that equation; if one divides D_i' into finitely many subwedges of width $\sim \epsilon x^{M_i}$ for small ϵ and then does a coordinate change of the form $(x, y - cx^{M_i})$ on each subwedge that transfers its lower boundary to the x-axis, then if the subwedges are narrow enough, equation (3.4) of [9] implies that (2.1) holds. Decreasing ϵ ensures that δ can be made as small as one would like. As for the last sentence of part b), although it is not in the statement of Theorem 3.1 of [9] it is shown in the proof.

Part d) is a consequence of the fact that in the version of the algorithm here, for a D_i' with $\beta_i = 0$ one starts with a coordinate change of the form $(x,y) \rightarrow (x, y + k_i x)$, $k_i \neq 0$ if needed. If additional coordinate changes are needed, then the second coordinate change is either of the form $(x, \pm y + l_i x^{M_i})$ with $M_i = s_i$ and we are done, or it is of the form $(x, \pm y + l_i x^{s_i} +$ possible higher order terms$)$ in such a way that the domains eventually giving a $\beta_i = 0$ wedge already are of width cx^m for some $m > s_i$. Further iterations of the resolution of singularities process will only add terms of degree greater than s_i and narrow the wedge further, resulting in an $M_i > s_i$.

The next lemma is a consequence of Theorem 2.1 we will need for our arguments.

Lemma 2.2. Suppose D_i' is such that $\beta_i = 0$. Then on $[0,b] \times [0,H_i]$ we may write

$$f \circ \eta_i(x, x^{M_i} y) = x^{\alpha_i} r_i(y) + E_i(x,y) \tag{2.3}$$

Here $r(y)$ is a polynomial that doesn't vanish on $[0,H_i]$ and there is a $\delta > 0$ such that for any $l \geq 0$ there is a constant C_{il} such that $E_i(x,y)$ satisfies

$$|\partial_x^l E_i(x,y)| \leq C_{il} x^{\alpha_i + \delta - l} \tag{2.4}$$

Proof. Again write $f \circ \eta_i(x,y) = \sum_{\alpha,\beta} F_{\alpha,\beta} x^\alpha y^\beta$. By part b) of Theorem 2.1, the minimum of $\alpha + M_i\beta$ in the sum above is α_i and furthermore $F_{\alpha_i,0} \neq 0$. Let $q_i(x,y)$ be the polynomial $\sum_{\alpha + M_i\beta = \alpha_i} F_{\alpha,\beta} x^\alpha y^\beta$. Then by mixed homogeneity we may write $q_i(x,y) = x^{\alpha_i} q_i(1, \frac{y}{x^{M_i}})$. We now do a partial Taylor expansion of $f \circ \eta_i(x,y)$ in the form

$$f \circ \eta_i(x,y) = q_i(x,y) + \sum_{\alpha_i < \alpha + M_i\beta < K} F_{\alpha,\beta} x^\alpha y^\beta + O(x^K) \tag{2.5}$$

Here K is a large number determined by our arguments. We have an $O(x^K)$ and not an $O(x^K) + O(y^K)$ remainder term here because $0 < y < H_i x^{M_i}$ on D_i'. Next, note that (2.5) implies

$$f \circ \eta_i(x, x^{M_i} y) = x^{\alpha_i} q_i(1,y) + \sum_{\alpha_i < \alpha + M_i\beta < K} F_{\alpha,\beta} x^{\alpha + M_i\beta} y^\beta + O(x^K) \tag{2.6}$$

By Theorem 2.1 b), there are positive constants e_i and E_i such that on $(0,b] \times [0,H_i]$ one has

$$e_i \leq \frac{|f \circ \eta_i(x, x^{M_i} y)|}{x^{\alpha_i}} \leq E_i \tag{2.7}$$

So by dividing by x^{α_i} and taking limits as $x \to 0$ in (2.6) we have that $q_i(1,y) \neq 0$ for $0 \leq y \leq H_i$. Thus Lemma 2.2 holds if we take $r_i(y) = q_i(1,y)$ and $\alpha_i + \delta$ to be the least value of $\alpha + M_i\beta$ for which $F_{\alpha,\beta}$ is nonzero other than α_i; each time one takes an x derivative each term in the sum of (2.6) loses a degree in x, as does the $O(x^K)$ term. Thus as long as K is chosen sufficiently large (depending on l) the conclusions of Lemma 2.2 follow.

3. Proof of Theorem 1.1

Let $k \geq 2$ denote the order of the zero of $S(x,y)$ at the origin. Doing a linear coordinate change if necessary, we may assume that we have $\frac{1}{2}|\mu_1| < |\mu_2| < 2|\mu_1|$ and also that the Taylor expansion $\partial_y S(x,y) = \sum_{\alpha\beta} S_{\alpha\beta} x^\alpha y^\beta$ has a nonvanishing $S_{k-1\,0} x^{k-1}$ term and a nonvanishing $S_{0\,k-1} y^{k-1}$ term, and that the same as true for $\partial_x S(x,y)$. We divide a small rectangle centered at the origin into 8 regions via the lines $y = mx$ and the x and y axes as in the beginning of section 2, and then after reflections about the x or y axes and/or the line $y = \pm x$ as necessary we assume we are working on 8 domains of the form $\{(x,y) : 0 < x < b,\, 0 < y < ax\}$.

We now apply Theorem 2.1 to each $\partial_y S(x,y)$, where $S(x,y)$ now refers to the phase in the possibly reflected coordinates of its domain. Let $\{D_i\}_{i=1}^n$ denote the domains resulting from applying Theorem 2.1 on these domains; we include the D_i from all 8 domains in a single list. Where ϕ is a cutoff function supported on a small neighborhood of the origin, define $T_i(\lambda_1,\mu_1,\mu_2)$ by

$$T_i(\lambda_1,\mu_1,\mu_2) = \int_{D_i} e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,\phi(x,y)\,dx\,dy \tag{3.1}$$

To be perfectly clear, we are still abusing notation slightly in (3.1); $S(x,y)$ denotes the phase function in the reflected coordinates. Since $|\mu_2| \sim |\mu_1|$ (in both the original and reflected coordinates), to prove Theorem 1.1 it suffices to show that if the support of ϕ is sufficiently small, then for each i there is a constant C depending on S such that for $|\mu| > 2$ we have

$$|T_i(\lambda_1,\mu_1,\mu_2)| < C|\mu_2|^{-\frac{1}{2}}\|\phi\|_{C^1(V)} \tag{3.2}$$

(If $|\mu| \leq 2$ one may just take absolute values and integrate to get the result). The i for which $\eta_i(x)$ in Theorem 2.1 is of the form $(x, -y+\psi_i(x))$ are dealt with the same way as the i for which $\eta_i(x)$ is of the form $(x, y+\psi_i(x))$, so we always assume $\eta_i(x)$ is of the latter form.

Write $T_i(\lambda_1,\mu_1,\mu_2) = T_i^1(\lambda_1,\mu_1,\mu_2) + T_i^2(\lambda_1,\mu_1,\mu_2)$, where

$$T_i^1(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_i : |\mu_2| > 2|\lambda_1\partial_y S(x,y)|\}} e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,\phi(x,y)\,dx\,dy \tag{3.3a}$$

$$T_i^2(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_i : |\mu_2| \leq 2|\lambda_1\partial_y S(x,y)|\}} e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,\phi(x,y)\,dx\,dy \tag{3.3b}$$

We bound $T_i^1(\lambda_1,\mu_1,\mu_2)$ first. We rewrite the right-hand side of (3.3a) as

$$\int_{\{(x,y)\in D_i : |\mu_2| > 2|\lambda_1\partial_y S(x,y)|\}} (i\lambda_1\partial_y S(x,y) + i\mu_2)e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}$$

$$\times\frac{1}{i\lambda_1\partial_y S(x,y) + i\mu_2}\,\phi(x,y)\,dx\,dy \tag{3.4}$$

Note that since $|\mu_2| > 2|\lambda_1\partial_y S(x,y)|$ in the domain of integration, in the above integration we have $|i\lambda_1\partial_y S(x,y) + i\mu_2| > \frac{1}{2}|\mu_2|$. This implies that we may integrate by parts in (3.4), integrating $(i\lambda_1\partial_y S(x,y) + i\mu_2)e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}$ and differentiating the other two factors. If the derivative lands on $\phi(x,y)$, we take absolute values and integrate, using that $|\frac{1}{i\lambda_1\partial_y S(x,y)+i\mu_2}| < \frac{2}{|\mu_2|}$. The result is a bound of $C\frac{1}{|\mu_2|}\|\phi\|_{C^1(V)}$, a better bound than what we need. If the derivative lands on $\frac{1}{i\lambda_1\partial_y S(x,y)+i\mu_2}$, we obtain a term bounded in absolute value by

$$\|\phi\|_{C^1(V)}\int_{\{(x,y)\in D_i:|\mu_2|>2|\lambda_1\partial_y S(x,y)|\}}\frac{|\partial_{yy}S(x,y)|}{(\partial_y S(x,y)+\mu_2)^2}\,dx\,dy \tag{3.5}$$

Because of the linear coordinate change performed at the beginning of the argument, $|\partial_y^k S(x,y)|\neq 0$ on the domain of integration of (3.5). Thus for fixed x, there are boundedly many segments on which $\frac{|\partial_{yy}S(x,y)|}{(\partial_y S(x,y)+\mu_2)^2}=\pm\frac{\partial_{yy}S(x,y)}{(\partial_y S(x,y)+\mu_2)^2}$. On each such segment one can integrate back $\pm\frac{\partial_{yy}S(x,y)}{(\partial_y S(x,y)+\mu_2)^2}$ to obtain $\mp\frac{1}{\partial_y S(x,y)+\mu_2}$, similar to in the proof of the Van der Corput lemma. Since $|\frac{1}{\partial_y S(x,y)+\mu_2}|\leq 2\frac{1}{|\mu_2|}$, we get that (3.5) is bounded by $C\frac{1}{|\mu_2|}\|\phi\|_{C^1(V)}$, the same bound as we had for the other term. Lastly, we observe that the endpoint terms in the integration by parts also give a bound of $C\frac{1}{|\mu_2|}\|\phi\|_{C^1(V)}$. Thus $T_i^1(\lambda_1,\mu_1,\mu_2)$ satisfies the bounds we seek.

We now proceed to bounding $T_i^2(\lambda_1,\mu_1,\mu_2)$. The argument from this point on is done somewhat differently if $\beta_i>0$ or $\beta_i=0$ for the domain D_i, where β_i is as in Theorem 2.1, which we recall we are applying to $\partial_y S(x,y)$.

Case 1. $\beta_i>0$.

We decompose $D_i=\cup_{j,k}D_{ijk}$, where $D_{ijk}=\{(x,y)\in D_i : 2^{-j-1}<x\leq 2^{-j},\ 2^{-k-1}<y-\psi_i(x)\leq 2^{-k}\}$, and we correspondingly define

$$T_{ijk}^2(\lambda_1,\mu_1,\mu_2)=\int_{\{(x,y)\in D_{ijk}:|\mu_2|\leq 2|\lambda_1\partial_y S(x,y)|\}}e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,\phi(x,y)\,dx\,dy \tag{3.6}$$

The second y derivative of the phase in (3.6) is $\lambda_1 S_{yy}(x,y)$, which by part c) of Theorem 2.1 can be written as $\lambda_1\beta_i d_i x^{\alpha_i}(y-\psi_i(x))^{\beta_i-1}+o(|\lambda_1 x^{\alpha_i}(y-\psi_i(x))^{\beta_i-1}|)$. It is here that we use the real-analyticity condition; if the function is not real-analytic then the error term might not be $o(|\lambda_1 x^{\alpha_i}(y-\psi_i(x))^{\beta_i-1}|)$ in the event that the lower boundary of D_i' is the x-axis. We now apply the measure version of the Van der Corput lemma (see [4]) in the y direction, integrate the result in x, and we get that

$$|T_{ijk}^2(\lambda_1,\mu_1,\mu_2)|\leq C\|\phi\|_{C^1(V)}2^{-j}(|\lambda_1|^{-\frac{1}{2}}2^{\frac{j\alpha_i}{2}}2^{\frac{k(\beta_i-1)}{2}}) \tag{3.7}$$

In order for (3.6) to be nonzero, there must be at least one point in D_{ijk} for which $|\mu_2|\leq 2|\lambda_1\partial_y S(x,y)|$. Since $|\lambda_1\partial_y S(x,y)|$ doesn't vary by more than a constant factor on D_{ijk}, this means there exists a C such that if (3.6) is nonzero then on all of D_{ijk} one has

$$\begin{aligned}|\mu_2|&\leq C|\lambda_1\partial_y S(x,y)|\\&\leq C'|\lambda_1|2^{-j\alpha-k\beta}\end{aligned} \tag{3.8}$$

Substituting this into (3.7), we get that

$$|T^2_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C\|\phi\|_{C^1(V)}2^{-j-\frac{k}{2}}|\mu_2|^{-\frac{1}{2}} \tag{3.9}$$

We now add (3.9) over all (j,k), resulting in a bound of a constant multiple of $\|\phi\|_{C^1(V)}|\mu_2|^{-\frac{1}{2}}$. Since $|\mu_2| \sim |\mu_1|$, this gives us the needed bound of a constant times $\|\phi\|_{C^1(V)}|\mu|^{-\frac{1}{2}}$.

Case 2. $\beta_i = 0$.

This time we decompose $D_i = \cup_j D_{ij}$ where $D_{ij} = \{(x,y) \in D_i : 2^{-j-1} < x \leq 2^{-j}\}$, and we correspondingly define

$$T^2_{ij}(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_{ij}:|\mu_2|\leq 2|\lambda_1\partial_y S(x,y)|\}} e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,\phi(x,y)\,dx\,dy \tag{3.10}$$

Let $\gamma = \frac{-M_i+\alpha_i}{2}$. We write $T^2_{ij}(\lambda_1,\mu_1,\mu_2) = T^3_{ij}(\lambda_1,\mu_1,\mu_2)+T^4_{ij}(\lambda_1,\mu_1,\mu_2)$, where

$$T^3_{ij}(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_{ij}:|\mu_2|\leq 2|\lambda_1\partial_y S(x,y)|,\ |\lambda_1\partial_y S(x,y)+\mu_2|\geq|\lambda_1|^{\frac{1}{2}}2^{-j\gamma}\}} \phi(x,y)$$
$$\times e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,dx\,dy \tag{3.11a}$$

$$T^4_{ij}(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_{ij}:|\mu_2|\leq 2|\lambda_1\partial_y S(x,y)|,\ |\lambda_1\partial_y S(x,y)+\mu_2|<|\lambda_1|^{\frac{1}{2}}2^{-j\gamma}\}} \phi(x,y)$$
$$\times e^{i\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}\,dx\,dy \tag{3.11b}$$

For $T^3_{ij}(\lambda_1,\mu_1,\mu_2)$ we integrate by parts in y exactly as we did in (3.4) − (3.5), using that $|\lambda_1\partial_y S(x,y)+\mu_2| \geq |\lambda_1|^{\frac{1}{2}}2^{-j\gamma}$ in place of $|\lambda_1\partial_y S(x,y)+\mu_2| \geq \frac{1}{2}|\mu_2|$. Instead of a bound of $C\frac{1}{|\mu_2|}\|\phi\|_{C^1(V)}$, this time we get the bound

$$|T^3_{ij}(\lambda_1,\mu_1,\mu_2)| \leq C2^{-j}\frac{1}{|\lambda_1|^{\frac{1}{2}}2^{-j\gamma}}\|\phi\|_{C^1(V)} \tag{3.12}$$

$$= C2^{-j}|\lambda_1|^{-\frac{1}{2}}2^{j(\frac{-M_i+\alpha_i}{2})}\|\phi\|_{C^1(V)} \tag{3.13}$$

Here the $\frac{1}{|\lambda_1|^{\frac{1}{2}}2^{-j\gamma}}\|\phi\|_{C^1(V)}$ factor is from the y integration and the 2^{-j} factor is from the subsequent x integration. Like in Case 1, if (3.11a) is nonzero then on the domain of integration we have $|\mu_2| \leq C|\lambda_1\partial_y S(x,y)|$. By Theorem 2.1 c) $|\partial_y S(x,y)| \sim x^{\alpha_i} \sim 2^{-j\alpha_i}$ here (since $\beta_i = 0$). So in (3.13), the $2^{\frac{j\alpha_i}{2}}$ factor is bounded by $C|\lambda_1|^{\frac{1}{2}}|\mu_2|^{-\frac{1}{2}}$, and therefore (3.13) is bounded by

$$C'|\mu_2|^{-\frac{1}{2}}2^{j(-\frac{M_i}{2}-1)}\|\phi\|_{C^1(V)} \tag{3.14}$$

Adding over all j gives a bound of $C''|\mu_2|^{-\frac{1}{2}}\|\phi\|_{C^1(V)}$, the desired bound since $|\mu_2| \sim |\mu_1|$. We next show that $T^4_{ij}(\lambda_1,\mu_1,\mu_2)$ is also bounded by (3.13), so that

$T^4_{ij}(\lambda_1,\mu_1,\mu_2)$ is also bounded by a constant times $|\mu_2|^{-\frac{1}{2}}\|\phi\|_{C^1(V)}$. Taking absolute values in (3.11b) and integrating, we get that $|T^4_{ij}(\lambda_1,\mu_1,\mu_2)|$ is at most

$$\|\phi\|_{C^1(V)} \times |\{(x,y)\in D_{ij} : |\mu_2| \leq 2|\lambda_1\partial_y S(x,y)|,\ |\lambda_1\partial_y S(x,y)+\mu_2| < |\lambda_1|^{\frac{1}{2}}2^{-j\gamma}\}| \tag{3.15}$$

We now shift y by $\psi_i(x,y)$, so that where η_i is in Theorem 2.1 we have that $|T^4_{ij}(\lambda_1,\mu_1,\mu_2)|$ is at most $\|\phi\|_{C^1(V)}$ times

$$|\{(x,y)\in D'_{ij} : |\mu_2| \leq 2|\lambda_1\partial_y(S\circ\eta_i)(x,y)|,\ |\lambda_1\partial_y(S\circ\eta_i)(x,y)+\mu_2| < |\lambda_1|^{\frac{1}{2}}2^{-j\gamma}\}| \tag{3.16}$$

Here D'_{ij} is the shift of D_{ij} by $\psi_i(x)$ in the y variable. The condition that $|\mu_2| \leq 2|\lambda_1\partial_y(S\circ\eta_i)(x,y)|$ is used only to go from (3.13) to (3.14), and we use only the $|\lambda_1\partial_y(S\circ\eta_i)(x,y)+\mu_2| < |\lambda_1|^{\frac{1}{2}}2^{-j\gamma}$ condition in proving (3.13). So as to be able to use Lemma 2.2, we change variables from y to $x^{M_i}y$ in (3.16) and get a term bounded by $\|\phi\|_{C^1(V)}$ times

$$2^{-jM_i}|\{(x,y)\in[2^{-j-1},2^{-j}]\times[0,H_i] : |\lambda_1\partial_y(S\circ\eta_i)(x,x^{M_i}y)+\mu_2| < |\lambda_1|^{\frac{1}{2}}2^{-j\gamma}\}| \tag{3.17}$$

Our use of $[2^{-j-1},2^{-j}]\times[0,H_i]$ here follows from parts a) and b) of Theorem 2.1. By Lemma 2.2, we have that

$$\begin{aligned}|\partial_x\big(\lambda_1\partial_y(S\circ\eta_i)(x,x^{M_i}y)+\mu_2)\big)| &> C|\lambda_1|x^{\alpha_i-1}\\ &> C'|\lambda_1|2^{-j\alpha_i+j}\end{aligned} \tag{3.18}$$

Thus for a fixed y, the x-measure of the set in (3.17) is at most $C|\lambda_1|^{-\frac{1}{2}}2^{-j\gamma+j\alpha_i-j}$. Thus $\|\phi\|_{C^1(V)}$ times the quantity in (3.17) is bounded by

$$C|\lambda_1|^{-\frac{1}{2}}2^{-j\gamma+j\alpha_i-j-jM_i}\|\phi\|_{C^1(V)} \tag{3.19}$$

Substituting back in for γ, this becomes

$$C|\lambda_1|^{-\frac{1}{2}}2^{-j\frac{M_i}{2}+j\frac{\alpha_i}{2}-j}\|\phi\|_{C^1(V)} \tag{3.20}$$

This is exactly (3.13). The condition that $|\mu_2| \leq 2|\lambda_1\partial_y S(x,y)|$ is now used exactly as it was when going from (3.13) to (3.14). This again leads to the bound (3.14) for $|T^4_{ij}(\lambda_1,\mu_1,\mu_2)|$, and after summing this in j we are done.

4. Proof of Theorem 1.2

In the proof of Theorem 1.2 we will make use of sublevel set estimates that are analogous to the oscillatory integral estimates we have been using. Specifically, if $f(x,y)$ is real analytic on a neighborhood of the origin such that $f(0,0)=0$ and $\nabla f(0,0)=0$, for a given U contained in the domain of $f(x,y)$ and an $0<r<\frac{1}{2}$ we define

$$A_U(r) = |\{(x,y)\in U : |f(x,y)| < r\}| \tag{4.1}$$

Using resolution of singularities (see [1] Ch. 6 for details), in the real-analytic case if U is a sufficiently small ball centered at the origin then as $r \to 0$ one has an asymptotic expansion of the form

$$A_U(r) = C_U r^\epsilon |\ln(r)|^m + o(r^\epsilon |\ln(r)|^m) \tag{4.2}$$

Here $C_U > 0$ and (ϵ, m) is the same as in (1.2), unless $(\epsilon, m) = (1, 0)$, in which case (ϵ, m) could be $(1, 0)$ or $(1, 1)$. In [8] it is shown that in the general smooth case, an analogue of (4.2) holds. Namely, there is a C_U such that $A_U(r) \leq C_U r^\epsilon |\ln(r)|^m$, and often (4.2) still holds. In [8] it is shown that in the cases where (4.2) does not hold, m is always 0 and for all $\epsilon' > \epsilon$ there is a constant $C_{U,\epsilon'} > 0$ such that $A_U(r) \geq C_{U,\epsilon'} r^{\epsilon'}$. This extension to the smooth case does use the notion of adapted coordinate systems, and is the only way in which this paper relies on them. However, one can avoid relying on the use of adapted coordinate systems entirely by doing arguments very similar to those of [8] on the constructions of Theorem 2.1.

The above discussion leads to the following lemma.

Lemma 4.1. Let (ϵ, m) be as above, and let $\{D_i'\}_{i=1}^n$ be the domains obtained by applying Theorem 2.1 to $f(x, y)$, and let (α_i, β_i) be as in that theorem. Then there exists a constant C such that for each i and all $0 < r < \frac{1}{2}$ we have

$$|\{(x, y) \in D_i' : x^{\alpha_i} y^{\beta_i} < r\}| \leq C r^\epsilon |\ln(r)|^m \tag{4.3}$$

Proof. In the case that $g_i(x)$ is not identically zero in Theorem 2.1, some sliver $\{(x, y) : 0 < x < b : 0 < y < C x^{M_i}\}$ is disjoint from D_i', so $x^{\alpha_i} y^{\beta_i}$ is bounded below by $C' x^{\alpha_i + M_i \beta_i}$ on D_i'. In part c) of Theorem 2.1, since f_2 has a zero of infinite order at the origin, for any N one has an estimate of the form $|f_2(x, y)| < C_N x^N$. Thus in (2.2) one can replace f_1 by f, which implies $f \circ \eta_i$ is within a constant factor of $x^{\alpha_i} y^{\beta_i}$ on D_i. Since the Jacobian of η_i is everywhere equal to 1, the measure of the sublevel sets of $|f \circ \eta_i|$ will be no greater than the measure of the corresponding sublevel sets of $|f|$. Thus (4.3) holds.

Now consider the case where $g_i(x)$ is identically zero. Define $D_i^N = \{(x, y) \in D_i' : y > x^N\}$. Then exactly as above one has that $|\{(x, y) \in D_i^N : x^{\alpha_i} y^{\beta_i} < r\}| \leq C_N r^\epsilon |\ln(r)|^m$. In the case that $\alpha_i \geq \beta_i$ this is enough; a direct calculation reveals that for large enough N, $|\{(x, y) \in D_i^N : x^{\alpha_i} y^{\beta_i} < r\}|$ is within a constant factor of $|\{(x, y) \in D_i' : x^{\alpha_i} y^{\beta_i} < r\}|$. In the case where $\alpha_i < \beta_i$, a direct calculation reveals that $|\{(x, y) \in D_i^N : x^{\alpha_i} y^{\beta_i} < r\}|$ is of the form $C_N r^{\delta_N} +$ lower order terms, and that $|\{(x, y) \in D_i' : x^{\alpha_i} y^{\beta_i} < r\}|$ is of the form $C r^\delta +$ lower order terms where $\lim_{N \to \infty} \delta_N = \delta$. So like in the previous paragraph, $\delta_N \geq \epsilon$ for each N, and taking limits as $N \to \infty$ we get that $\delta \geq \epsilon$. So regardless of whether or not $m = 0$ or 1, (4.3) will hold and we are done.

We now are in a position to prove Theorem 1.2.

Proof of Theorem 1.2.

Let $k > 0$ be the order of the zero of $S(x,y)$ at the origin. Rotating coordinates if necessary, we assume that the Taylor expansion $\sum_{\alpha\beta} S_{\alpha\beta}x^\alpha y^\beta$ of $S(x,y)$ has nonvanishing $S_{0k}y^k$ and $S_{k0}x^k$ terms. We perform the reflections at the beginning of section 2 and then apply Theorem 2.1 to (the reflected) $S(x,y)$. Note this is a different function from the previous section. Let $\{D_i\}_{i=1}^n$ be all of the resulting regions. We will bound the portion of $T(\lambda_1,\lambda_2,\lambda_3) = T(\lambda_1,\mu_1,\mu_2)$ coming from a given D_i and sum over all i. We will slightly abuse notation in the following and refer to a reflected $S(x,y)$ as just $S(x,y)$. The argument naturally breaks into three cases. The first is when β_i in Theorem 2.1 is greater than 1 and the lower boundary of D_i' is the x-axis (in other words, $g_i(x)$ is identically zero). The second case is when either $\beta_i = 0$ or $\beta_i > 1$ and the lower boundary of D_i' is not the x-axis. The third case is when β_i is zero.

Case 1. $\beta_i > 1$ and $g_i(x)$ is identically zero.

Consider (2.2) when $l = 0$ and $m = \beta_i$. Because in (2.2) the function f_2 has a zero of infinite order at the origin, on D_i' one has $|\partial_y^{\beta_i}(f_2 \circ \eta_i)(x,y)| < C_N x^N$ for any N. Thus we may replace f_1 by f (which is S here) to obtain that for some constant C we have

$$|\partial_y^{\beta_i}(S \circ \eta_i)(x,y)| > Cx^{\alpha_i} \tag{4.4}$$

Denote by $T_i(\lambda_1,\mu_1,\mu_2)$ portion of the integral (1.1) coming from D_i. In this integral, we do the coordinate change $\eta_i(x,y)$ given by Theorem 2.1, obtaining

$$T_i(\lambda_1,\mu_1,\mu_2) = \int_{D_i'} e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}\phi_i(x,y)\,dx\,dy \tag{4.5}$$

The $i\mu_2 y$ term might have a minus sign in front, but since that case is done exactly the same way we will assume $T_i(\lambda_1,\mu_1,\mu_2)$ is of the form (4.5). Here $\phi_i(x,y)$ is a compactly supported function such that $\phi_i(x^N,y)$ is smooth for some N. We now dyadically decompose $T_i = \cup_j T_{ij}$. Denoting by D_{ij} the set $\{(x,y) \in D_i' : 2^{-j-1} \leq x < 2^{-j}\}$, we define

$$T_{ij}(\lambda_1,\mu_1,\mu_2) = \int_{D_{ij}} e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}\phi_i(x,y)\,dx\,dy \tag{4.6}$$

We now apply the standard Van der Corput lemma (see [17] ch 8) in (4.6) in the y direction, using (4.4), and then integrate the result in x. One gets

$$|T_{ij}(\lambda_1,\mu_1,\mu_2)| \leq C\|\phi\|_{C^1(V)}|\lambda_1|^{-\frac{1}{\beta_i}}2^{j\frac{\alpha_i}{\beta_i}} \times 2^{-j} \tag{4.7}$$

In (4.6) we can get a crude estimate by taking absolute values and integrating, obtaining a constant times $\|\phi\|_{C^1(V)}2^{-j-jM_i}$, M_i as in Theorem 2.1a). Thus one can extend (4.7) to

$$|T_{ij}(\lambda_1,\mu_1,\mu_2)| \leq C\|\phi\|_{C^1(V)}\min(2^{-j-jM_i}, |\lambda_1|^{-\frac{1}{\beta_i}}2^{j\frac{\alpha_i}{\beta_i}} \times 2^{-j}) \tag{4.8}$$

An elementary calculation reveals that the measure of $\{(x,y) \in D_{ij} : x^{\alpha_i}y^{\beta_i} < \frac{1}{|\lambda_1|}\}$ is within a constant factor of $\min(2^{-j-jM_i}, |\lambda_1|^{-\frac{1}{\beta_i}}2^{j\frac{\alpha_i}{\beta_i}} \times 2^{-j})$. Thus we have

$$|T_{ij}(\lambda_1,\mu_1,\mu_2)| \leq C'\|\phi\|_{C^1(V)}\big|\{(x,y) \in D_{ij} : x^{\alpha_i}y^{\beta_i} < |\lambda_1|^{-1}\}\big| \tag{4.9}$$

Adding over all j then gives

$$|T_i(\lambda_1,\mu_1,\mu_2)| \leq C'\|\phi\|_{C^1(V)}\big|\{(x,y) \in D_i' : x^{\alpha_i}y^{\beta_i} < |\lambda_1|^{-1}\}\big| \tag{4.10}$$

By Lemma 4.1, the right-hand side of (4.10) is bounded by a constant times $\|\phi\|_{C^1(V)}|\lambda_1|^{-\epsilon}(\ln|\lambda_1|)^m$ and we are done. Note that the Case 1 argument did not use any restrictions on the value of ϵ.

Case 2. $\beta_i = 1$ or $\beta_i > 1$ and $g_i(x)$ is not identically zero.

Similar to in the previous case, when $l = m = 1$ we can replace f_1 by f in (2.2). When $\beta_i = 1$ this is because $x^{\alpha_i-1}y^{\beta_i-1}$ is a power of x, so since $|\frac{\partial^2}{\partial x\partial y}(f_2 \circ \eta_i)(x,y)| < C_N x^N$ for any given N, changing from f_1 to f (which is S here) will not interfere with the validity of (2.2). When $\beta_i > 1$ and $g_i(x)$ is not identically zero, we may do this replacement since a sliver $\{(x,y) : 0 < x < b : 0 < y < Cx^{M_i}\}$ is disjoint from the domain, so that $x^{\alpha_i-1}y^{\beta_i-1}$ is bounded below by $C'x^{\alpha_i-1-M_i\beta_i-M_i}$ and similar considerations apply. Thus we may replace f_1 by $f = S$ in (2.2), and on D_i' we have a lower bound of the form

$$\left|\frac{\partial^2}{\partial x\partial y}(S \circ \eta_i)(x,y)\right| > Cx^{\alpha_i-1}y^{\beta_i-1} \tag{4.11}$$

Note that $\frac{\partial^2}{\partial x\partial y}(i\lambda_1(S\circ\eta_i)(x,y) + i\mu_1 x + i\mu_2 y + i\mu_2\psi_i(x)) = i\lambda_1\frac{\partial^2}{\partial x\partial y}(S\circ\eta_i)(x,y)$. Thus (4.11) is relevant to $T_{ijk}(\lambda_1,\mu_1,\mu_2)$.

This time we dyadically decompose (4.5) in both the x and y directions. Specifically, let $D_{ijk} = \{(x,y) \in D_i' : 2^{-j-1} \leq x < 2^{-j},\ 2^{-k-1} \leq y < 2^{-k}\}$ and define $T_{ijk}(\lambda_1,\mu_1,\mu_2)$ by

$$T_{ijk}(\lambda_1,\mu_1,\mu_2) = \int_{D_{ijk}} e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}\phi_i(x,y)\,dx\,dy \tag{4.12}$$

We now proceed similarly to in Case 2 of Theorem 1.1. Write $T_{ijk}(\lambda_1,\mu_1,\mu_2) = T_{ijk}^1(\lambda_1,\mu_1,\mu_2) + T_{ijk}^2(\lambda_1,\mu_1,\mu_2)$, where

$$T_{ijk}^1(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_{ijk}:|\lambda_1\partial_y S+\mu_2|<|\lambda_1|^{\frac{1}{2}}2^{k-\frac{j\alpha_i+k\beta_i}{2}}\}}\phi_i(x,y)$$

$$\times e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}\,dx\,dy \tag{4.13a}$$

$$T_{ijk}^2(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y)\in D_{ijk}:|\lambda_1\partial_y S+\mu_2|\geq|\lambda_1|^{\frac{1}{2}}2^{k-\frac{j\alpha_i+k\beta_i}{2}}\}}\phi_i(x,y)$$

$$\times e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}\,dx\,dy \tag{4.13b}$$

For (4.13a), we simply take absolute values and integrate, obtaining

$$|T^1_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C|\{(x,y)\in D_{ijk} : |\lambda_1\partial_y S+\mu_2| < |\lambda_1|^{\frac{1}{2}}2^{k-\frac{j\alpha_i+k\beta_i}{2}}\}|\ \|\phi\|_{C^1(V)} \tag{4.14}$$

By (4.11), the absolute value of the x derivative of $\lambda_1\partial_y S+\mu_2$ is bounded below by $C|\lambda_1|2^{-j(\alpha_i-1)-k(\beta_i-1)}$. So for a given y, the x-measure of the set in (4.14) is at most $C|\lambda_1|^{-\frac{1}{2}}2^{k-\frac{j\alpha_i+k\beta_i}{2}}\times 2^{j(\alpha_i-1)+k(\beta_i-1)} = C|\lambda_1|^{-\frac{1}{2}}2^{-j+\frac{j\alpha_i+k\beta_i}{2}}$. Inserting this into (4.14) and then integrating in y, we get that

$$|T^1_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C|\lambda_1|^{-\frac{1}{2}}2^{-j-k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)} \tag{4.15}$$

This is the estimate we will need. Moving on to $T^2_{ijk}(\lambda_1,\mu_1,\mu_2)$, we integrate by parts in y in (4.13b). We write $e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}$ as

$$(i\lambda_1\partial_y(S\circ\eta_i)(x,y)+i\mu_2)e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1 x+i\mu_2 y+i\mu_2\psi_i(x)}$$

$$\times\frac{1}{i\lambda_1\partial_y(S\circ\eta_i)(x,y)+i\mu_2} \tag{4.16}$$

We integrate by parts in (4.13b) by integrating the left factor of (4.16) and differentiating the rest. There are two places the derivative may land, namely on the $\frac{1}{i\lambda_1\partial_y(S\circ\eta_i)(x,y)+i\mu_2}$ factor or on the $\phi_i(x,y)$ factor. In the first case, the differentiation gives $\frac{-\lambda_1\partial_{yy}(S\circ\eta_i)(x,y)}{i(\partial_y(S\circ\eta_i)(x,y)+\mu_2)^2}$. We then take absolute values and integrate in y, very similar to in (3.5). As in that situation, the end result of the y integration is a bound of $C\|\phi\|_{C^1(V)}$ times the maximum of $|\frac{1}{i\lambda_1\partial_y(S\circ\eta_i)(x,y)+i\mu_2}|$ on the domain of integration, or $C|\lambda_1|^{-\frac{1}{2}}2^{-k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)}$. We then do the x integration to get an overall factor of $C|\lambda_1|^{-\frac{1}{2}}2^{-j-k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)}$, which is the same as in (4.15).

The second place the derivative may land is the $\phi_i(x,y)$ term. In this case we take absolute values, bound $|\frac{1}{i\lambda_1\partial_y S(x,y)+i\mu_2}|$ by $|\lambda_1|^{-\frac{1}{2}}2^{-k+\frac{j\alpha_i+k\beta_i}{2}}$, and integrate. The result is $C|\lambda_1|^{-\frac{1}{2}}2^{-j-2k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)}$, better than what we need. Lastly, we have the endpoint terms of the integration by parts, which will give the same bounds as in the last paragraph, or $C|\lambda_1|^{-\frac{1}{2}}2^{-j-k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)}$.

Putting the above together, we conclude that

$$|T_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C|\lambda_1|^{-\frac{1}{2}}2^{-j-k+\frac{j\alpha_i+k\beta_i}{2}}\|\phi\|_{C^1(V)} \tag{4.17}$$

We can rewrite this as

$$|T_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C'\|\phi\|_{C^1(V)}\int_{D_{ijk}}\frac{1}{(|\lambda_1|x^{\alpha_i}y^{\beta_i})^{\frac{1}{2}}} \tag{4.18}$$

One can take absolute values in (4.12) and integrate to get another (crude) bound for $|T_{ijk}(\lambda_1,\mu_1,\mu_2)|$. Incorporating this into (4.18) gives

$$|T_{ijk}(\lambda_1,\mu_1,\mu_2)| \leq C'\|\phi\|_{C^1(V)}\int_{D_{ijk}} \min\left(1, \frac{1}{(|\lambda_1|x^{\alpha_i}y^{\beta_i})^{\frac{1}{2}}}\right) \tag{4.19}$$

Adding this over all j and k then gives

$$|T_i(\lambda_1,\mu_1,\mu_2)| \leq C'\|\phi\|_{C^1(V)}\int_{D_i'} \min\left(1, \frac{1}{(|\lambda_1|x^{\alpha_i}y^{\beta_i})^{\frac{1}{2}}}\right) \tag{4.20}$$

$$= C'\|\phi\|_{C^1(V)}\Bigg(\Big|\{(x,y)\in D_i' : x^{\alpha_i}y^{\beta_i} < \frac{1}{|\lambda_1|}\}\Big|$$

$$+\frac{1}{|\lambda_1|^{\frac{1}{2}}}\int_{\{(x,y)\in D_i':\, x^{\alpha_i}y^{\beta_i}\geq \frac{1}{|\lambda_1|}\}} \frac{1}{(x^{\alpha_i}y^{\beta_i})^{\frac{1}{2}}}\Bigg) \tag{4.21}$$

By Lemma 4.1, the measure of $\{(x,y)\in D_i' : x^{\alpha_i}y^{\beta_i} < \frac{1}{|\lambda_1|}\}$ is bounded by $C|\lambda_1|^{-\epsilon}\ln|\lambda_1|$, the desired estimate. As for the second term, we write the integral in terms of distribution functions. Namely, we have

$$\int_{\{(x,y)\in D_i':x^{\alpha_i}y^{\beta_i}\geq \frac{1}{|\lambda_1|}\}} \frac{1}{(x^{\alpha_i}y^{\beta_i})^{\frac{1}{2}}} = \int_{\frac{1}{|\lambda_1|}}^{\infty} \frac{t^{-\frac{3}{2}}}{2}|\{(x,y)\in D_i' : \frac{1}{|\lambda_1|}\leq x^{\alpha_i}y^{\beta_i}\leq t\}|\,dt \tag{4.22}$$

By Lemma 4.1, this is at most

$$C\int_{\frac{1}{|\lambda_1|}}^{1} \frac{1}{2}t^{-\frac{3}{2}}t^{\epsilon}\ln(t)^m\,dt \tag{4.23}$$

Since ϵ is being assumed to be at most $\frac{1}{3}$ here, the integral converges and is bounded by $C|\lambda_1|^{\frac{1}{2}-\epsilon}(\ln|\lambda_1|)^m$. Thus the second term of (4.21) is bounded by

$$C\|\phi\|_{C^1(V)}|\lambda_1|^{-\frac{1}{2}}|\lambda_1|^{\frac{1}{2}-\epsilon}(\ln|\lambda_1|)^m$$

$$= C\|\phi\|_{C^1(V)}|\lambda_1|^{-\epsilon}(\ln|\lambda_1|)^m \tag{4.24}$$

This is the desired estimate and we are done. Note that the Case 2 argument only required that $\epsilon < \frac{1}{2}$, which was used to say that (4.23) converges.

Case 3. $\beta_i = 0$.

For this case, it will be helpful to use the following consequence of the Van der Corput lemma from [2] which was also used in [11][13].

Lemma 4.2. Suppose f is a smooth real-valued function on an interval I such that for some integer $n\geq 2$ and some constants $C, C' > 0$, for all $t\in I$ one has that $C' \leq \sum_{i=2}^n |f^{(i)}(t)| \leq C$. Then there is a constant C'' depending only on C and C' such that for all $\lambda \in \mathbb{R}$ one has

$$\left|\int_I e^{i\lambda f(t)}\phi(t)\,dt\right| \leq C''(\|\phi\|_{L^\infty(I)} + \|\phi'\|_{L^1(I)})(1+|\lambda|)^{-\frac{1}{n}} \tag{4.25}$$

We now start the Case 3 argument. By theorem 2.1a)-b), when $\beta_i = 0$ the domain D_i' is of the form $\{(x,y) : 0 < x < b,\ 0 < y < H_i x^{M_i}\}$. Thus we have

$$T_i(\lambda_1,\mu_1,\mu_2) = \int_{\{(x,y):0<x<b,\ 0<y<H_ix^{M_i}\}} e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_1x+i\mu_2y+i\mu_2\psi_i(x)}$$
$$\times\phi_i(x,y)\,dx\,dy \tag{4.26}$$

By part d) of Theorem 2.1, either $\psi_i(x) = k_ix$ for some k_i (possibly zero), or $\psi_i(x) = k_ix + l_ix^{s_i}+$ higher order terms (if any), where $l_i \neq 0$ and $1 < s_i \leq M_i$. Thus if we write $\xi(x) = \psi_i(x) - k_ix$, then either $\xi(x) = 0$ or $\xi(x)$ as a zero of order $s_i \leq M_i$ at the origin. Letting $\mu_3 = \mu_1 + k_i\mu_2$, we correspondingly write the expression (4.26) for $T_i(\lambda_1,\mu_1,\mu_2)$ as

$$\int_{\{(x,y):0<x<b,\ 0<y<H_ix^{M_i}\}} e^{i\lambda_1(S\circ\eta_i)(x,y)+i\mu_3x+i\mu_2(\xi(x)+y)}\phi_i(x,y)\,dx\,dy \tag{4.27}$$

By Theorem 2.1b) there is some $(\alpha,\beta) \neq (\alpha_i,0)$, β a positive integer, for which the Taylor expansion of $S\circ\eta_i(x,y)$ has a nonzero $S_{\alpha\beta}x^\alpha y^\beta$ term and for which $\alpha + M_i\beta = \alpha_i$. So $\alpha_i \geq M_i$. Since $S(x,y)$ is assumed to have a zero of order at least 2 at the origin, (α,β) cannot be $(0,1)$ and the statement $\alpha + M_i\beta = \alpha_i$ in fact implies that $\alpha_i > M_i$.

We first prove Theorem 1.2 under the assumption that $\xi(x)$ is not identically zero and $\epsilon < \frac{1}{3}$; the modifications needed in the cases where $\epsilon = \frac{1}{3}$ or where $\xi(x)$ is identically zero will be described afterwards.

So assuming $\xi(x)$ is not identically zero, for fixed y the phase in (4.27) can be written as the sum of three terms. The first is $\lambda_1(S\circ\eta_i)(x,y)$, which is of the form $\lambda_1d_ix^{\alpha_i}$ plus a small error term by Theorem 2.1b), with corresponding expressions for its x derivatives. The second term is $\mu_2(\xi(x)+y)$, where $\mu_2\xi(x)$ is of the form $\mu_2\big(l_ix^{s_i}+O(x^{s_i+\delta})\big)$, and the third is μ_3x. It is the first two terms that concern us here. Note that $\alpha_i > M_i \geq s_i > 1$, so that the exponents α_i and s_i are distinct. As a result, the 2 by 2 matrix A_i with rows $(\alpha_i(\alpha_i-1), s_i(s_i-1))$ and $(\alpha_i(\alpha_i-1)(\alpha_i-2), s_i(s_i-1)(s_i-2))$ has determinant $\alpha_i(\alpha_i-1)s_i(s_i-1)(\alpha_i-s_i) \neq 0$. Thus for some constants c and c', for any vector v one has $c'\|v\| \geq \|A_iv\| \geq c\|v\|$. In particular, letting $v = (\lambda_1d_ix^{\alpha_i}, \mu_2l_ix^{s_i})$, we have

$$|\lambda_1\alpha_i(\alpha_i-1)d_ix^{\alpha_i} + s_i(s_i-1)\mu_2l_ix^{s_i}|$$
$$+|\lambda_1\alpha_i(\alpha_i-1)(\alpha_i-2)d_ix^{\alpha_i} + s_i(s_i-1)(s_i-2)\mu_2l_ix^{s_i}|$$
$$\geq c|(\lambda_1d_ix^{\alpha_i}, \mu_2l_ix^{s_i})| \tag{4.28}$$

Restating, (4.28) implies that for $f(x) = d_i x^{\alpha_i} + \mu_2 l_i x^{s_i} + \mu_3 x$ we have

$$|x^2 f''(x)| + |x^3 f'''(x)| \geq c'(|\lambda_1 d_i x^{\alpha_i}| + |\mu_2 l_i x^{s_i}|) \tag{4.29}$$

Adjusting for error terms, if x is sufficiently small and the δ coming from Theorem 2.1b) is sufficiently small (recall it can be chosen independent of the s_i and α_i), then independent of the parameters λ_1, μ_2, and μ_3, if $p_y(x)$ denotes the phase function $\lambda_1(S \circ \eta_i)(x, y) + \mu_3 x + \mu_2(\xi(x) + y)$ we similarly have

$$|x^2 p_y''(x)| + |x^3 p_y'''(x)| \geq c''(|\lambda_1 d_i x^{\alpha_i}| + |\mu_2 l_i x^{s_i}|) \tag{4.30}$$

Next we dyadically decompose (4.27), writing $T_i = \sum_j T_{ij}$, where T_{ij} is defined by

$$T_{ij}(\lambda_1, \mu_1, \mu_2) = \int_{\{(x,y):2^{-j-1}<x<2^{-j},\ 0<y<H_i x^{M_i}\}} e^{ip_y(x)} \phi_i(x, y)\, dx\, dy \tag{4.31}$$

We scale (4.31) in x, obtaining

$$T_{ij}(\lambda_1, \mu_1, \mu_2) = 2^{-j} \int_{\{(x,y):\frac{1}{2}<x<1,\ 0<y<H_i x^{M_i}\}} e^{ip_y(2^{-j}x)} \phi_i(2^{-j}x, y)\, dx\, dy \tag{4.32}$$

By (4.30), the phase function $q(x) = p_y(2^{-j}x)$ satisfies

$$|q''(x)| + |q'''(x)| \geq c(|\lambda_1 2^{-j\alpha_i}| + |\mu_2 2^{-js_i}|) \tag{4.33}$$

We now apply the Van der Corput-type lemma, Lemma 4.2, in the x direction letting $f(t) = (|\lambda_1 2^{-j\alpha_i}| + |\mu_2 2^{-js_i}|)q(x)$, and letting $|\lambda_1 2^{-j\alpha_i}| + |\mu_2 2^{-js_i}|$ be what is called λ in that lemma. The cutoff function $\phi_i(2^{-j}x, y)$ is equal to $\phi(2^{-j}x, y + \psi_i(2^{-j}x))$ where $\psi_i(x)$ is of the form $\zeta(x^{\frac{1}{N}})$ for a smooth ζ, for some large N. Thus the effect of this cutoff function in an application of Lemma 4.2 in the x direction is to an introduce a factor bounded by (something slightly better than) $C\|\phi\|_{C^1(V)}$. After applying Lemma 4.2 in the x direction with $n = 3$ and then integrating in y we get

$$|T_{ij}(\lambda_1, \mu_1, \mu_2)| \leq C 2^{-j(M_i+1)} \|\phi\|_{C^1(V)} (|\lambda_1 2^{-j\alpha_i}| + |\mu_2 2^{-js_i}|)^{-\frac{1}{3}} \tag{4.34}$$

We are only interested in the first of the two terms in the right hand side of (4.34), so we use the bound

$$|T_{ij}(\lambda_1, \mu_1, \mu_2)| \leq C\|\phi\|_{C^1(V)} 2^{-j(M_i+1)} |\lambda_1 2^{-j\alpha_i}|^{-\frac{1}{3}} \tag{4.35}$$

By simply taking absolute values and integrating in (4.31), one has

$$|T_{ij}(\lambda_1, \mu_1, \mu_2)| \leq C 2^{-j(M_i+1)} \|\phi\|_{C^1(V)}$$

Combining this with (4.35), we get that

$$|T_{ij}(\lambda_1, \mu_1, \mu_2)| \leq C 2^{-j(M_i+1)} \|\phi\|_{C^1(V)} \min(1, |\lambda_1 2^{-j\alpha_i}|^{-\frac{1}{3}}) \tag{4.36}$$

By Lemma 2.2, $|S\circ\eta(x,y)|\sim x^{\alpha_i}\sim 2^{-j\alpha_i}$ on D_i', and furthermore the portion of D_i' between 2^{-j-1} and 2^{-j} has measure $\sim 2^{-j}\times 2^{-jM_i}$. So (4.36) implies that

$$|T_{ij}(\lambda_1,\mu_1,\mu_2)|\leq C'\|\phi\|_{C^1(V)}\int_{\{(x,y)\in D_i':\, x\in[2^{-j-1},2^{-j}]\}}\min(1,|\lambda_1 x^{\alpha_i}|^{-\frac{1}{3}}) \tag{4.37}$$

We now argue as in (4.19) – (4.24). Adding (4.37) over all j gives

$$|T_i(\lambda_1,\mu_1,\mu_2)|\leq C'\|\phi\|_{C^1(V)}\int_{D_i'}\min(1,|\lambda_1 x^{\alpha_i}|^{-\frac{1}{3}}) \tag{4.38}$$

$$= C'\|\phi\|_{C^1(V)}\bigg(\Big|\{(x,y)\in D_i': x^{\alpha_i}<\frac{1}{|\lambda_1|}\}\Big| + \frac{1}{|\lambda_1|^{\frac{1}{3}}}\int_{\{(x,y)\in D_i': x^{\alpha_i}\geq\frac{1}{|\lambda_1|}\}}\frac{1}{(x^{\alpha_i})^{\frac{1}{3}}}\bigg) \tag{4.39}$$

For the first term in (4.39), by Lemma 4.1 the measure of $\{(x,y)\in D_i': x^{\alpha_i}<\frac{1}{|\lambda_1|}\}$ is bounded by $C|\lambda_1|^{-\epsilon}\ln|\lambda_1|$, the desired estimate. In view of the form of this sublevel set, we won't have a logarithmic factor so we even have that

$$|\{(x,y)\in D_i': x^{\alpha_i}<\frac{1}{|\lambda_1|}\}|\leq C|\lambda_1|^{-\epsilon} \tag{4.40}$$

For the second term of (4.39), like before we write the integral in terms of distribution functions. We get

$$\int_{\{(x,y)\in D_i': x^{\alpha_i}\geq\frac{1}{|\lambda_1|}\}}\frac{1}{(x^{\alpha_i})^{\frac{1}{3}}}=\int_{\frac{1}{|\lambda_1|}}^{\infty}\frac{1}{3}t^{-\frac{4}{3}}|\{(x,y)\in D_i':\frac{1}{|\lambda_1|}\leq x^{\alpha_i}\leq t\}|\,dt$$

Inserting (4.40), this is at most

$$C\int_{\frac{1}{|\lambda_1|}}^{\infty}\frac{1}{3}t^{-\frac{4}{3}}t^{\epsilon}\,dt \tag{4.41}$$

Since we are assuming $\epsilon<\frac{1}{3}$ for now, the integral (4.41) is absolutely integrable and is bounded by $C'|\lambda_1|^{-\epsilon+\frac{1}{3}}$. Thus the second term in the parentheses of (4.39) is bounded by $C'|\lambda_1|^{-\epsilon}$. Adding together with the first term we see that we have the estimate

$$|T_i(\lambda_1,\mu_1,\mu_2)|\leq C\|\phi\|_{C^1(V)}|\lambda_1|^{-\epsilon} \tag{4.42}$$

This gives the desired bounds (and even an extra logarithm when $m=1$). This concludes the proof where $\epsilon<\frac{1}{3}$ and $\xi(x)$ does not have a zero of infinite order at $x=0$.

We now assume $\xi(x)$ is identically zero, for any $\epsilon\leq\frac{1}{3}$. Then for fixed y the phase in (4.27) is the sum $\lambda_1(S\circ\eta_i(x,y))$ and a linear function of x. Thus by (2.1) we can just use Lemma 4.2 for second derivatives here, since the second x-derivative of a linear function is zero. So if $p_y(x)$ again is the phase function of (4.27), by (2.1) we have

$$|x^2 p_y''(x)|\geq c|\lambda_1 d_i x^{\alpha_i}| \tag{4.43}$$

Then one can apply Lemma 4.2 for $n = 2$ instead of $n = 3$, and in place of (4.39) we get a bound for $|T_i(\lambda_1, \mu_1, \mu_2)|$ of the form

$$C'\|\phi\|_{C^1(V)}\left(\left|\{(x,y) \in D_i' : x^{\alpha_i} < \frac{1}{|\lambda_1|}\}\right| + \frac{1}{|\lambda_1|^{\frac{1}{2}}}\int_{\{(x,y)\in D_i' :\, x^{\alpha_i} \geq \frac{1}{|\lambda_1|}\}} \frac{1}{(x^{\alpha_i})^{\frac{1}{2}}}\right) \tag{4.44}$$

Then performing the argument analogous to before once again gives (4.42), this time only using that $\epsilon < \frac{1}{2}$.

Lastly, we consider the case where $\epsilon = \frac{1}{3}$ and $\xi(x)$ does not have a zero of infinite order at $x = 0$. If we had $|x^2 p_y''(x)| \geq c|x^{\alpha_i}|$, then we could proceed as in the case where $\xi(x)$ is identically zero since the argument required only that $\epsilon < \frac{1}{2}$. However this does not necessarily hold; this is because there can be an x for which $\lambda_1\alpha_i(\alpha_i - 1)d_i x^{\alpha_i} + \mu_2 s_i(s_i - 1)\mu_2 l_i x^{s_i} = 0$ and then the two main terms of $x^2 p_y''(x)$ cancel. However, since $s_i \neq \alpha_i$ there will always be an integer j_0 such that so long as x is not in an interval $[2^{-j_0-1}, 2^{-j_0+1}]$ then one does have $|x^2 p_y''(x)| \geq c|x^{\alpha_i}|$. So we may apply the argument of the $\xi(x) = 0$ case for T_{ij} with $j \neq j_0$ or $j_0 - 1$. Adding these in j gives the bounds $C\|\phi\|_{C^1(V)}|\lambda_1|^{-\epsilon}$ of (4.42).

For $j = j_0$ or $j_0 - 1$, we apply the argument leading to (4.37) unchanged. Then the steps leading to (4.41) lead to the bound

$$|T_{ij}(\lambda_1, \mu_1, \mu_2)| \leq C\|\phi\|_{C^1(V)}\frac{1}{|\lambda_1|^{\frac{1}{3}}}\int_{\{t : t > \frac{1}{|\lambda_1|},\ 2^{-j-1} < t < 2^{-j}\}} \frac{1}{3}t^{-\frac{4}{3}}t^{\epsilon}\,dt \tag{4.45}$$

Since $\epsilon = \frac{1}{3}$, (4.45) implies that we have

$$T_{ij}|(\lambda_1, \mu_1, \mu_2)| \leq C'\|\phi\|_{C^1(V)}\frac{1}{|\lambda_1|^{\frac{1}{3}}} \tag{4.46}$$

This is exactly the right-hand side of (4.42). Since there are only two such j, if we combine with the earlier estimate for the the sum of the T_{ij} for which $j \neq j_0$ or $j_0 - 1$, we see that (4.42) once again holds and we are done.

This completes the proof of Theorem 1.2.

5. Proofs of PDE Theorems

We start with the proof of Theorem 1.3.

Proof of Theorem 1.3.

Written on the Fourier transform side in the x variables, (1.6) becomes

$$\frac{\partial \hat{f}}{\partial t}(t, \xi_1, \xi_2) = iS(\xi_1, \xi_2)\hat{f}(t, \xi_1, \xi_2)$$

$$\hat{f}(0,\xi_1,\xi_2) = \hat{g}(\xi_1,\xi_2) \tag{5.1}$$

This is solved by $\hat{f}(t,\xi_1,\xi_2) = e^{itS(\xi_1,\xi_2)}\hat{g}(\xi_1,\xi_2)$. We are looking for solutions for when the support of $\hat{g}(\xi_1,\xi_2)$ is in a sufficiently small neighborhood of the origin. So we may fix a $\phi(\xi_1,\xi_2)$ supported in a neighborhood of the origin on which Theorems 1.1 and 1.2 hold such that $\phi(\xi_1,\xi_2) = 1$ on a neighborhood B of the origin, and we may assume that $\hat{g}(\xi_1,\xi_2)$ is supported on B. Thus we may write

$$\hat{f}(t,\xi_1,\xi_2) = \hat{g}(\xi_1,\xi_2)e^{itS(\xi_1,\xi_2)}\phi(\xi_1,\xi_2) \tag{5.2}$$

Thus if $T(t,x_1,x_2)$ is as in Theorems 1.1 and 1.2 we have

$$f(t,x_1,x_2) = (g * T)(t,x_1,x_2) \tag{5.3}$$

Here the convolution is in the x variables for fixed t. By Theorems 1.1 and 1.2, one has

$$|T(t,x_1,x_2)| \leq C\min\left((|t|+2)^{-\epsilon}(\ln(|t|+2))^m, |x|^{-\frac{1}{2}}\right) \tag{5.4}$$

(We can add 2 to $|t|$ because $|T(t,x_1,x_2)|$ uniformly bounded simply by taking absolute values of the integrand and then integrating.) In view of (5.4), for a given t it is natural to break up $T(t,x_1,x_2)$ into $(|t|+2)^{-\epsilon}(\ln(|t|+2))^m < |x|^{-\frac{1}{2}}$ and $(|t|+2)^{-\epsilon}(\ln(|t|+2))^m \geq |x|^{-\frac{1}{2}}$ pieces. To this end, for a given t let j_0 be the nearest nonnegative integer to the j for which $2^j = (|t|+2)^{2\epsilon}(\ln(|t|+2))^{-2m}$. As usual, let $\chi_{\{x:|x|<2^{j_0}\}}(x)$ denote the characteristic function of the ball centered at the origin of radius 2^{j_0} and let $\chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x)$ be the characteristic function of the annulus. Then we have

$$|T(t,x_1,x_2)| \leq C2^{-\frac{j_0}{2}}\chi_{\{x:|x|<2^{j_0}\}}(x_1,x_2) + C\sum_{j=j_0+1}^{\infty} 2^{-\frac{j}{2}}\chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x_1,x_2) \tag{5.5}$$

So by (5.3) we have

$$|f(t,x_1,x_2)| \leq C2^{-\frac{j_0}{2}}\left||g| * \chi_{\{x:|x|<2^{j_0}\}}(x_1,x_2)\right|$$
$$+C\sum_{j=j_0+1}^{\infty} 2^{-\frac{j}{2}}\left||g| * \chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x_1,x_2)\right| \tag{5.6}$$

And therefore for any q, where the L^q norm is in the x variables we have

$$\|f(t,x_1,x_2)\|_q \leq C2^{-\frac{j_0}{2}}\||g| * \chi_{\{x:|x|<2^{j_0}\}}(x)\|_q$$
$$+C\sum_{j=j_0+1}^{\infty} 2^{-\frac{j}{2}}\||g| * \chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x)\|_q \tag{5.7}$$

By Young's inequality, if $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, the above is bounded by

$$C2^{-\frac{j_0}{2}+\frac{2j_0}{r}}\|g\|_p + C\sum_{j=j_0+1}^{\infty} 2^{-\frac{j}{2}+\frac{2j}{r}}\|g\|_p \tag{5.8}$$

In the case that $\frac{2}{r} < \frac{1}{2}$, the above converges and we get

$$\|f(t, x_1, x_2)\|_q \leq C_q 2^{-\frac{j_0}{2} + \frac{2j_0}{r}} \|g\|_p \tag{5.9}$$

The condition that $\frac{2}{r} < \frac{1}{2}$ translates into $\frac{1}{p} - \frac{1}{q} > \frac{3}{4}$, which can occur when $p < \frac{4}{3}$. Given the definition of j_0, (5.9) can be rewritten as

$$\|f(t, x_1, x_2)\|_q \leq C_q \big((|t| + 2)^{2\epsilon} (\ln(|t| + 2))^{-2m}\big)^{\frac{2}{r} - \frac{1}{2}} \|g\|_p \tag{5.10}$$

This is the same as

$$\|f(t, x_1, x_2)\|_q \leq C_q (|t| + 2)^{4\epsilon(\frac{1}{q} - \frac{1}{p} + \frac{3}{4})} (\ln(|t| + 2))^{-4m(\frac{1}{q} - \frac{1}{p} + \frac{3}{4})} \|g\|_p \tag{5.11}$$

This gives Theorem 1.3, except in the case where the exponent $\frac{1}{q} - \frac{1}{p} + \frac{3}{4}$ is zero. In this case, the exponent $-\frac{j_0}{2} + 2\frac{j_0}{r}$ is always zero, so (5.8) diverges. In this case, one may use the Hardy-Littlewood-Sobolev inequality instead, directly using the bound (5.5). For (5.5) says that $|T(t, x_1, x_2)| \leq C|x|^{-\frac{1}{2}}$, and since $\frac{1}{p} = \frac{1}{q} + \frac{3}{4}$ the Hardy-Littlewood-Sobolev inequality gives that $\|f(t, x_1, x_2)\|_q = \|g * T(t, x_1, x_2)\|_q \leq C_q \|g\|_p$, as long as $p \neq 1$ and $q \neq \infty$. This concludes the proof of Theorem 1.3.

In order to prove Theorems 1.4 and 1.5, in place of Theorems 1.1 and 1.2 we use the following, which generalizes estimates for the $\mu_1 = \mu_2 = 0$ case in [1].

Lemma 5.1. Let $S(x, y)$ be as in Theorems 1.4 and 1.5, and define $R(\lambda_1, \mu_1, \mu_2)$ by

$$R(\lambda_1, \mu_1, \mu_2) = \int_{\mathbb{R}^2} e^{-\lambda_1 S(x,y) + i\mu_1 x + i\mu_2 y} \phi(x, y) \, dx \, dy \tag{5.12}$$

There is a neighborhood of the origin V that if the support of ϕ is contained in V then for $\lambda_1 > 2$ one has the estimate

$$|R(\lambda_1, \mu_1, \mu_2)| \leq C_S \|\phi\|_{C^1(V)} \min\big((\lambda_1)^{-\epsilon} (\ln \lambda_1)^m, |\mu|^{-1}\big) \tag{5.13}$$

Here (ϵ, m) is as in (4.2).

Proof. We will first prove that $|R(\lambda_1, \mu_1, \mu_2)| \leq C_S \|\phi\|_{C^1(V)} (\lambda_1)^{-\epsilon} (\ln \lambda_1)^m$ and afterwards that $|R(\lambda_1, \mu_1, \mu_2)| \leq C_S \|\phi\|_{C^1(V)} |\mu|^{-1}$. For the first estimate, we take absolute values on (5.12) and integrate, obtaining

$$|R(\lambda_1, \mu_1, \mu_2)| \leq C_V \|\phi\|_{C^1(V)} \int_V e^{-\lambda_1 S(x,y)} \, dx \, dy \tag{5.14}$$

Note that

$$\int_V e^{-\lambda_1 S(x,y)} \, dx \, dy = \int_0^\infty \lambda_1 e^{-\lambda_1 t} |\{(x, y) \in V : S(x, y) < t\} \, dt \tag{5.15}$$

We can truncate the integral here at $t = \frac{1}{2}$ since the $t > \frac{1}{2}$ portion gives an estimate much better than what we need. Thus inserting (4.2) we must bound

$$\int_0^{\frac{1}{2}} \lambda_1 e^{-\lambda_1 t} t^\epsilon |\ln t|^m \, dt \tag{5.16}$$

Changing variables to $\lambda_1 t$ here, the integral in (5.15) is at most

$$\lambda_1^{-\epsilon}\int_0^{\frac{\lambda_1}{2}} e^{-t}(|\ln t| + \ln\lambda_1)^m \tag{5.17}$$

Regardless of whether $m = 0$ or 1, equation (5.17) is bounded by $C\lambda_1^{-\epsilon}(\ln\lambda_1)^m$. Putting this back into (5.14) gives the desired estimate.

We now prove that $|R(\lambda_1,\mu_1,\mu_2)| \leq C_{S,\phi}|\mu|^{-1}$. As before we can assume $|\mu| > 2$ as the $|\mu| \leq 2$ case is obtained simply by taking absolute values and integrating. Rotating coordinates and shrinking our neighborhood V of the origin if necessary, we assume that for some $k \geq 2$ we have $\partial_y^k S$ and $\partial_x^k S$ are nonzero on V. Since the x and y axes are interchangeable here, without loss of generality we assume $|\mu_2| \geq |\mu_1|$. We write the phase in (5.12) in the form

$$e^{-\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y} = (-\lambda_1\partial_y S(x,y) + i\mu_2)e^{-\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}$$
$$\times\left(\frac{1}{-\lambda_1\partial_y S(x,y) + i\mu_2}\right) \tag{5.18}$$

Next, we integrate by parts in (5.12), integrating the factor $(-\lambda_1\partial_y S(x,y) + i\mu_2)$ $e^{-\lambda_1 S(x,y)+i\mu_1 x+i\mu_2 y}$ in (5.18) and differentiating the rest. The derivative can land in two places. First, it can land on the $\phi(x,y)$ factor. For this term, we take absolute values and integrate, using the bound $|\frac{1}{-\lambda_1\partial_y S(x,y)+i\mu_2}| \leq \frac{1}{|\mu_2|} \leq \frac{2}{|\mu|}$, and we obtain the needed bound of $C\|\phi\|_{C^1(V)}|\mu|^{-1}$. The second place the derivative can land is the $\frac{1}{-\lambda_1\partial_y S(x,y)+i\mu_2}$ factor, which becomes a factor of $\frac{\lambda_1\partial_{yy}S(x,y)}{(-\lambda_1\partial_y S(x,y)+i\mu_2)^2}$. The resulting term is bounded in absolute value by

$$\int_{\mathbb{R}^2}\frac{|\lambda_1\partial_{yy}S(x,y)|}{|-\lambda_1\partial_y S(x,y)+i\mu_2|^2}|\phi(x,y)|\,dx\,dy \tag{5.19}$$

We split this into two terms, depending on whether or not $|\lambda_1\partial_y S(x,y)| \geq |\mu_2|$. We get that (5.19) is bounded by

$$\int_{|\lambda_1\partial_y S(x,y)|\geq|\mu_2|}\frac{|\lambda_1\partial_{yy}S(x,y)|}{(\lambda_1\partial_y S(x,y))^2}|\phi(x,y)|\,dx\,dy$$
$$+\int_{|\lambda_1\partial_y S(x,y)|<|\mu_2|}\frac{|\lambda_1\partial_{yy}S(x,y)|}{\mu_2^2}|\phi(x,y)|\,dx\,dy \tag{5.20}$$
$$\leq C\|\phi\|_{C^1(V)}\int_{\{(x,y)\in V:|\lambda_1\partial_y S(x,y)|\geq|\mu_2|\}}\frac{|\lambda_1\partial_{yy}S(x,y)|}{(\lambda_1\partial_y S(x,y))^2}\,dx\,dy$$
$$+C\|\phi\|_{C^1(V)}\frac{1}{\mu_2^2}\int_{\{(x,y)\in V:|\lambda_1\partial_y S(x,y)|<|\mu_2|\}}|\lambda_1\partial_{yy}S(x,y)|\,dx\,dy \tag{5.21}$$

In the first term of (5.21) we integrate in y for fixed x. We use the fact $\partial_y^k S(x,y) \neq 0$ on V to split the interval of integration into boundedly many subintervals on which

$\partial_{yy}S(x,y)$ has constant sign. Integrating the $\frac{|\lambda_1\partial_{yy}S(x,y)|}{(\lambda_1\partial_y S(x,y))^2}$ leads to $\pm\frac{1}{\lambda_1\partial_y S(x,y)}$ at the boundary points, which is bounded in absolute value by $\frac{1}{|\mu_2|}$ given that on the domain of integration one has $|\lambda_1\partial_y S(x,y)| \geq |\mu_2|$. Thus the overall term is bounded by $C\|\phi\|_{C^1(V)}\frac{1}{|\mu_2|} \leq C'\|\phi\|_{C^1(V)}\frac{1}{|\mu|}$ as needed.

In the second term of (5.21), we do the analogous argument, and the resulting integration leads to $\pm\lambda_1\partial_y S(x,y)$ at the boundary points of the subintervals of integration. This time the condition that $|\lambda_1\partial_y S(x,y)| < |\mu_2|$ on the domain of integration leads to the term being bounded by bounded by $C\frac{1}{\mu_2^2}\|\phi\|_{C^1(V)}|\mu_2| \leq C'\|\phi\|_{C^1(V)}\frac{1}{|\mu|}$ and we are done.

Proof of Theorem 1.4.

The proof will proceed much like the proof of Theorem 1.3. This time, on the Fourier transform side the PDE becomes

$$\frac{\partial \hat{f}}{\partial t}(t,\xi_1,\xi_2) = -S(\xi_1,\xi_2)\hat{f}(t,\xi_1,\xi_2)$$

$$\hat{f}(0,\xi_1,\xi_2) = \hat{g}(\xi_1,\xi_2) \tag{5.22}$$

So if $R(t,x_1,x_2)$ is as in Lemma 5.1, for $t \geq 0$ the equation is solved by

$$f(t,x_1,x_2) = (g * R)(t,x_1,x_2) \tag{5.23}$$

By Lemma 5.1, we have

$$|R(t,x_1,x_2)| \leq C\min\left((t+2)^{-\epsilon}(\ln(t+2))^m, |x|^{-1}\right) \tag{5.24}$$

This time we break into $T(t,x_1,x_2)$ into $(t+2)^{-\epsilon}(\ln(t+2))^m < |x|^{-1}$ and $(t+2)^{-\epsilon}(\ln(t+2))^m \geq |x|^{-1}$ pieces. We let j_1 be the nearest nonnegative integer to the j for which $2^j = (t+2)^{\epsilon}(\ln(t+2))^{-m}$. Then in analogy to (5.5) we have

$$|R(t,x_1,x_2)| \leq C2^{-j_1}\chi_{\{x:|x|<2^{j_1}\}}(x_1,x_2) + C\sum_{j>j_1} 2^{-j_1}\chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x_1,x_2) \tag{5.25}$$

So in analogy to (5.7), for any q we have that $\|f(t,x_1,x_2)\|_q$ is bounded by

$$C2^{-j_1}\||g| * \chi_{\{x:|x|<2^{j_1}\}}(x)\|_q + C\sum_{j>j_1} 2^{-j_1}\||g| * \chi_{\{x:2^{j-1}\leq|x|<2^j\}}(x)\|_q \tag{5.26}$$

By Young's inequality, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, this is bounded by

$$C2^{-j_1+\frac{2j_1}{r}}\|g\|_p + C\sum_{j>j_1} 2^{-j+\frac{2j}{r}}\|g\|_p \tag{5.27}$$

So when $r > 2$ this converges and this time we have the bound

$$\|f(t,x_1,x_2)\|_q \leq C_q 2^{-j_1+\frac{2j_1}{r}}\|g\|_p \tag{5.28}$$

The condition that $r > 2$ translates into $\frac{1}{p} - \frac{1}{q} > \frac{1}{2}$, which can occur when $p < 2$. By definition of j_1, (5.28) is the same as

$$\|f(t,x_1,x_2)\|_q \leq C_q\big((t+2)^{\epsilon}(\ln(t+2))^{-m}\big)^{\frac{2}{r}-1}\|g\|_p \tag{5.29}$$

This in turn is the same as

$$\|f(t,x_1,x_2)\|_q \leq C_q\big((t+2)^{2\epsilon}(\ln(t+2))^{-2m}\big)^{\frac{1}{q}-\frac{1}{p}+\frac{1}{2}}\|g\|_p \tag{5.30}$$

This gives Theorem 1.4, except in the case where the exponent $\frac{1}{q} - \frac{1}{p} + \frac{1}{2}$ is zero. In this case, (5.25) gives that $|R(t,x_1,x_2)| \leq C|x|^{-1}$, and then the Hardy-Littlewood-Sobolev theorem gives that $\|f(t,x_1,x_2)\|_q = \|g*R(t,x_1,x_2)\|_q \leq C_{p,q,S}\|g\|_p$ as long as $p \neq 1$ and $q \neq \infty$ and we are done.

Proof of Theorem 1.5.

On the Fourier transform side in the x_1 and x_2 variables, (1.12) becomes

$$\hat{f}(\xi_1,\xi_2) = S(\xi_1,\xi_2)^{-\delta}\hat{g}(\xi_1,\xi_2) \tag{5.31a}$$

Like in the previous two theorems, the support condition on $\hat{g}$ means we can insert a cutoff function in (5.31a), turning the equation into

$$\hat{f}(\xi_1,\xi_2) = \hat{g}(\xi_1,\xi_2)S(\xi_1,\xi_2)^{-\delta}\phi(\xi_1,\xi_2) \tag{5.31b}$$

Thus if we define $Q(x_1,x_2)$ by

$$Q(x_1,x_2) = \int_{\mathbb{R}^2} S(\xi_1,\xi_2)^{-\delta}e^{ix_1\xi_1+ix_2\xi_2}\phi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2 \tag{5.32}$$

Then we have

$$f = g * Q \tag{5.33}$$

Next, note that if $t > 0$ and $\delta > 0$ one has

$$\int_0^{\infty} e^{-ut}u^{\delta-1}du = c_\delta t^{-\delta} \tag{5.34}$$

Inserting this into (5.32) gives

$$\begin{aligned} Q(x_1,x_2) &= c_\delta \int_{\mathbb{R}^2}\left(\int_0^{\infty} e^{-uS(\xi_1,\xi_2)}u^{\delta-1}\,du\right)e^{ix_1\xi_1+ix_2\xi_2}\phi(\xi_1,\xi_2)\,du\,d\xi_1\,d\xi_2 \\ &= c_\delta \int_0^{\infty} u^{\delta-1}\left(\int_{\mathbb{R}^2} e^{-uS(\xi_1,\xi_2)+ix_1\xi_1+ix_2\xi_2}\phi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\right)du \end{aligned} \tag{5.35}$$

We perform the (ξ_1,ξ_2) integration in (5.35), use the bounds from Lemma 5.1, then integrate the result in u. The result is

$$|Q(x_1,x_2)| \leq C\int_0^{\infty} u^{\delta-1}\min\big((u+2)^{-\epsilon}(\ln(u+2))^m, |x|^{-1}\big)\,du \tag{5.36}$$

We now bound $|Q(x_1, x_2)|$. For $|x| < 4$, we just use the bound obtained by taking absolute values and integrating in (5.32), and get $|Q(x_1, x_2)| < C$. Note that here we use that $\delta < \epsilon$; the fact that (4.2) holds ensures that $|S(\xi_1, \xi_2)|^{-\delta}$ is integrable.

Now assume $|x| > 4$. If $m = 0$, let $\epsilon' = \epsilon$, and if $m = 1$, let ϵ' be any number satisfying $\delta < \epsilon' < \epsilon$. Then (5.36) gives

$$|Q(x_1, x_2)| \leq C' \int_0^\infty u^{\delta - 1} \min\left((u+2)^{-\epsilon'}, |x|^{-1}\right) du \tag{5.37}$$

$$= C' \frac{1}{|x|} \int_0^{|x|^{\frac{1}{\epsilon'}} - 2} u^{\delta-1}\, du + C' \int_{|x|^{\frac{1}{\epsilon'}} - 2}^\infty u^{\delta - 1}(u+2)^{-\epsilon'}\, du \tag{5.38}$$

The first term is bounded by $C' \frac{1}{|x|} \int_0^{|x|^{\frac{1}{\epsilon'}}} u^{\delta - 1}\, du$, or $C''|x|^{\frac{\delta}{\epsilon'} - 1}$. For the second term, we have

$$\int_{|x|^{\frac{1}{\epsilon'}} - 2}^\infty u^{\delta-1}(u+2)^{-\epsilon'}\, du \leq \int_{|x|^{\frac{1}{\epsilon'}} - 2}^\infty u^{\delta - \epsilon' - 1} du \tag{5.39}$$

Given our assumption that $\delta < \epsilon'$, (5.39) converges, and since $|x| > 4$, this is at most

$$C \int_{|x|^{\frac{1}{\epsilon'}}}^\infty u^{\delta - \epsilon' - 1} du \tag{5.40}$$

This integrates to a term bounded by a constant times $|x|^{\frac{\delta}{\epsilon'} - 1}$. Combining with the first term, we conclude that for $|x| \geq 4$ we have

$$|Q(x_1, x_2)| < C|x|^{\frac{\delta}{\epsilon'} - 1} \tag{5.41}$$

Combining with the $|x| < 4$, bound we have

$$|Q(x_1, x_2)| < C \min(1, |x|^{\frac{\delta}{\epsilon'} - 1}) \tag{5.42}$$

Note that the right-hand side of (5.42) is in L^r for $r > \frac{2\epsilon'}{\epsilon' - \delta}$. Thus for such r, by Young's inequality, if $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ one has a bound

$$\|f\|_q = \|g * Q\|_q \leq C_{p,q,S}\|g\|_p \tag{5.43}$$

One also has this bound when $r = \frac{2\epsilon'}{\epsilon' - \delta}$ by the Hardy-Littlewood-Sobolev inequality, as long as $p \neq 1$ and $q \neq \infty$. Stated in terms of p and q alone, we have that an estimate of the form (5.43) holds whenever $\frac{1}{q} \leq \frac{1}{p} - \frac{\delta}{2\epsilon'} - \frac{1}{2}$, unless $\frac{1}{q} = \frac{1}{p} - \frac{\delta}{2\epsilon'} - \frac{1}{2}$ and $p = 1$ or $q = \infty$.

Given how ϵ' was defined, our conclusions are therefore as follows. When $m = 0$, there is an estimate of the form (5.43) whenever $\frac{1}{q} \leq \frac{1}{p} - \frac{\delta}{2\epsilon} - \frac{1}{2}$, except when $\frac{1}{q} = \frac{1}{p} - \frac{\delta}{2\epsilon} - \frac{1}{2}$ and $p = 1$ or $q = \infty$. When $m = 1$, there is an estimate of the form (5.43) whenever $\frac{1}{q} < \frac{1}{p} - \frac{\delta}{2\epsilon} - \frac{1}{2}$. This concludes the proof of Theorem 1.5.

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