HIGHER RELATIVE INDEX THEOREMS FOR FOLIATIONS, A SUPERCONNECTION APPROACH October 14, 2025

MOULAY-TAHAR BENAMEUR 1965-2025‡ AND JAMES L. HEITSCH

Abstract. We continue our extension of the groundbreaking results of Gromov and Lawson, [GL83], to Dirac operators defined along the leaves of foliations of non-compact complete Riemannian manifolds. Given two leafwise Dirac operators on two foliated manifolds which agree near infinity, we have the topological indices of [BH24], and using Bismut superconnections, we define analytic Connes-Chern characters, all in Haefliger cohomologies. We show that they are pairwise equal.

In this paper we do not assume that the foliations are Riemannian. Thus, in order to relate our invariants to the invariants of the so called "index bundles" of the operators, we must strengthen the assumptions in [BH24] on the Novikov-Shubin invariants of the foliations and require that the manifolds satisfy a stronger growth condition. This allows us to use results in [HL99] to extend our higher relative index bundle theorem for Riemannian foliations to a much broader class of foliations.

We construct examples of non-Riemannian foliations and use our results to show that their spaces of leafwise metrics of positive scalar curvature have infinitely many path-connected components.

Contents

1. Introduction

This paper is a continuation of our program, [BH19, BH21, BH24], of extending the groundbreaking relative index theorem, Theorem 4.18, [GL83], of Gromov and Lawson, to Dirac operators defined along the leaves of foliations whose holonomy groupoids are Hausdorff. We assume that the foliations are on (possibly non-compact) complete Riemannian manifolds, and that all of their structures have bounded geometry. In [BH24], we sometimes assumed that the foliations were Riemannian. Here we make no such assumption. This necessitates strengthening the conditions on the Novikov-Shubin invariants of the foliations. The reason we need more restrictive assumptions is that, in the Riemannian case, the results we use from [BH08] are valid only for foliations whose holonomy maps are isometries, which is precluded in the general case. Our assumptions do allow us to use the more general results of [HL99].

Our approach is quite similar to that in [BH24]. In particular, we assume that we have a foliated manifold (M, F) as above, and a Clifford bundle $E_M \to M$ over the Clifford algebra of the co-tangent bundle to F, along with a Hermitian connection $\nabla^{F,E}$ compatible with Clifford multiplication. This determines a leafwise Dirac operator, denoted D_F. We assume that we have a second foliated manifold (M', F') with the same structures. We further assume that there are compact subspaces $\mathcal{K}_M = M \smallsetminus V_M$ and $\mathcal{K}'_{M'} = M' \smallsetminus V'_{M'}$ so

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that the situations on V_M and $V'_{M'}$ are identical. Objects which are identified "near infinity", that is off compact neighborhoods of $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$, are called φ compatible.

As in [BH24], we work on the holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$ of F and F', with their canonical foliations F_s and F'_s. Thus we lift everything to $\mathcal{G}$ using the map $r : \mathcal{G} \to M$, which maps the leaves of F_s to those of F, and similarly for M'. In particular, we have the $\mathcal{G}$ invariant leafwise Dirac operator D for F_s, and similarly D' for F'_s.

For each U_i in a good cover of M by foliation charts, let $T_i \subset U_i$ be a transversal and set $T = \bigcup T_i$. The Haefliger forms $\mathcal{A}^*_c(M/F)$ associated to F are uniformly bounded smooth forms on T which have compact support in each T_i, modulo forms minus their holonomy images. Their dual spaces are the Haefliger currents. The Haefliger cohomology of F, denoted $\mathrm{H}^*_c(M/F)$, is the associated cohomology, which is independent of the choice of good cover.

The relative Haefliger cohomology for the pair $((M, F), (M', F'))$, denoted $\mathrm{H}^*_c(M/F, M'/F'; \varphi)$ is the cohomology of pairs of φ compatible Haefliger forms (that is, on T_i far enough away from $\mathcal{K}_M$, and similarly for the T'_i), modulo pairs of forms minus their holonomy images, which are φ compatible. More precise definitions are given in Section 2 below.

Using the Atiyah-Singer characteristic forms $AS(D_F)$ and $AS(D_{F'})$, which agree near infinity, we define topological indices

$$\mathrm{Ind}_t(D) \in \mathrm{H}^*_c(M/F), \;\; \mathrm{Ind}_t(D') \in \mathrm{H}^*_c(M'/F'), \;\text{ and }\; \mathrm{Ind}_t(D, D') \in \mathrm{H}^*_c(M/F, M'/F'; \varphi).$$

The usual heat operator definition of the superconnection analytic Chern character gives

$$\mathrm{ch}_a(D) \in \mathrm{H}^*_c(M/F) \;\;\text{and}\;\; \mathrm{ch}_a(D') \in \mathrm{H}^*_c(M'/F'),$$

and previous results give,

$$\mathrm{ch}_a(D) = \mathrm{Ind}_t(D) \in \mathrm{H}^*_c(M/F) \quad \text{ and } \quad \mathrm{ch}_a(D') = \mathrm{Ind}_t(D') \in \mathrm{H}^*_c(M'/F').$$

Using a Bismut superconnection $\mathbb{B} = \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2$, [B86], and appropriate finite propagation even functional calculi we are able to define the relative analytic Connes-Chern character,

$$\mathrm{ch}_a(D, D') \in \mathrm{H}^*_c(M/F, M'/F'; \varphi).$$

Our first result is the following.

Theorem 3.2 *Suppose that (M, F), (M', F'), and (D, D') are as above. Then the relative index formula holds in $\mathrm{H}^*_c(M/F, M'/F'; \varphi)$, i.e.*

$$\mathrm{ch}_a(D, D') = \mathrm{Ind}_t(D, D').$$

Next we relate these relative (and absolute) Chern characters with the so-called index bundle, so as to be able to apply the classical vanishing principle of the topological indices whenever the kernel projections are trivial, and to deduce important corollaries, see for instance [BH19, BH21]. As is already well known for closed foliated manifolds, this is a non-trivial issue and one cannot expect such results without imposing extra assumptions, see for instance [HL99, BH08, BHW14].

Denote by P_0 the leafwise spectral projection to the kernel of D^2, and by $P_{(0,\epsilon)}$ the spectral projection associated to the interval $(0, \epsilon)$. In general these are not transversely smooth (although they are always leafwise smooth), so that in general we cannot define their Connes-Chern characters in Haefliger forms. When they are transversely smooth, their characters are well defined Haefliger forms and the form for P_0 is closed, so we have

$$\mathrm{ch}_a(P_0) \in \mathrm{H}^*_c(M/F) \;\;\text{and}\;\; \mathrm{ch}_a(P_{(0,\epsilon)}) \in \mathcal{A}^*_c(M/F),$$

and similarly for P'_0 and $P'_{(0,\epsilon)}$.

We have two main results, begining with the (non-Riemannian) gap at zero Relative Index Theorem.

Theorem 3.3 *Assume that the holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Hausdorff, and that for sufficiently small ϵ, $P_{(0,\epsilon)} = P'_{(0,\epsilon)} = 0$. Assume moreover that:*

(1) *P_0 is transversely smooth and satisfies* $\displaystyle\int_M \operatorname{tr}(P_0)\,dx < \infty$, *and similarly for* P_0';
(2) $\mathbb{B}_1(P_0)$ *and* $\mathbb{B}_1(P_0')$ *are bounded leafwise smoothing operators;*
(3) *M, so also M', has sub-exponential growth.*

Then, for $\mathcal{C}$ and $\mathcal{C}'$ closed bounded φ compatible continuous holonomy invariant Haefliger currents, the pairings $\langle \mathrm{ch}_a(P_0), \mathcal{C}\rangle$ and $\langle \mathrm{ch}_a(P_0'), \mathcal{C}'\rangle$ are well defined, and

$$\langle \mathrm{ch}_a(D, D'), (\mathcal{C}, \mathcal{C}')\rangle = \langle \mathrm{ch}_a(P_0), \mathcal{C}\rangle - \langle \mathrm{ch}_a(P_0'), \mathcal{C}'\rangle = \langle \Big[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D_{F'})\Big], (\mathcal{C}, \mathcal{C}')\rangle.$$

The element dx is the volume form on M. The finite integral assumption in the first item is for instance satisfied when the locally defined zeroth order operator in the associated Bochner Identity for D_F is uniformly positive near infinity on M, so also on M'. This is true for the leafwise spin-Dirac operator when the foliations have positive scalar curvature near infinity in the ambient manifolds.

When there is no gap in the spectra near zero, the Novikov-Shubin invariants $NS(D)$ of D provide a measure of the density of the image of $P_{(0,\epsilon)}$. The larger $NS(D)$ is, the sparser the image of $P_{(0,\epsilon)}$ is as $\epsilon \to 0$, and they are infinite in the gap at zero case.

The general non-Riemannian Relative Index Theorem is the following.

Theorem 3.5 *Assume that the holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Hausdorff, and that P_0, P_0', $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$, (for ϵ sufficiently small), are transversely smooth. Assume further that,*

(1) *$NS(D)$ and $NS(D')$ are greater than* $3\,\mathrm{codim}(F)$;
(2) $\mathbb{B}_1(P_0)$, $\mathbb{B}_1(P_{(0,\epsilon)})$ *are bounded leafwise smoothing, and similarly for P_0' and $P'_{(0,\epsilon)}$;*
(3) $P_{[0,\epsilon)}$ *satisfies* $\displaystyle\int_M \operatorname{tr}\big(P_{[0,\epsilon)}\big)\,dx < \infty$, *and similarly for* $P'_{[0,\epsilon)}$;
(4) *M has sub-super polynomial growth, so also does M'.*

Then, for $\mathcal{C}$ and $\mathcal{C}'$ closed bounded φ compatible continuous holonomy invariant Haefliger currents, the pairings $\langle \mathrm{ch}_a(P_0), \mathcal{C}\rangle$ and $\langle \mathrm{ch}_a(P_0'), \mathcal{C}'\rangle$ are well defined, and

$$\langle \mathrm{ch}_a(D, D'), (\mathcal{C}, \mathcal{C}')\rangle = \langle \mathrm{ch}_a(P_0), \mathcal{C}\rangle - \langle \mathrm{ch}_a(P_0'), \mathcal{C}'\rangle = \langle \Big[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D_{F'})\Big], (\mathcal{C}, \mathcal{C}')\rangle.$$

The notion of sub-super polynomial growth is introduced in Definition 3.4.

Remarks 1.1.

(1) *The reader should note carefully that, for simplicity, the equalities in our theorems are up to universal non-zero constants.*
(2) *Under the assumptions of Theorems 3.3 or 3.5, the proofs actually show that*

$$\mathrm{ch}_a(D) = \mathrm{ch}_a(P_0) = \Big[\int_F \mathrm{AS}(D_F)\Big] \text{ in } \mathrm{H}_c^*(M/F),$$

and similarly for $\mathrm{ch}_a(D')$.
(3) *The growth conditions on the ambient manifolds are needed so that that the pairings will be as claimed.*
(4) *As above, the finite integral assumption is satisfied when the locally defined zeroth order operator in the associated Bochner Identity is uniformly positive near infinity on M, so also on M'. This is proven in Theorem 4.6 of* [BH24].

As we shall see, the proof of Theorem 3.2 is standard and follows the classical steps for the heat equation proof of index theorems. The idea behind the proofs of Theorems 3.3 and 3.5 is to use the results and techniques of [HL99] coupled with finite propagation techniques, see also [Roe87, BGV92]. The family of foliation Bismut superconnections associated to D is

$$\mathbb{B}(t) = \sqrt{t}\mathbb{B}_0 + \mathbb{B}_1 + \frac{1}{\sqrt{t}}\mathbb{B}_2.$$

The Connes-Chern character $\mathrm{ch}_a(D)$ is the Haefliger class denoted $\mathrm{S}\mathfrak{Tr}\left(e^{-\mathbb{B}(t)^2/2}\right)$ determined by the Haefliger form $\int_F \mathrm{Str}(e^{-\mathbb{B}(t)^2/2})$, where Str is the supertrace of the Schwartz kernel of the leafwise operator $e^{-\mathbb{B}(t)^2/2}$. See pp. 188-89 of [HL99]. Denote by Q_ϵ the spectral projection of D^2 associated to the interval $[\epsilon,\infty)$. Set,

$$\mathbb{B}(t) \;=\; \bar{\mathbb{B}}_{\epsilon,t} \;+\; A_{\epsilon,t}, \quad \text{where} \quad \bar{\mathbb{B}}_{\epsilon,t} = (P_0+Q_\epsilon)\mathbb{B}(t)(P_0+Q_\epsilon) + P_{(0,\epsilon)}\mathbb{B}(t)P_{(0,\epsilon)}.$$

As in [HL99], we show that the Haefliger cohomology class determined by $\lim_{t\to\infty}\int_F \mathrm{Str}(e^{-\bar{\mathbb{B}}^2_{\epsilon,t}/2})$ is $\mathrm{ch}_a(P_0)$. If $P_{(0,\epsilon)} = 0$, then $A_{\epsilon,t} = 0$ also, and we have the essential result we need for the proof of Theorem 3.3. To extend that proof to Theorem 3.5, we then show that the term $A_{\epsilon,t}$ does not affect the limit as $t\to\infty$.

The main technical challenges are to define $\mathrm{ch}_a(D,D')$, and then to extend the existing arguments to the new definition. This entails a number of subtle difficulties. The problem is that, even though D and D' are φ compatible, $e^{tD^2/2}$ and $e^{tD'^2/2}$ are not in general, since what D is on say $\mathcal{K}_M$ can effect what $e^{-tD^2/2}$ is near infinity. So, if D and D' are not φ compatible globally, this can cause $e^{-tD^2/2}$ and $e^{-tD'^2/2}$ to not be φ compatible. The solution is to approximate $e^{-tD^2/2}$ and $e^{tD'^2/2}$ by operators which have finite propagations, and which converge to $e^{-tD^2/2}$ and $e^{-tD'^2/2}$ as $t\to 0$ and ∞. This is accomplished by using families of Schwartz functions denoted χ^t approximating e^{-tz^2} which have Fourier transforms which are compactly supported. Then $\chi^t(D)$ and $\chi^t(D')$ will be φ compatible, and as $t\to 0$ and ∞, they converge to $e^{tD^2/2}$ and $e^{tD'^2/2}$.

The final section contains the application of the main results to non-Riemannian foliations which admit positive scalar curvature. In particular we show how to construct families of such foliations whose spaces of positive scalar curvature metrics have infinitely many path connected components.

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2. The Setup

In order for this paper to be self contained we repeat some of Section 2 of [BH24].

Denote by M a smooth (if non-compact, then complete) Riemannian manifold of dimension n, and by F an oriented foliation (with the induced metric) of M of dimension p, assumed to be even, and codimension $q = n-p$. The tangent and cotangent bundles of M and F are denoted TM, T^*M, TF and T^*F. The normal and dual normal bundles of F are denoted ν and ν^*. A leaf of F is denoted by L. In [BH24], we assumed that F is Riemannian. Here we make no such assumption.

If M is non-compact, we assume that both M and F are of bounded geometry, that is, the injectivity radii on M and on all the leaves of F are uniformly bounded below, and the curvatures and all of their covariant derivatives on M and on all the leaves of F are uniformly bounded (the bound may depend on the order of the derivative).

Let $\mathcal{U}$ be a good cover of M by foliation charts as defined in [HL90]. In particular, denote by $\mathbb{D}^p(r) = \{x\in\mathbb{R}^p, ||x|| < r\}$, and similarly for $\mathbb{D}^q(r)$. An open uniformly locally finite cover $\{(U_i,\psi_i)\}$ of M by foliation coordinate charts $\psi_i : U_i \to \mathbb{D}^p(1)\times\mathbb{D}^q(1)\subset\mathbb{R}^n$ is a good cover for F provided that

1. For each $y\in\mathbb{D}^q(1)$, $P_y = \psi_i^{-1}(\mathbb{D}^p(1)\times\{y\})$ is contained in a leaf of F. P_y is called a plaque of F.
2. If $U_i\cap U_j\neq\varnothing$, then it is contractible, and if $\overline{U}_i\cap\overline{U}_j\neq\varnothing$, then $U_i\cap U_j\neq\varnothing$.
3. Each ψ_i extends to a diffeomorphism $\psi_i : V_i \to \mathbb{D}^p(2)\times\mathbb{D}^q(2)$, so that the cover $\{(V_i,\psi_i)\}$ satisfies (1) and (2), with $\mathbb{D}^p(1)$ and $\mathbb{D}^q(1)$ replaced by $\mathbb{D}^p(2)$ and $\mathbb{D}^q(2)$.

(4) Each plaque of V_i intersects at most one plaque of V_j and a plaque of U_i intersects a plaque of U_j if and only if the corresponding plaques of V_i and V_j intersect.

(5) There are global positive upper and lower bounds on the norms of each of the derivatives of the ψ_i.

Bounded geometry foliated manifolds always admit good covers. In particular, we may assume that the V_i have a uniform bound on their diameters significantly less than the bound on the injectivity radii.

For each $U_i \in \mathcal{U}$, let $T_i \subset U_i$ be a transversal, and set $T = \bigcup T_i$. We may assume that the closures of the T_i are disjoint. Given (U_i, T_i) and (U_j, T_j), suppose that $\gamma_{ij\ell} : [0,1] \to M$ is a path whose image is contained in a leaf with $\gamma_{ij\ell}(0) \in T_i$ and $\gamma_{ij\ell}(1) \in T_j$. Then $\gamma_{ij\ell}$ induces a local diffeomorphism $h_{\gamma_{ij\ell}} : T_i \to T_j$, with domain $\mathrm{Dom}_{\gamma_{ij\ell}}$ and range $\mathrm{Ran}_{\gamma_{ij\ell}}$. The space $\mathcal{A}_c^k(T)$ consists of all uniformly bounded smooth k-forms on T which have compact support in each T_i. The Haefliger k-forms for F, denoted $\mathcal{A}_c^k(M/F)$, consists of elements in the quotient of $\mathcal{A}_c^k(T)$ by the closure of the vector subspace W generated by elements of the form $\alpha_{ij\ell} - h^*_{\gamma_{ij\ell}}\alpha_{ij\ell}$ where $\alpha_{ij\ell} \in \mathcal{A}_c^k(T)$ has support contained in $\mathrm{Ran}_{\gamma_{ij\ell}}$. We need to take care as to what this means. Members of W consist of possibly infinite sums of elements of the form $\alpha_{ij\ell} - h^*_{\gamma_{ij\ell}}\alpha_{ij\ell}$, with the following restrictions: each member of W has a bound on the leafwise length of all the $\gamma_{ij\ell}$ for that member, and each $\gamma_{ij\ell}$ occurs at most once. Note that these conditions plus bounded geometry imply that for each member of W, there is $n \in \mathbb{N}$ so that the number of elements of that member having $\mathrm{Dom}_{\gamma_{ij\ell}}$ contained in any T_i is less than n, and that each U_i and each U_j appears at most a uniformly bounded number of times. The projection map is denoted

$$[\cdot] : \mathcal{A}_c^*(T) \;\to\; \mathcal{A}_c^*(M/F).$$

Denote the exterior derivative by $d_T : \mathcal{A}_c^k(T) \to \mathcal{A}_c^{k+1}(T)$, which induces $d_H : \mathcal{A}_c^k(M/F) \to \mathcal{A}_c^{k+1}(M/F)$. Note that $\mathcal{A}_c^k(M/F)$ and d_H are independent of the choice of cover $\mathcal{U}$. The cohomology $\mathrm{H}_c^*(M/F)$ of the complex $\{\mathcal{A}_c^*(M/F), d_H\}$ is the Haefliger cohomology of F.

Denote by $\mathcal{A}_u^{p+k}(M)$ the space of $p+k$-forms on M which are smooth and uniformly C^∞ bounded, and denote its exterior derivative by d_M. Its cohomology is denoted $H_u^{p+k}(M;\mathbb{R})$. As the bundle TF is oriented, there is a continuous open surjective linear map, called integration over F,

$$\int_F : \mathcal{A}_u^{p+k}(M) \to \mathcal{A}_c^k(T),$$

which commutes with the exterior derivatives. This map is given by choosing a partition of unity $\{\phi_i\}$ subordinate to the cover $\mathcal{U}$, and setting $\displaystyle\int_F \omega$ to be the class of $\displaystyle\sum_i \int_{U_i} \phi_i\omega$. It is a standard result, [Ha80], that the image of this form $\displaystyle\Big[\int_F \omega\Big] \in \mathcal{A}_c^k(M/F)$ is independent of the partition of unity and of the cover $\mathcal{U}$. As $\displaystyle\int_F$ commutes with d_M and d_H, it induces the map $\displaystyle\int_F : H_u^{p+k}(M;\mathbb{R}) \to H_c^k(M/F)$.

Note that $\displaystyle\int_{U_i}$ is integration over the fibers of the projection $U_i \to T_i$, and that each integration $\omega \to \displaystyle\int_{U_i} \phi_i\omega$ is essentially integration over a compact fibration, so $\displaystyle\int_F$ satisfies the Dominated Convergence Theorem on each $U_i \in \mathcal{U}$.

The holonomy groupoid $\mathcal{G}$ of F consists of equivalence classes of paths $\gamma : [0,1] \to M$ such that the image of γ is contained in a leaf of F. Two such paths γ_1 and γ_2 are equivalent if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the holonomy germ along them is the same. Two classes may be composed if the first ends where the second begins, and the composition is just the juxtaposition of the two paths. This makes $\mathcal{G}$ a groupoid. The space $\mathcal{G}^{(0)}$ of units of $\mathcal{G}$ consists of the equivalence classes of the constant paths, and we identify $\mathcal{G}^{(0)}$ with M.

$\mathcal{G}$ is $2p+q$ dimensional (in general, non-Hausdorff) manifold. We restrict to foliations for which $\mathcal{G}$ is Hausdorff. The maps $r, s : \mathcal{G} \to M$ are given by $s([\gamma]) = \gamma(0)$ and $r([\gamma]) = \gamma(1)$. $\mathcal{G}$ has two natural foliations, F_s and F_r, whose leaves are the fibers of s and r. We will primarily use F_s, whose leaves are denoted $\widetilde{L}_x = s^{-1}(x)$, for $x \in M$. Note that $r : \widetilde{L}_x \to L$ is the holonomy covering map.

If E is a bundle, the smooth sections are denoted by $C^\infty(E)$, and those with compact support by $C_c^\infty(E)$. We assume that any connection or any metric on E, and all their derivatives, are uniformly bounded. See [Sh92] for material about bounded geometry bundles and their properties.

In [BH23] we worked on M, while in [H95, HL99, BH04, BH08, BH24], we worked on $\mathcal{G}$. The results in [BH23] extend readily to $\mathcal{G}$ with the only change being that the spectral projections used on $\mathcal{G}$ are for the operator lifted to F_s. This represents another extension, in the spirit of Connes' extensions in [C79, C81], of the classical Atiyah L^2 covering index theorem, [A76]. In addition, the results in the above cited papers where M was assumed to be compact still hold provided both M and F are of bounded geometry.

Our basic data will be taken from (M, F). In particular, denote by D_F a generalized leafwise Dirac operator for the even dimensional foliation F. It is defined as follows. Let E_M be a complex vector bundle over M with Hermitian metric and connection, which is of bounded geometry. Assume that the tangent bundle TF is spin with a fixed spin structure. Denote by $\mathcal{S}_F = \mathcal{S}_F^+ \oplus \mathcal{S}_F^-$ the bundle of spinors along the leaves of F. Denote by ∇^F the Levi-Civita connection on each leaf L of F. ∇^F induces a connection ∇^F on $\mathcal{S}_F|L$, and we denote by $\nabla^{F,E}$ the tensor product connection on $\mathcal{S}_F \otimes E_M|L$. These data determine a smooth family $D_F = \{D_L\}$ of leafwise Dirac operators, where D_L acts on sections of $\mathcal{S}_F \otimes E_M|L$ as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of TL, and set

$$D_L \;=\; \sum_{i=1}^{p} \rho(X_i)\nabla^{F,E}_{X_i}$$

where $\rho(X_i)$ is the Clifford action of X_i on the bundle $\mathcal{S}_F \otimes E_M|L$. Then D_L does not depend on the choice of the X_i, and it is an odd operator for the $\mathbb{Z}_2$ grading of $\mathcal{S}_F \otimes E_M = (\mathcal{S}_F^+ \otimes E_M) \oplus (\mathcal{S}_F^- \otimes E_M)$. Thus $D_F : C_c^\infty(\mathcal{S}_F^\pm \otimes E_M) \to C_c^\infty(\mathcal{S}_F^\mp \otimes E_M)$, and $D_F^2 : C_c^\infty(\mathcal{S}_F^\pm \otimes E_M) \to C_c^\infty(\mathcal{S}_F^\pm \otimes E_M)$. For more on generalized Dirac operators, see [LM89].

All the data above may be lifted to $(\mathcal{G}, F_s)$ using the map $r : \mathcal{G} \to M$. The notation we will use is obtained from that above by:

$$E_M \to E; \quad \mathcal{S}_F \to \mathcal{S}; \quad \nabla^{F,E} \to \nabla; \quad L \to \widetilde{L}_x; \quad D_F \to D; \quad D_L \to D_x.$$

Thus the smooth family $D = \{D_x\}$ of $\mathcal{G}$ invariant leafwise Dirac operators acting on sections of $\mathcal{S} \otimes E|\widetilde{L}_x$ is given as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of $T\widetilde{L}_x$. Then,

$$D_x = \sum_{i=1}^{p} \rho(X_i)\nabla_{X_i}, \;\; \text{where} \;\; D : C_c^\infty(\mathcal{S}^\pm \otimes E) \to C_c^\infty(\mathcal{S}^\mp \otimes E), \;\; \text{and} \;\; D^2 : C_c^\infty(\mathcal{S}^\pm \otimes E) \to C_c^\infty(\mathcal{S}^\pm \otimes E).$$

Denote by $\wedge\nu_s^*$, the exterior powers of the dual normal bundle ν_s^* of F_s which we identify with $s^*(T^*M) = s^*(T^*F) \oplus s^*(\nu^*)$. We extend D to

$$D : C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*) \longrightarrow C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*),$$

by using the leafwise flat connection on $\wedge\nu_s^*$ determined by the pull-back of the Levi-Civiti connection on T^*M.

Given a leafwise operator A on $\mathcal{S} \otimes E \otimes \wedge\nu_s^*$, denote its leafwise Schwartz kernel by k_A. Then we have the usual pointwise trace $\operatorname{tr}(k_A(\overline{x}, \overline{x})$ and supertrace $\operatorname{Str}(k_A(\overline{x}, \overline{x})$ defined on $M \subset \mathcal{G}$. The element $\overline{x} \in \widetilde{L}_x$ is the class of the constant path at $x \in L \subset M$.

We also have the Haefliger traces, $\operatorname{Tr}(A)$, and if $d_H \operatorname{STr}(A) = 0$, $\mathfrak{Tr}(A)$ which are,

$$\operatorname{Tr}(A) \;=\; \int_F \operatorname{tr}(k_A(\overline{x}, \overline{x}))dx_F \in \mathcal{A}_c^*(M/F) \;\; \text{and} \;\; \mathfrak{Tr}(A) \;=\; \left[\int_F \operatorname{tr}(k_A(\overline{x}, \overline{x}))dx_F\right] \in \mathrm{H}_c^*(M/F),$$

where dx_F is the leafwise volume form associated with the fixed orientation of the foliation F. If A is an even $\mathbb{Z}_2$ graded operator, that is $A = A^+ \oplus A^-$, where

$$A^{\pm} : C_c^{\infty}(\mathcal{S}^{\pm} \otimes E \otimes \wedge \nu_s^*) \to C_c^{\infty}(\mathcal{S}^{\pm} \otimes E \otimes \wedge \nu_s^*),$$

we have the supertraces,

$$\operatorname{STr}(A) = \operatorname{Tr}(A^+) - \operatorname{Tr}(A^-), \quad \text{and if } d_H \operatorname{STr}(A) = 0,\ \operatorname{S}\mathfrak{Tr}(A) = \mathfrak{Tr}(A^+) - \mathfrak{Tr}(A^-).$$

Now suppose that we have the situation in Section 4 of [BH23]. That is, we have:

- foliated manifolds (M, F) and (M', F');
- Clifford bundles $E_M \to M$ and $E_{M'} \to M'$, with Clifford compatible Hermitian connections;
- leafwise Dirac operators D_F and $D_{F'}$;
- compact subspaces $\mathcal{K}_M = M \smallsetminus V_M$ and $\mathcal{K}'_{M'} = M' \smallsetminus V'_{M'}$;
- an isometry $\varphi : V_M \to V'_{M'}$ with $\varphi^{-1}(F') = F$;
- an isomorphism $\phi : E_M|_{V_M} \to E'_{M'}|_{V'_{M'}}$ with $\phi^*(\nabla^{F',E'}|_{V'_{M'}}) = \nabla^{F,E}|_{V_M}$.

The pair $\Phi = (\phi, \varphi)$ is called a bundle morphism from $E_M|V_M$ to $E'_{M'}|V'_{M'}$. The well defined (since they are differential operators) restrictions of D_F and $D_{F'}$ to the sections over V_M and $V'_{M'}$ agree through Φ, i.e.

$$(\Phi^{-1})^* \circ D_F \circ \Phi^* |_{V'_{M'}} = D_{F'} |_{V'_{M'}}.$$

Such operators are called φ compatible. Without loss of generality, we may assume that $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$ are the closures of open subsets.

Recall the following material from [BH23]. Denote by $g : M \to [0, \infty)$ and $g' : M' \to [0, \infty)$ compatible smooth approximations to the distance functions $\mathfrak{d}_M(\mathcal{K}_M, x)$ and $\mathfrak{d}_{M'}(\mathcal{K}'_{M'}, x')$. So we assume that g and g' are 0 on $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$ respectively and they satisfy $g' \circ \varphi = g$. Hence, for $s \geqslant 0$, the open submanifolds $M(s) = \{g > s\}$ and $M'(s) = \{g' > s\}$ agree through φ, that is $\varphi(M(s)) = M'(s)$ and $g|_{M(s)} = g' \circ \varphi|_{M(s)}$. For $s \geqslant 0$ denote by T_s the set

$$T_s \;=\; \{T_i \subset T \mid T_i \cap M(s) \neq \varnothing\},$$

and similarly for T'_s.

Suppose that $(\zeta, \zeta') \in W \times W' \subset \mathcal{A}_c^*(T) \times \mathcal{A}_c^*(T')$, with $\zeta = \sum_{(\alpha,\gamma)} \alpha - h_\gamma^* \alpha$ and $\zeta' = \sum_{(\alpha',\gamma')} \alpha' - h_{\gamma'}^* \alpha'$. For simplicity, we have dropped the subscripts. The vector subspace $W \times_\varphi W' \subset W \times W'$ consists of elements (ζ, ζ') which are φ compatible. This means that all but a finite number of the (α, γ) and (α', γ') are paired, that is

$$\alpha \;=\; \varphi^*(\alpha') \quad \text{and} \quad \gamma' \;=\; \varphi \circ \gamma, \quad \text{so} \quad \alpha - h_\gamma^* \alpha \;=\; \varphi^*(\alpha' - h_{\gamma'}^* \alpha').$$

Definition 2.1. *Given $\beta \in \mathcal{A}_c^*(T)$ and $\beta' \in \mathcal{A}_c^*(T')$, the pair (β, β') is φ compatible if there exists $s \geqslant 0$ so that $\beta = \varphi^*(\beta')$ on T_s. Set*

$$\mathcal{A}_c^*(M/F, M'/F'; \varphi) \;=\; \{(\beta, \beta') \in \mathcal{A}_c^*(T) \times \mathcal{A}_c^*(T') \,|\, (\beta, \beta') \text{ is } \varphi \text{ compatible}\}/(\overline{W \times_\varphi W'}).$$

The de Rham differentials on $\mathcal{A}_c^*(T)$ and $\mathcal{A}_c^*(T')$ yield a well defined relative Haefliger complex, whose homology spaces are denoted

$$\mathrm{H}_c^*(M/F, M'/F'; \varphi) = \oplus_{0 \leqslant k \leqslant q} H_c^k(M/F, M'/F'; \varphi).$$

Definition 2.2. *Suppose $(\xi, \xi') \in \mathcal{A}_c^*(M/F, M'/F'; \varphi)$, and $\mathcal{C}$ and $\mathcal{C}'$ are closed bounded φ compatible continuous holonomy invariant Haefliger currents. Set*

$$\langle (\xi, \xi'), (\mathcal{C}, \mathcal{C}') \rangle \;=\; \lim_{s \to \infty} \left(\mathcal{C}(\xi|_{T \smallsetminus T_s}) - \mathcal{C}'(\xi'|_{T' \smallsetminus T'_s}) \right).$$

This is well defined because any representative in (ξ, ξ') is φ compatible, so the right hand side is eventually constant. In addition, every $(\zeta, \zeta') \in W \times_\varphi W'$ is φ compatible, so satisfies

$$\lim_{s \to \infty} \left(\mathcal{C}(\xi|_{T \smallsetminus T_s}) - \mathcal{C}'(\xi'|_{T' \smallsetminus T'_s}) \right) \;=\; 0.$$

To see this, recall that there is a global bound on the leafwise length of the γ and γ' in ζ and ζ'. This, and the fact that there are only finitely many unpaired (α,γ) and (α',γ'), insures that for large s, every unpaired (α,γ) will have both Dom_γ and $\mathrm{Ran}_\gamma \subset T \smallsetminus T_s$, so $\mathcal{C}(\alpha - h_\gamma^*\alpha)$ will be zero, and similarly for every unpaired (α',γ'). Those (α,γ) and (α',γ') which are paired and appear in the integration, will have Dom_γ and/or $\mathrm{Ran}_\gamma \subset T \smallsetminus T_s$ with corresponding $\mathrm{Dom}_{\gamma'}$ and/or $\mathrm{Ran}_{\gamma'} \subset T' \smallsetminus T'_s$. In both cases, their integrals will cancel.

Remark 2.3. *Examples of such currents include the following.*

(1) *Invariant transverse measures Λ and Λ' on T and T' which are φ compatible as in* [BH23].
(2) *Suppose $\omega \in C^\infty(\wedge^*\nu^*)$ and $\omega' \in C^\infty(\wedge^*\nu'^*)$ are closed holonomy invariant forms on M and M' which are $\varphi-$compatible, i.e. $\varphi-$compatible basic cohomology classes,* [R58]. *They determine $\varphi-$compatible closed holonomy invariant Haefliger currents, also denoted ω and ω'. In particular,*

$$\langle(\xi,\xi'),(\omega,\omega')\rangle \;=\; \lim_{s\to\infty}\left(\int_{T\smallsetminus T_s}\xi\wedge\omega_T - \int_{T'\smallsetminus T'_s}\xi'\wedge\omega'_{T'}\right).$$

Here $\omega_T = \omega\,|_T$, which is well defined and is holonomy invariant, as is $\omega'_{T'}$.

Next, consider the 'infinite dimensional fiber bundle' $\mathcal{W} = C_c^\infty(\mathcal{S}\otimes E)$ over M whose 'fiber' over $x \in M$ is the space $\mathcal{W}_x = C_c^\infty(\widetilde{L}_x;\mathcal{S}\otimes E)$. Filter the space $\mathcal{M}$ of all sections of $\wedge T^*M \otimes End(\mathcal{W},\mathcal{W})$ over M by the subspaces $\mathcal{M}_i$ of sections of $\sum_{j\geqslant i}\wedge^j T^*M\otimes End(\mathcal{W},\mathcal{W})$. Filter the space $\mathcal{N}$ of all sections of $\wedge T^*M\otimes End_S(\mathcal{W},\mathcal{W})$ similarly, where $End_S(\mathcal{W},\mathcal{W})$ is the space of leafwise smoothing operators. An element $A \in \mathcal{M}$ assigns to each k vector $X \in \wedge^k TM_x$ an operator $A(X)$ on the space $\mathcal{W}_x$, that may be viewed as an (unbounded) operator on its L^2 completion or between any Sobolev completions. We say $A \in \mathcal{M}$ is bounded if

$$\|A\| \equiv \sup\{\|A(X)\|\,|\,X \in \wedge^*TM,\ \|X\|=1\} < \infty,$$

and it is smooth if it is a smooth section of $\wedge T^*M\otimes End(\mathcal{W},\mathcal{W})$. Denote by $\|\ \ \|_{r,s}$ the Sobolev r,s norm on $End(\mathcal{W}_x,\mathcal{W}_x)$. We say $A \in \mathcal{N}$ is bounded if for *each* r and s,

$$\|A\|_{r,s} \equiv \sup\{\|A(X)\|_{r,s}\,|\,X \in \wedge^*TM,\ \|X\|=1\} < \infty.$$

Finally, recall the Bismut superconnection $\mathbb{B}$ for F and E associated to the metric on M. See [B86, BV87, H95] for the details of its construction. It is a Dirac type odd operator on $C_c^\infty(\wedge\nu_s^*\otimes\mathcal{S}\otimes E)$, and $\mathbb{B}$ satisfies the usual properties (suitably interpreted) for a Bismut superconnection. In particular, the operator $\mathbb{B}$ may be written as

$$\mathbb{B} \;=\; \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2,$$

where each $\mathbb{B}_i$ is a uniformly bounded smooth differential operator, $\mathbb{B}_i \in \mathcal{M}_i$ and $\mathbb{B}_0 = D$.

For $t > 0$, denote by $\mathbb{B}(t)$ the Bismut superconnection associated to the metric obtained by scaling the original metric by $1/t$. By [BV87, H95],

2.4. $$\mathbb{B}(t) \;=\; \sqrt{t}D + \mathbb{B}_1 + \tfrac{1}{\sqrt{t}}\mathbb{B}_2 \;=\; \sqrt{t}D + B_{\sqrt{t}},$$

so,

2.5. $$\mathbb{B}(t)^2 \;=\; tD^2 + \sqrt{t}[D,\mathbb{B}_1] + \mathbb{B}_1^2 + \cdots \;=\; tD^2 - C_t,$$

where C_t is a smooth leafwise differential operator of order at most one, with uniformly bounded coefficients. As such, its Sobolev norm $||C_t||_{\ell,\ell-1}$ is uniformly bounded, see [H95] and [Sh92]. Notice also that C_t is nilpotent since it is in $\mathcal{M}_1$.

Remark 2.6. *Note that P_0, and $P_{(0,\epsilon)}$ are bounded elements of $\mathcal{N}$, and we are assuming that P_0 and, for sufficiently small ϵ, $P_{(0,\epsilon)}$ are smooth elements of $\mathcal{N}$.*

3. The Theorems

Denote by $AS(D_F)$ the Atiyah-Singer characteristic form for D_F, and similarly for $D_{F'}$. Then integrating over the leaves we get the topological index classes

$$\mathrm{Ind}_t(D) \;=\; \left[\int_F \mathrm{AS}(D_F)\right] \;\in\; \mathrm{H}_c^*(M/F), \;\; \mathrm{Ind}_t(D') \;=\; \left[\int_{F'} AS(D'_{F'})\right] \;\in\; \mathrm{H}_c^*(M'/F').$$

Note carefully that the elements of $\mathrm{H}_c^*(M/F)$ and $\mathrm{H}_c^*(M'/F')$ are not generally represented by compactly supported forms since they can have non-zero contributions from smooth compactly supported forms on infinitely many transversals. Because of our compatibility assumptions near infinity, the forms $AS(D_F)$ and $AS(D'_{F'})$ agree near infinity on M and M'. Hence integrating over the leaves yields a φ compatible pair $\left(\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'})\right)$ and the corresponding relative topological class

$$\mathrm{Ind}_t(D, D') \;=\; \left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'})\right] \;\in\; \mathrm{H}_c^*(M/F, M'/F'; \varphi).$$

Next, we introduce the analytic superconnection relative index class using the metric-rescaled Bismut superconnection $\mathbb{B}(t)$. For technical reasons, we will replace the variable $t \in [0, \infty)$ by $\beta(t)$, a smooth function with domain $[0, 1)$, which is increasing, with $\beta(t) = t$ near 0, and $\beta(t) = (1 - t)^{-1}$ near 1.

Definition 3.1. *The Bismut relative analytic Connes-Chern character is*

$$\mathrm{ch}_a(D, D') \;=\; \left[\mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right), \mathrm{STr}\left(\chi^t(\mathbb{B}'(\beta(t)))\right)\right] \;\in\; \mathrm{H}_c^*(M/F, M'/F'; \varphi).$$

For the definition of the Schwartz function χ^t, see Section 4. It is an approximation of $e^{-z^2/2}$, whose Fourier transform is compactly supported. That insures that $\mathrm{ch}_a(D, D')$ is defined since the operators $\chi^t(\mathbb{B}(\beta(t)))$ and $\chi^t(\mathbb{B}'(\beta(t)))$ will then have finite propagation, so will be φ related. We cannot use $e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}$ and $e^{-\frac{1}{2}\mathbb{B}'(\beta(t))^2}$ because, in general, they are not φ related. So the pair

$$\left(\mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right) \mathrm{STr}\left(\chi^t(\mathbb{B}'(\beta(t)))\right)\right)$$

is an element of $\mathcal{A}_c^*(M/F, M'/F'; \varphi)$ whose relative Haefliger class is the class $\mathrm{ch}_a(D, D')$. The notation is legitimate as this class will be shown to be independent of all the choices used to define the superconnections $\mathbb{B}$ and $\mathbb{B}'$.

For the Bismut absolute analytic Connes-Chern characters, we can use the usual heat operator and set (no need of finite propagation here)

$$\mathrm{ch}_a(D) = \mathrm{S}\mathfrak{Tr}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right) \in \mathrm{H}_c^*(M/F) \;\text{ and }\; \mathrm{ch}_a(D') = \mathrm{S}\mathfrak{Tr}\left(e^{-\frac{1}{2}\mathbb{B}'(\beta(t))^2}\right) \in \mathrm{H}_c^*(M'/F').$$

Indeed, in this absolute case, we also show in Section 4 that these characters are all well defined, but can as well be defined using such function χ^t, i.e. that we have

$$\mathrm{S}\mathfrak{Tr}\left(\chi^t(\mathbb{B}(\beta(t)))\right) = \mathrm{S}\mathfrak{Tr}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right) \;\in\; \mathrm{H}_c^*(M/F), \text{ and similarly on } (M', F').$$

The higher relative index theorem for (non-riemannian) foliations is as follows.

Theorem 3.2. *Suppose that (M, F), (M', F'), D and D' are as above. Then,*

$$\mathrm{ch}_a(D, D') = \mathrm{Ind}_t(D, D') \;\in\; \mathrm{H}_c^*(M/F, M'/F'; \varphi).$$

In the process of the proof, we shall also obtain the more familiar index formulae in absolute Haefliger cohomologies, which is now valid for all bounded geometry foliations, namely

$$\mathrm{ch}_a(D) = \mathrm{Ind}_t(D) \;\in\; \mathrm{H}_c^*(M/F) \;\text{ and }\; \mathrm{ch}_a(D') = \mathrm{Ind}_t(D') \;\in\; \mathrm{H}_c^*(M'/F').$$

Next, we relate of our higher relative index formula with the index bundles P_0 and P'_0, which we are assuming are ransversely smooth. Thus, we have

$$\mathrm{ch}_a(P_0) \;=\; \mathrm{S}\mathfrak{Tr}\Big(P_0 \exp\left(\frac{-(\delta(P_0))^2}{2\pi i}\right)\Big) \;\in\; \mathrm{H}_c^*(M/F),$$

where δ is a certain differential operator, and similarly for P_0'. For details see [BH08], Section 3. For simplicity of notation, we will uniformly suppress the constant $2\pi i$ in what follows.

The gap at zero non-Riemannian Foliation Higher Relative Index Theorem is the following.

Theorem 3.3. *Assume that M, so also M', has sub-exponential growth, and that the holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Hausdorff. Assume furthermore that:*

(1) *P_0 and P_0' are transversely smooth with transverse derivatives $\mathbb{B}_1(P_0)$ and $\mathbb{B}_1'(P_0')$ bounded and leafwise smoothing;*
(2) *(gap condition) for sufficiently small ϵ, $P_{(0,\epsilon)} = P'_{(0,\epsilon)} = 0$;*
(3) *P_0 satisfies $\displaystyle\int_M \operatorname{tr}(P_0)\,dx \;<\; \infty$, and similarly for P_0';*

Then, for $\mathcal{C}$ and $\mathcal{C}'$ closed bounded φ compatible continuous holonomy invariant Haefliger currents, the pairings $\langle \operatorname{ch}_a(P_0), \mathcal{C}\rangle$ and $\langle \operatorname{ch}_a(P_0'), \mathcal{C}'\rangle$ are well defined, and

$$\langle \operatorname{ch}_a(P_0), \mathcal{C}\rangle - \langle \operatorname{ch}_a(P_0'), \mathcal{C}'\rangle \;=\; \big\langle \Big[\int_F \operatorname{AS}(D_F), \int_{F'} AS(D_{F'})\Big], (\mathcal{C}, \mathcal{C}')\big\rangle.$$

We will actually prove that under the our assumptions,

$$\big\langle \operatorname{ch}_a(D, D'), (\mathcal{C}, \mathcal{C}')\big\rangle \;=\; \langle \operatorname{ch}_a(P_0), \mathcal{C}\rangle - \langle \operatorname{ch}_a(P_0'), \mathcal{C}'\rangle,$$

and then apply Theorem 3.2. Note that the leafwise smoothness of the projections is automatic because of the bounded geometry. As an immediate corollary, we have that if $P_0 = P_0' = 0$, then,

$$\big\langle [\int_F \operatorname{AS}(D_F), \int_{F'} AS(D_{F'})], (\mathcal{C}, \mathcal{C}')\big\rangle = 0.$$

The Novikov-Shubin invariants $NS(D)$ of D are greater than $k \geqslant 0$ provided that there is $\tau > k$ so that

$$\operatorname{Tr}(P_{(0,\epsilon)}) \text{ is } \mathcal{O}(\epsilon^\tau) \text{ as } \epsilon \to 0,$$

and similarly for D'. A Haefliger form Ψ depending on ϵ is $\mathcal{O}(\epsilon^\tau)$ as $\epsilon \to 0$ means that there is a representative $\psi \in \Psi$ defined on a transversal T, and a constant $c_\psi > 0$, so that the function on T, $\|\psi\|_T \leqslant c_\psi \epsilon^\tau$ as $\epsilon \to 0$. Here $\|\;\;\|_T$ is the pointwise norm on forms on the transversal T induced from the metric on M.

Definition 3.4. *A function $f(t)$ has super polynomial decay if for all $m \in \mathbb{Z}_+$,*

$$f(t) \;=\; \mathcal{O}(t^{-m}), \quad \text{as } t \to \infty.$$

A function $g(t)$ has sub-super polynomial growth, if $\lim_{t\to\infty} f(t)g(t) \;=0$, for all $f(t)$ with super polynomial decay.

The general non-Riemannian Foliation Higher Relative Index Theorem is the following.

Theorem 3.5. *Assume that M, so also M', has sub-super polynomial growth, and that the holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$ are Hausdorff. Assume furthermore that:*

(1) *for sufficiently small ϵ, P_0 and $P_{(0,\epsilon)}$ are transversely smooth, with bounded leafwise smoothing transverse derivatives $\mathbb{B}_1(P_0)$ and $\mathbb{B}_1(P_{(0,\epsilon)})$, and similarly for P_0' and $P'_{(0,\epsilon)}$;*
(2) *for ϵ sufficiently small ϵ, $P_{[0,\epsilon)}$ satisfies $\displaystyle\int_M \operatorname{tr}\left(P_{[0,\epsilon)}\right) \;<\; \infty$, and similarly for $P'_{[0,\epsilon)}$;*
(3) *The Novikov-Shubin invariants $NS(D)$ and $NS(D')$ are greater than $3\operatorname{codim}(F)$.*

Then, for $\mathcal{C}$ and $\mathcal{C}'$ closed bounded φ compatible continuous holonomy invariant Haefliger currents, the pairings $\langle \operatorname{ch}_a(P_0), \mathcal{C}\rangle$ and $\langle \operatorname{ch}_a(P_0'), \mathcal{C}'\rangle$ are well defined, and

$$\langle \operatorname{ch}_a(P_0), \mathcal{C}\rangle - \langle \operatorname{ch}_a(P_0'), \mathcal{C}'\rangle \;=\; \big\langle \Big[\int_F \operatorname{AS}(D_F), \int_{F'} AS(D_{F'})\Big], (\mathcal{C}, \mathcal{C}')\big\rangle.$$

The condition on the Novikov-Shubin invariants can be improved, that is the constant 3 can be replaced by a smaller positive number. See the comments before Lemma 10 of [HL99]. It cannot be less that 1/2, unless additional conditions are imposed, [BHW14].

The growth condition on M and M' are needed for the proof that the pairings will be as claimed. As above, we will prove that under our assumptions,

$$\langle \mathrm{ch}_a(D, D'), (\mathcal{C}, \mathcal{C}') \rangle \;=\; \langle \mathrm{ch}_a(P_0), \mathcal{C} \rangle - \langle \mathrm{ch}_a(P'_0), \mathcal{C}' \rangle,$$

and then apply Theorem 3.2. The proofs of Theorems 3.3 and 3.5 also show the following absolute index formula for bounded geometry foliations.

Corollary 3.6. *Under the assumptions of Theorems 3.3 or 3.5 on the foliation (M, F),*

$$\mathrm{ch}_a(D) \;=\; \mathrm{ch}_a(P_0) \;=\; \left[\int_F \mathrm{AS}(D_F)\right] \;\textit{in}\; \mathrm{H}^*_c(M/F).$$

4. Proofs of the Theorems

A basic fact we will use repeatedly is that uniformly bounded geometry and standard estimates insure that any bounded $A \in \mathcal{N}$ has Schwartz kernel K_A which is leafwise smooth and pointwise uniformly bounded, as are all its leafwise derivatives. In particular, for sufficiently large ℓ, and for any $x \in M$ and $\gamma, \xi \in \widetilde{L}_x$, the Schwartz kernel K_A of A satisfies $\|K_A(\gamma, \xi)\| \leqslant c_\ell \|A\|_{-\ell,\ell}$ where c_ℓ is a constant depending only on ℓ. Thus estimates on $\|A\|_{-\ell,\ell}$ translate directly into uniform pointwise estimates on $\mathrm{Tr}(A)$ and $\mathrm{STr}(A)$. See the proof of Theorem 2.3.9 and the statement of Theorem 2.3.13 of [HL90]. In particular, if a family $A_t \in \mathcal{N}$ satisfies $\|A_t\|_{-\ell,\ell} = \mathcal{O}(t^\alpha)$ as $t \to a$, this implies $|| \mathrm{Tr}(A_t)|| = \mathcal{O}(t^\alpha)$ and $|| \mathrm{STr}(A_t)|| = \mathcal{O}(t^\alpha)$, uniformly pointwise as $t \to a$.

Recall that $\beta(t)$ a smooth function with domain $[0, 1)$, which is increasing, with $\beta = t$ near 0, and $\beta(t) = (1 - t)^{-1}$ near 1. We consider the operator (see Equation 2.5),

$$e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2} \;=\; e^{(-\beta(t)D^2 + C_{\beta(t)})/2},$$

and proceed as in [HL99, BH24] with the Volterra series for the exponential of a perturbed operator to write as in [BGV92] p. 78,

$$\textbf{4.1.}\qquad e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2} = \sum_{k=0}^{\infty} \int_{\Delta_k} 2^{-k} e^{-x_1\beta(t)D^2/2} C_{\beta(t)} e^{-x_2\beta(t)D^2/2} C_{\beta(t)} \cdots C_{\beta(t)} e^{-x_{k+1}\beta(t)D^2/2} dx_k \ldots dx_1,$$

where Δ_k is the standard k simplex $\Delta_k = \{(x_1, \ldots, x_{k+1}) \,|\, x_i \geqslant 0, \sum x_i = 1\}$. Recall that $C_{\beta(t)}$ is a smooth leafwise differential operator of order at most one, whose Sobolev norm $||C_{\beta(t)}||_{\ell,\ell-1}$ is uniformly bounded, which is also nilpotent since it is in $\mathcal{M}_1$. Thus the sum is finite, in particular $k \leqslant n = \dim M$, (in fact, $k \leqslant q = \mathrm{codim}(F)$, see the proof of Lemma 13 of [HL99]). This series allows us to extend results for D to $\mathbb{B}(\beta(t))$.

Next, we construct a family of operators with finite propagations which converges to $e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}$. To do so, we follow Section 5 of [BH24]. Denote the Fourier Transform of a real function g by $\widehat{g}$ and $\mathrm{FT}(g)$, and its inverse transform by $\widetilde{g}$ and $\mathrm{FT}^{-1}(g)$. Set $g_\lambda(z) = g(\lambda z)$, for non-zero $\lambda \in \mathbb{R}$. We have the following facts:

$$\mathrm{FT}(g_\lambda) = \frac{1}{\lambda}\,\mathrm{FT}(g)_{\frac{1}{\lambda}};\; \mathrm{FT}(g \star h) = \sqrt{2\pi}\,\mathrm{FT}(g)\,\mathrm{FT}(h);\; \text{and}\; \mathrm{FT}(\widehat{g}) = \mathrm{FT}^{-1}(\widehat{g}) = g,\; \text{if } g \text{ is even.}$$

As usual, we denote by $\star$ the convolution product of integrable functions. Fix a smooth even non-negative function ψ supported in $[-1, 1]$, which equals 1 on $[-1/4, 1/4]$, is non-increasing on $\mathbb{R}_+$, and whose integral over $\mathbb{R}$ is 1. Note that $\mathrm{FT}(\widehat{\psi}) = \psi$ since ψ is even. The family $\frac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}}$ is an approximate identity when acting on a Schwartz function f by convolution since, up to the constant $\sqrt{2\pi}$ which we systematically ignore,

$$\textbf{4.2.}\qquad \tfrac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f \;=\; \mathrm{FT}^{-1}(\mathrm{FT}(\tfrac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f)) \;=\; \mathrm{FT}^{-1}(\psi_{\sqrt{t}}\widehat{f}) \to \widetilde{\widehat{f}} \;=\; f,$$

in the Schwartz topology as $t \to 0$. Denote by $\alpha(t)$ a smooth function with domain $[0,1]$, with $\alpha(t) = t$ near 0, $\alpha(t) = 1-t$ near 1, is increasing on $[0,1/2]$ and symmetric about $t = 1/2$. Set $e(z) = e^{-z^2/2}$, and for $t \in (0,1)$, set

$$\chi^t \;=\; \frac{1}{\sqrt{\alpha(t)}} \widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star e.$$

Note that $e^{-\mathbb{B}(\beta(t))^2/2}$, by [H95] and the Volterra series, and $\chi^t(\mathbb{B}(\beta(t)))$, by [Roe87], and all their derivatives are bounded smooth elements of $\mathcal{N}$. Recall that for any even function f, the operator $f(\mathbb{B}(\beta(t)))$ is defined by the Fourier integral $f(\mathbb{B}(\beta(t))) = \displaystyle\int_{\mathbb{R}} \widehat{f}(x)\cos(x\mathbb{B}(\beta(t)))dx$, where for a given section g_0, $g_t(x) = \cos(x\mathbb{B}(\beta(t)))g_0$ is the unique solution of the equation $(\partial_x^2 + \mathbb{B}(\beta(t))^2)g_t(x) = 0$ with initial data $g_t(0) = g_0$ and $(\partial_x g_t)(0) = 0$.

Lemma 4.3. *The limits in the Schwartz topology as $t \to 0,1$ of $\chi^t(z) \;-\; e^{-z^2/2}$ are zero. So, as $t \to 0,1$, the Schwartz kernel of $\chi^t(\mathbb{B}(\beta(t))) \;-\; e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}$ converges uniformly pointwise to zero.*

In addition, $\chi^t(\mathbb{B}(\beta(t)))$ has finite propagation bounded by a multiple of $\sqrt{\beta(t)/\alpha(t)}$.

By the remarks above we have the first two statements. The proof of finite propagation proceeds as in [BH24], using the Volterra series.

Thus, we have the following.

Proposition 4.4. *Under the assumptions in Theorem 3.2,*

$$\lim_{t\to 0} \operatorname{Str}(\chi^t(\mathbb{B}(\beta(t)))) \;=\; \lim_{t\to 0} \operatorname{Str}(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}) \;=\; \operatorname{AS}(D_F),$$

uniformly pointwise on M, and similarly for $\mathbb{B}'(\beta(t))$ associated to D'. Under the assumptions in Theorems 3.3 and 3.5,

$$\lim_{t\to 1} \operatorname{Str}(\chi^t(\mathbb{B}(\beta(t)))) \;=\; \lim_{t\to 1} \operatorname{Str}(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}) \;=\; \operatorname{Str}(P_0 e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0),$$

uniformly pointwise on M, and similarly for $\mathbb{B}'(\beta(t))$.

These equalities are from Lemma 4.3 and [H95] and [HL99], respectively. From Theorem 5.2 of [BH08] we have

$$\operatorname{STr}(P_0 e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0) = \operatorname{STr}\Big(P_0 \exp\big(-(\delta P_0)^2\big)\Big) \;\in\; \mathcal{A}_c^*(T),$$

and

$$\Big[\operatorname{STr}\Big(P_0 \exp\big(-(\delta P_0)^2\big)\Big)\Big] = \operatorname{S}\mathfrak{Tr}\Big(P_0 \exp\big(-(\delta P_0)^2\big)\Big) = \operatorname{ch}_a(P_0) \in H_c^*(M/F).$$

As $\displaystyle\int_F$ involves integrating over compact subsets, we may interchange the limits with $\displaystyle\int_F$, thus extending Proposition 4.4 to $\operatorname{STr}(\chi^t(\mathbb{B}(\beta(t))))$, where $\operatorname{AS}(D_F)$ is replaced by $\displaystyle\int_F \operatorname{AS}(D_F)$, and $\operatorname{Str}(P_0 e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0)$ is replaced by $\operatorname{STr}(P_0 e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0)$.

We show below that $\operatorname{STr}(\chi^t(\mathbb{B}(\beta(t))))$ is a closed Haefliger form and that $\operatorname{S}\mathfrak{Tr}(\chi^t(\mathbb{B}(\beta(t))))$ is independent of t. The same proof works for $\operatorname{STr}(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2})$ and $\operatorname{S}\mathfrak{Tr}(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2})$. Thus we have Theorem 3.2.

Now for the proofs of Theorems 3.3 and 3.5.

Lemma 4.5. *If f_1 and f_2 are even Schwartz functions, the operators $\mathbb{B}(t)$, $f_1(\mathbb{B}(t))$ and $f_2(\mathbb{B}(t))$ commute, and*

$$(f_1 f_2)(\mathbb{B}(t)) = f_1(\mathbb{B}(t)) f_2(\mathbb{B}(t)).$$

Proof. Recall that $f(\mathbb{B}(t)) = \displaystyle\int_{\mathbb{R}} \widehat{f}(x)\cos(x\mathbb{B}(t))dx$. We first show this for $\cos(x\mathbb{B}(t))$ and $\cos(y\mathbb{B}(t))$. For a fixed $u_0 \in C_c^\infty(\nu_s^* \wedge \mathcal{S} \otimes E)$ and $y_0 \in \mathbb{R}$, the section v_t given by

$$v_t(x) = \cos(y_0\mathbb{B}(t))\cos(x\mathbb{B}(t))u_0,$$

satisfies the equation $(\partial_x^2 + \mathbb{B}(t)^2)v_t(x) = 0$ with initial data $v_t(0) = \cos(y_0\mathbb{B}(t))u_0$ and $(\partial_x v_t)(0) = 0$. By the uniqueness theorem for this equation, we have

$$\cos(y_0\mathbb{B}(t))\cos(x\mathbb{B}(t))u_0 = v_t(x) = \cos(x\mathbb{B}(t))\Big(v_t|_{x=0}\Big) = \cos(x\mathbb{B}(t))\cos(y_0\mathbb{B}(t))u_0.$$

A similar argument gives $\mathbb{B}(t)\cos(x\mathbb{B}(t)) = \cos(x\mathbb{B}(t))\mathbb{B}(t)$.

The Fourier transform of $f_1 f_2$ is $\widehat{f}_1 \star \widehat{f}_2$, so

$$(f_1 f_2)(\mathbb{B}(t)) \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}_1(y)\widehat{f}_2(x-y)\cos(x\mathbb{B}(t))dydx \;=\; \int_{\mathbb{R}} \widehat{f}_1(y)\left(\int_{\mathbb{R}} \widehat{f}_2(z)\cos((y+z)\mathbb{B}(t))dz\right)dy.$$

By the uniqueness of the solution to the wave equation with initial data,

$$\cos((y+z)\mathbb{B}(t)) \;=\; \cos(y\mathbb{B}(t))\cos(z\mathbb{B}(t)) - \sin(y\mathbb{B}(t))\sin(z\mathbb{B}(t)).$$

Since sin is odd and $\widehat{f}_1$ and $\widehat{f}_2$ are even, we have

$$\int_{\mathbb{R}} \widehat{f}_1(y)\left(\int_{\mathbb{R}} \widehat{f}_2(z)\cos((y+z)\mathbb{B}(t))dz\right)dy \;=$$

$$\int_{\mathbb{R}} \widehat{f}_1(y)\cos(y\mathbb{B}(t))\left(\int_{\mathbb{R}} \widehat{f}_2(z)\cos(z\mathbb{B}(t))dz\right)dy \;-\; \int_{\mathbb{R}} \widehat{f}_1(y)\sin(y\mathbb{B}(t))\left(\int_{\mathbb{R}} \widehat{f}_2(z)\sin(z\mathbb{B}(t))dz\right)dy \;=$$

$$\int_{\mathbb{R}} \widehat{f}_1(y)\cos(y\mathbb{B}(t))\left(\int_{\mathbb{R}} \widehat{f}_2(z)\cos(z\mathbb{B}(t))dz\right)dy \;=\; f_1(\mathbb{B}(t))f_2(\mathbb{B}(t)).$$

□

For definiteness and simplicity of notation, we will assume that our Haefliger currents are closed holonomy invariant forms $\omega \in C^\infty(\wedge^*\nu^*)$ and $\omega' \in C^\infty(\wedge^*\nu'^*)$, which are $\varphi-$compatible. In that case, for example,

$$\langle \mathrm{ch}_a(P_0), \omega\rangle \;=\; \int_T \mathrm{ch}_a(P_0)\wedge\omega_T,$$

where ω_T is the Haefliger form determined by ω.

Proposition 4.6. *For $t \in (0,1)$, the Haefliger forms $\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))))$ and $\mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))$ are closed and φ compatible, so represent the class*

$$\big[\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))), \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))\big] \;\in\; \mathrm{H}_c^*(M/F, M'/F'; \varphi).$$

Moreover, the pairing,

$$\Big\langle\big[\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))), \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))\big], [\omega,\omega']\Big\rangle$$

is independent of the parameter t.

Proof. Note that we may assume that $\chi^t(z) = \widetilde{\chi}^t(z)^2$, where $\widetilde{\chi}^t(z)$ is an even positive Schwartz function, and has the same properties as $\chi^t(z)$. If this is not the case, we may systematically replace $\chi^t(z)$ by $\widetilde{\chi}^t(z)^2$ where

$$\widetilde{\chi}^t(z) \;=\; \left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star \widetilde{e}\right](z),$$

and $\widetilde{e}(z) = e^{-z^2/4}$.

Equation 2.4 in this case is

$$\mathbb{B}(\beta(t)) \;=\; \beta(t)^{1/2}D + \mathbb{B}_1 + \beta(t)^{-1/2}\mathbb{B}_2,$$

where D and $\mathbb{B}_2$ are leafwise differential operators. From (Lemma 4.5), we have,

$$\big[\mathbb{B}(\beta(t)), \chi^t(\mathbb{B}(\beta(t)))\big] = 0.$$

Note that,

$$[\beta(t)^{1/2}D + \beta(t)^{-1/2}\mathbb{B}_2, \chi^t(\mathbb{B}(\beta(t)))] =$$

$$\Big[(\beta(t)^{1/2}D + \beta(t)^{-1/2}\mathbb{B}_2)\tilde{\chi}^t(\mathbb{B}(\beta(t))) + \tilde{\chi}^t(\mathbb{B}(\beta(t)))(\beta(t)^{1/2}D + \beta(t)^{-1/2}\mathbb{B}_2)\;,\; \tilde{\chi}^t(\mathbb{B}(\beta(t)))\Big].$$

Since this is a graded commutator of leafwise smoothing operators, its supertrace is zero. So,

$$\mathrm{STr}([\mathbb{B}_1, \chi^t(\mathbb{B}(\beta(t)))] = \mathrm{STr}(\big[\mathbb{B}(\beta(t)), \chi^t(\mathbb{B}(\beta(t)))\big]) = 0.$$

Denote by ∇^B a Bott connection on ν. Then $\mathbb{B}_1 = \sum_{i=1}^q \omega_i \wedge \nabla^B_{X_i}$, where $(X_1, .., X_q)$ and $(\omega_1, .., \omega_q)$ are dual bases of ν and ν^*. See [H95]. It is immediate that for a smooth form κ,

$$d_T \int_F \kappa \;=\; \int_F \mathbb{B}_1(\kappa) \;=\; \int_F [\mathbb{B}_1, \kappa].$$

Thus,

$$d_T(\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))))) = \mathrm{STr}([\mathbb{B}_1, \chi^t(\mathbb{B}(\beta(t)))]) = 0,$$

and we have the first result.

For the t independence of the pairing, consider the smooth foliation $F_{\mathbb{R}}$ of $M \times (0,1)$ whose leaves are $L \times \{t\}$. The metric on $M \times (0,1)$ is the metric corresponding to $\beta(t)$ on $M \times \{t\}$ product with the usual metric on $\mathbb{R}$. The resulting Bismut superconnection for $F_{\mathbb{R}}$ is

$$\mathbb{B}_{\mathbb{R}} = \mathbb{B}\left(\beta(t)\right) + dt\frac{\partial}{\partial t},$$

and $\mathbb{B}^2_{\mathbb{R}} = \mathbb{B}\left(\beta(t)\right)^2 + dt \wedge \frac{\partial \mathbb{B}(\beta(t))}{\partial t}$. Now the smooth bounded Haefliger form $\mathrm{STr}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ in $\mathcal{A}^*_c(T_{\mathbb{R}})$ can be written for each $t \in (0,1)$ as $\gamma_t + dt \wedge \delta_t$ with γ_t and δ_t Haefliger forms for F. Here we obviously take $T_{\mathbb{R}} = T \times (0,1)$. It is clear that γ_t is given by killing all the terms appearing in $\mathrm{STr}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ containing dt, so $\gamma_t = \mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right)$. Since $\mathrm{STr}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ and γ_t are closed Haefliger forms for the foliations $F_{\mathbb{R}}$ and F, respectively, we have

$$0 = d_{T_{\mathbb{R}}} \mathrm{STr}(\chi^t(\mathbb{B}_{\mathbb{R}})) = d_{T_{\mathbb{R}}}(\gamma_t + dt \wedge \delta_t) = dt \wedge \frac{\partial \gamma_t}{\partial t} \;-\; dt \wedge d_T\delta_t = dt \wedge (\frac{\partial \gamma_t}{\partial t} \;-\; d_T\delta_t).$$

So,

4.7. $$d_T\delta_t = \frac{\partial \gamma_t}{\partial t} = \frac{\partial}{\partial t} \mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right).$$

It is also clear that,

4.8. $$\delta_t \;=\; i_{\partial/\partial t} \mathrm{STr}(\chi^t(\mathbb{B}_{\mathbb{R}})) \;=\; \int_F i_{\partial/\partial t} \mathrm{Str}(\chi^t(\mathbb{B}_{\mathbb{R}})),$$

where $i_{\partial/\partial t}$ is the interior product of the vector field $\partial/\partial t$. We have the same results for γ'_t, and δ'_t. Note that $i_{\partial/\partial t} \mathrm{Str}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ and $i_{\partial/\partial t} \mathrm{Str}(\chi^t(\mathbb{B}'_{\mathbb{R}}))$ are $\varphi-$related, so also are $d_T\delta_t$ and $d_{T'}\delta'_t$.

Since ω and ω' are closed $\varphi-$related forms, we have,

$$\frac{\partial}{\partial t}\langle(\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))), \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))), (\omega, \omega')\rangle \;=$$

$$\lim_{r\to\infty}\left(\int_{T\smallsetminus T_r} \frac{\partial}{\partial t} \mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))) \wedge \omega_T \;-\; \int_{T'\smallsetminus T'_r} \frac{\partial}{\partial t} \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t)))) \wedge \omega_{T'}\right) \;=$$

$$\lim_{r\to\infty}\left(\int_{T\smallsetminus T_r} d_T\delta_t \wedge \omega_T \;-\; \int_{T'\smallsetminus T'_r} d_{T'}\delta'_t \wedge \omega_{T'}\right) \;=\; \lim_{r\to\infty}\left(\int_{T\smallsetminus T_r} d_T(\delta_t \wedge \omega_T) \;-\; \int_{T'\smallsetminus T'_r} d_{T'}(\delta'_t \wedge \omega_{T'})\right) \;=$$

$$\lim_{r\to\infty}\left(\int_{\partial T_r} \delta_t \wedge \omega_T \;-\; \int_{\partial T'_r} \delta'_t \wedge \omega_{T'}\right) \;=\; 0,$$

as the two integrals agree for r sufficiently large. $\square$

Note that $\mathrm{ch}_a(D,D')$ is now well defined. This is because $\mathrm{Str}(\chi^t(\mathbb{B}(\beta(t))))$ and $\mathrm{Str}(\chi^t(\mathbb{B}'(\beta(t))))$, $\mathrm{Str}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ and $\mathrm{Str}(\chi^t(\mathbb{B}'_{\mathbb{R}}))$, and $i_{\partial/\partial t}\,\mathrm{Str}(\chi^t(\mathbb{B}_{\mathbb{R}}))$ and $i_{\partial/\partial t}\,\mathrm{Str}(\chi^t(\mathbb{B}'_{\mathbb{R}}))$ are all $\varphi-$related. So, their corresponding Haefliger forms are also $\varphi-$related, and Equations 4.7 and 4.8 give that

$$\mathrm{ch}_a(D,D') \;=\; \left[\mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right), \mathrm{STr}\left(\chi^t(\mathbb{B}'(\beta(t)))\right)\right] \;\in\; \mathrm{H}^*_c(M/F, M'/F';\varphi).$$

is independent of t, as was promised, completing the proof of Theorem 3.2.

Lemma 4.9.

$$\lim_{t\to 0}\left\langle\left[\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))), \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))\right], [\omega,\omega']\right\rangle \;=\; \left\langle\left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D_{F'})\right], [\omega,\omega']\right\rangle.$$

Proof. We may assume that the two integrands agree on the co-compact subsets $M(s_0)$ and $M'(s_0)$. Then we have,

$$\lim_{t\to 0}\left\langle\left(\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))), \mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))\right), (\omega,\omega')\right\rangle \;=$$

$$\lim_{t\to 0}\lim_{s\to\infty}\left(\int_{T\smallsetminus T_s}\mathrm{STr}\left(\chi^t(\mathbb{B}(\beta(t)))\right)\wedge\omega_T \;-\; \int_{T'\smallsetminus T'_s}\mathrm{STr}\left(\chi^t(\mathbb{B}'(\beta(t)))\right)\wedge\omega'_{T'}\right) \;=$$

$$\lim_{t\to 0}\left(\int_{M\smallsetminus M(s_0)}\mathrm{Str}\left(\chi^t(\mathbb{B}(\beta(t)))\right)\wedge\omega \;-\; \int_{M'\smallsetminus M'(s_0)}\mathrm{Str}\left(\chi^t(\mathbb{B}(\beta(t)))\right)\wedge\omega'\right) \;=$$

$$\int_{M\smallsetminus M(s_0)}\lim_{t\to 0}\mathrm{Str}\left(\chi^t(\mathbb{B}(\beta(t)))\right)\wedge\omega \;-\; \int_{M'\smallsetminus M'(s_0)}\lim_{t\to 0}\mathrm{Str}\left(\chi^t(\mathbb{B}(\beta(t)))\right)\wedge\omega' \;=$$

$$\int_{M\smallsetminus M(s_0)}\lim_{t\to 0}\mathrm{Str}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right)\wedge\omega \;-\; \int_{M'\smallsetminus M'(s_0)}\lim_{t\to 0}\mathrm{Str}\left(e^{-\frac{1}{2}\mathbb{B}'(\beta(t))^2/2}\right)\wedge\omega' \;=$$

$$\int_{\mathcal{K}} AS(D_F)\wedge\omega - \int_{\mathcal{K}'} AS(D'_{M'})\wedge\omega' \;=\; \langle([\int_F \mathrm{AS}(D_F), \int_{F'} AS(D_{F'}))], (\omega,\omega')\rangle.$$

As we are integrating over compact subsets, we may interchange the limits with the integrations, and we may change the integrands thanks to Lemma 4.3. ☐

Given Lemma 4.9, to prove Theorems 3.3 and 3.5, we need to show that

$$\lim_{t\to 1}\left\langle\left(\mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}(\beta(t)))), \mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}'(\beta(t))))\right), (\omega,\omega')\right\rangle \;=\; \langle(\mathrm{ch}_a(P_0), \mathrm{ch}_a(P'_0)), (\omega,\omega')\rangle.$$

Now for the proof of Theorem 3.3. Denote by Q_ϵ the spectral projection of D^2 associated to the interval $[\epsilon,\infty)$.

Proposition 4.10. *Under the assumptions in Theorem 3.3:*

(1) $\mathrm{STr}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right) \;=\; \mathrm{STr}(P_0 e^{-((P_0\mathbb{B}_1P_0)^2/2+\mathcal{O}(t^{-1/2}))}P_0) \;+\; \mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon)$;

(2) $\lim_{t\to 1}\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon) = 0$, *uniformly exponentially pointwise;*

(3) $\displaystyle\lim_{t\to 1}\int_{T\smallsetminus T_{\beta(t)}}\mathrm{STr}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right)\wedge\omega_T = \int_T \mathrm{STr}(P_0 e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0)\wedge\omega_T \;=\; \langle\mathrm{ch}_a(P_0),\omega\rangle.$

The same holds for $\mathbb{B}'(\beta(t))$ and $\omega'_{T'}$ associated to D'.

Proof. These follow directly from the arguments on pp. 188-195 of [HL99]. When discussing the material in [HL99], we will adopt the notation there, in particular, $t \in (0,\infty)$. Thus, we are assuming that $P_{(0,\epsilon)} = 0$ and that ϵ is fixed, that is, it is not a function of t. Next, first replace the term $t^{-1/a}$ by ϵ where appropriate, that is where ϵ had been replaced by $t^{-1/a}$. Remove any remaining term involving a, and replace the term α by $-1/2$. More specifically, now $||G_\epsilon||_{s,s} \leqslant \epsilon^{-1}$, $A_{\epsilon,t} = 0$,

$$\bar{B}_{\epsilon,t} = (P_0 + Q_\epsilon)\mathbb{B}(t)(P_0+Q_\epsilon) = \mathbb{B}(t),$$

and $T_{\epsilon,t} = Q_\epsilon\mathbb{B}(t)Q_\epsilon$.

The equation just before Lemma 10, where $\nabla = P_0\mathbb{B}_1P_0$, gives
$$\mathbb{B}(t)^2 = \psi_t g_\epsilon^{-1}\psi_t^{-1}A\psi_t g_\epsilon \psi_t^{-1},$$
where
$$A = \begin{bmatrix} (P_0\mathbb{B}_1P_0)^2 + \mathcal{O}(t^{-1/2}) & 0 \\ 0 & (Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2 + \mathcal{O}(t^0) \end{bmatrix} = \begin{bmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{bmatrix},$$
ψ_t multiplies a section of $\wedge^k T^*M \otimes \mathrm{End}(\mathcal{W},\mathcal{W})$ by $t^{-k/2}$, and g_ϵ is a measurable section of $\mathcal{M}$, with $g_\epsilon - \mathrm{I}$ and $g_\epsilon^{-1} - \mathrm{I} \in \mathcal{N}_1$.

Lemma 4.11. *(Lemma 10 of [HL99]). We may assume without loss of generality that* $\psi_t g_\epsilon \psi_t^{-1} = I = \psi_t g_\epsilon^{-1}\psi_t^{-1}$.

Proof. Using the notation of [HL99] we have
$$\psi_t g_{n+1}\psi_t^{-1} = \mathrm{I} + \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \quad \text{and} \quad \psi_t g_{n+1}^{-1}\psi_t^{-1} = \mathrm{I} + \begin{bmatrix} g_{1,1}^{-1} & g_{1,2}^{-1} \\ g_{2,1}^{-1} & g_{2,2}^{-1} \end{bmatrix}$$
where each matrix entry is $\mathcal{O}(t^{-\frac{1}{2}})$. We show below that the Schwartz kernel of $Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon \to 0$ exponentially pointwise as $t \to \infty$. So $e^{-A_{1,1}} = \mathcal{O}(t^0)$ and we may assume that $e^{-A_{2,2}} = \mathcal{O}(e^{-t})$. In addition, the matrix entries with a subscript 1 contain a P_0, and those with a subscript 2 contain a Q_ϵ. For example, $g_{1,2}$ contains both P_0 and Q_ϵ, but $g_{2,2}$ only contains Q_ϵ.

Next, the long equation on p. 193 becomes
$$\mathrm{STr}(e^{-\mathbb{B}_t^2}) = \mathrm{STr}(\psi_t g_{n+1}^{-1}\psi_t^{-1}e^{-A}\psi_t g_{n+1}\psi_t^{-1}) =$$
$$\mathrm{STr}\left(\left(\mathrm{I} + \begin{bmatrix} g_{1,1}^{-1} & g_{1,2}^{-1} \\ g_{2,1}^{-1} & g_{2,2}^{-1} \end{bmatrix}\right) e^{-A}\left(\mathrm{I} + \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix}\right)\right) =$$
$$\mathrm{STr}(e^{-A}) + \mathrm{STr}(\mathcal{O}(t^{-\frac{1}{2}})e^{-A}) + \mathrm{STr}(e^{-A}\mathcal{O}(t^{-\frac{1}{2}})) + \mathrm{STr}(\mathcal{O}(t^{-\frac{1}{2}})e^{-A}\mathcal{O}(t^{-\frac{1}{2}})) =$$
$$\mathrm{STr}(e^{-A}) + \mathrm{STr}\left(\begin{bmatrix} \mathcal{O}(t^{-\frac{1}{2}}) & \mathcal{O}(t^{-\frac{1}{2}})\mathcal{O}(e^{-t})) \\ \mathcal{O}(t^{-\frac{1}{2}}) & \mathcal{O}(t^{-\frac{1}{2}})\mathcal{O}(e^{-t})) \end{bmatrix}\right) + \mathrm{STr}\left(\begin{bmatrix} \mathcal{O}(t^{-\frac{1}{2}}) & \mathcal{O}(t^{-\frac{1}{2}}) \\ \mathcal{O}(e^{-t})\mathcal{O}(t^{-\frac{1}{2}}) & \mathcal{O}(e^{-t})\mathcal{O}(t^{-\frac{1}{2}}) \end{bmatrix}\right) +$$
$$\mathrm{STr}\left(\begin{bmatrix} \mathcal{O}(t^{-1}) + \mathcal{O}(t^{-1})\mathcal{O}(e^{-t}) & \mathcal{O}(t^{-1}) + \mathcal{O}(t^{-1})\mathcal{O}(e^{-t}) \\ \mathcal{O}(t^{-1}) + \mathcal{O}(t^{-1})\mathcal{O}(e^{-t}) & \mathcal{O}(t^{-1}) + \mathcal{O}(t^{-1})\mathcal{O}(e^{-t}) \end{bmatrix}\right).$$
The important thing to note here is that all the entries in the last three terms either are at least $\mathcal{O}(t^{-\frac{1}{2}})$ and contain the operator P_0, or have the entry $\mathcal{O}(e^{-t})$. We can ignore the terms of the first type because they are dominated by a multiple of the form $\mathcal{O}(t^{-\frac{1}{2}})P_0$ which is integrable. Then using the Dominated Convergence Theorem we have that they go to zero in the limit. We can ignore the terms containing $\mathcal{O}(e^{-t})$ because the sub-exponential growth condition insures that they will disappear when we integrate and then take the limit. □

Thus, we may assume that
$$\mathrm{STr}\left(e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}\right) = \mathrm{STr}(P_0 e^{-((P_0\mathbb{B}_1P_0)^2/2+\mathcal{O}(t^{-1/2}))}P_0) + \mathrm{STr}(Q_\epsilon e^{((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon)$$
so we have (1). Note that in [HL99], we abused notation by writing this equality as
$$\mathrm{STr}(e^{-\mathbb{B}(t)^2/2}) = \mathrm{STr}(e^{-((P_0\mathbb{B}_1P_0)^2/2+\mathcal{O}(t^{-1/2}))}) + \mathrm{STr}(e^{-((Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2/2+\mathcal{O}(t^0))}).$$

In the Volterra series proof of Proposition 11 involving $e^{-((Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2/2+\mathcal{O}(t^0))}$ near the bottom of p. 194, the equation now becomes,
$$t^{\frac{k}{2}}e^{-x_{k+1}Q_\epsilon tD^2Q_\epsilon} = t^{-m}(\epsilon G_\epsilon)^{m+\frac{k}{2}}(Q_\epsilon tD^2Q_\epsilon)^{m+\frac{k}{2}}e^{-x_{k+1}Q_\epsilon tD^2Q_\epsilon},$$

where G_ϵ is the Green's operator for $Q_\epsilon tD^2Q_\epsilon$, and m is a positive integer as large as we please. Denote by $\varrho_{[\epsilon,\infty)}$ the characteristic function for the interval $[\epsilon,\infty)$. The resulting estimate at the bottom of p. 194 then becomes, replacing $-\ell,\ell$ by r,s,

$$||(Q_\epsilon tD^2Q_\epsilon)^{m+\frac{k}{2}}e^{-x_{k+1}Q_\epsilon tD^2Q_\epsilon}||_{r,s} \;\leqslant\; \max_{z\geqslant 0}(1+z^2)^{(s-r)/2}\varrho^{2m+k}_{[\epsilon,\infty)}(z)(tz^2)^{m+\frac{k}{2}}e^{-x_{k+1}\varrho_{[\epsilon,\infty)}(z)tz^2\varrho_{[\epsilon,\infty)}(z)} \;=$$

$$\max_{z\geqslant\epsilon}(1+z^2)^{(s-r)/2}(tz^2)^{m+\frac{k}{2}}e^{-x_{k+1}tz^2}.$$

A straightforward computation shows that this maximum occurs when z satisfies an equation of the form

$$x_{k+1}t \;=\; \frac{az^2+b(1+z^2)}{z^2(1+z^2)},$$

for constants a and b. In particular, as $t\to\infty$, z must go to 0. Thus, for large t, the maximum must occur at $z=\epsilon$, and we have,

$$||(Q_\epsilon tD^2Q_\epsilon)^{m+\frac{k}{2}}e^{-x_{k+1}Q_\epsilon tD^2Q_\epsilon}||_{r,s} \;\leqslant\; (1+\epsilon^2)^{(s-r)/2}\epsilon^{2m+k}t^{m+\frac{k}{2}}e^{-x_{k+1}\epsilon^2 t} \;\to 0,$$

exponentially as $t\to\infty$. Then, the argument in the proof of Proposition 11 translates to,

$$\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon)\to 0,$$

uniformly exponentially pointwise as $t\to\infty$. So we have (2).

As for (3), since $\lim_{t\to 1}\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon) = 0$, uniformly exponentially pointwise and $T\smallsetminus T_t$ grows sub-exponentially,

$$\lim_{t\to 1}\int_{T\smallsetminus T_t}\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon)\wedge\omega_T \;=\; 0.$$

Finally, the family $(P_0\mathbb{B}_1P_0)^2+P_0\mathcal{O}(t^{-1/2})P_0$ consists of uniformly bounded elements of $\mathcal{N}$, so the Schwartz kernel of $e^{-((P_0\mathbb{B}_1P_0)^2+P_0\mathcal{O}(t^{-1/2})P_0)/2}$ is uniformly bounded pointwise due to bounded geometry. In addition, a multiple of $\mathrm{tr}(P_0(\overline{x},\overline{x}))$ dominates $\|\,\mathrm{Str}(P_0e^{-((P_0\mathbb{B}_1P_0)^2+P_0\mathcal{O}(t^{-1/2})P_0))}P_0))\|$ pointwise. This combined with assumption (3) in Theorem 3.3 and the Dominated Convergence Theorem gives,

$$\lim_{t\to\infty}\int_{T\smallsetminus T_t}\mathrm{STr}(P_0e^{-((P_0\mathbb{B}_1P_0)^2+P_0\mathcal{O}(t^{-1/2})P_0)/2}P_0)\wedge\omega_T \;=$$

$$\int_T\lim_{t\to\infty}\mathrm{STr}(P_0e^{-((P_0\mathbb{B}_1P_0)^2+P_0\mathcal{O}(t^{-1/2})P_0)/2}P_0)\wedge\omega_T \;=$$

$$\int_T\mathrm{STr}(P_0e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0)\wedge\omega_T = \langle\mathrm{ch}_a(P_0),\omega\rangle.$$

The last equality follows from Section 5 of [BH08]. Translating this to our notation here gives (3). □

To finish the proof of Theorem 3.3, we have the following.

Proposition 4.12. *Under the assumptions in Theorem 3.3,* $\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))))$ *and* $\mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))$ *satisfy the conclusions in Proposition 4.10, so*

$$\big\langle\big(\mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}(\beta(t)))),\mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}'(\beta(t))))\big),(\omega,\omega')\big\rangle \;=\; \langle([\mathrm{ch}_a(P_0)],[\mathrm{ch}_a(P_0')]),(\omega,\omega')\rangle.$$

Proof. We need to transfer the results above to χ^t. Please recall the material in the paragraph containing Equation 4.2. Note that $\chi^t(z)$ is a power series, in particular,

$$\chi^t(z) \;=\; \left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}}\star e\right](z) \;=\; \left[\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})\star e\right](z) \;=\; \int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-(z-x)^2/2}\,dx \;=$$

$$\left[\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}e^{zx}\,dx\right]e^{-z^2/2} \;=\; \sum_{k,\ell=0}^{\infty}\left[\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}x^\ell\,dx\right]\frac{(-1)^k z^{2k+\ell}}{2^k k!\ell!} \;=$$

$$\sum_{r=0}^{\infty}\left[\sum_{k=0}^{[r/2]}\frac{(-1)^k}{2^k k!(r-2k)!}\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}x^{r-2k}\,dx\right]z^r,$$

where $[r/2]$ is the greatest integer less than or equal to $r/2$. This is actually an even power series, since $\chi^t(z)$ is even because its Fourier transform is even. That is,

$$\chi^t(z) \;=\; \sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\frac{(-1)^k}{2^k k!(2r-2k)!}\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}x^{2r-2k}\,dx\right]z^{2r}.$$

Replace z^{2r} by z^r to get,

$$\widetilde{\chi}^t(z) \;=\; \sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\frac{(-1)^k}{2^k k!(2r-2k)!}\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}x^{2r-2k}\,dx\right]z^{r}.$$

Next, note that if B_1 and B_2 satisfy $B_1B_2 = B_2B_1 = 0$, then for any power series $f(z)$, $f(B_1 + B_2) = f(B_1) + f(B_2)$. Arguing just as we did above, we may assume that as $t \to 1$,

$$\mathbb{B}(\beta(t))^2 \;=\; \begin{bmatrix} (P_0\mathbb{B}_1P_0)^2 + \mathcal{O}(\beta(t)^{-1/2}) & 0 \\ 0 & (Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2 + \mathcal{O}(\beta(t)^0) \end{bmatrix}.$$

So, just as above, we may conclude that,

$$\begin{gathered}\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))) \;=\; \mathrm{STr}(\widetilde{\chi}^t(\mathbb{B}(\beta(t))^2)) \;=\\ \mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + \mathcal{O}(\beta(t)^{-1/2}))P_0) \;+\; \mathrm{STr}(Q_\epsilon\widetilde{\chi}^t((Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon)^2 + \mathcal{O}(\beta(t)^0))Q_\epsilon) \;=\\ \mathrm{STr}(P_0(P_0\mathbb{B}_1P_0 + \mathcal{O}(\beta(t)^{-1/2}))P_0) \;+\; \mathrm{STr}(Q_\epsilon(Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon + \mathcal{O}(\beta(t)^0))Q_\epsilon).\end{gathered}$$

Thus we have (1) of Proposition 4.10.

In the first equation for $\chi^t(z)$ above, the index ℓ must be even, so only the even powers of e^{zx} play a role, and it can be replaced by $\cosh(zx)$. Then, using the Volterra series on $\cosh(zx)$ and $e^{-z^2/2}$ in that equation, we get the following. See Equations 2.4, 2.5 and 4.1.

$$\begin{gathered}\chi^t(\mathbb{B}(\beta(t))) \;=\; \left(\left[\int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}\cosh(zx)\,dx\right]e^{-z^2/2}\right)(\mathbb{B}(\beta(t)) \;=\\ \int_{\mathbb{R}}\mathrm{FT}(\psi_{\sqrt{\alpha(t)}})(x)e^{-x^2/2}\left(\sum_{k=0}^{q}\int_{\Delta_k}2^{-k}\cosh(y_1x\sqrt{\beta(t)}D)B_{\sqrt{\beta(t)}}\cdots B_{\sqrt{\beta(t)}}\cosh(y_{k+1}x\sqrt{\beta(t)}D)\prod_{\ell=k}^{1}dy_\ell\right)dx\\ \times\sum_{k=0}^{q}\int_{\Delta_k}2^{-k}e^{-x_1\beta(t)D^2/2}C_{\beta(t)}\cdots C_{\beta(t)}e^{-x_{k+1}\beta(t)D^2/2}dx_k\ldots dx_1.\end{gathered}$$

Applying the techniques of the proof that $\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2/2+\mathcal{O}(t^0))}Q_\epsilon) \to 0$, uniformly exponentially pointwise as $t \to \infty$ to this Volterra expression shows that the same holds for

$$\mathrm{STr}(Q_\epsilon\chi^t(Q_\epsilon\mathbb{B}(\beta(t))Q_\epsilon + \mathcal{O}(\beta(t)^0))Q_\epsilon)$$

and its twin, as $t \to 1$, that is, we have (2) of Proposition 4.10 and we may ignore those terms when computing

$$\lim_{t\to1}\left[\int_{T\smallsetminus T_{\beta(t)}}\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))))\wedge\omega_T \;-\; \int_{T\smallsetminus T_{\beta(t)}}\mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))\wedge\omega'_{T'}\right].$$

For property (3), we are now reduced to computing,

$$\lim_{t\to1}\int_{T\smallsetminus T_{\beta(t)}}\mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2}))P_0)\wedge\omega_T.$$

As above, the family $(P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2})P_0$ consists of uniformly bounded elements of $\mathcal{N}$, so the Schwartz kernel of $\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2}))$ is uniformly bounded pointwise due to bounded geometry.

Thus, a multiple of $\mathrm{tr}(P_0(\overline{x},\overline{x}))$ dominates $\|\,\mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2}))P_0)\|$ pointwise. This combined with assumption (3) in Theorem 3.3 and the Dominated Convergence Theorem gives,

$$\lim_{t\to 1}\int_{T\smallsetminus T_{\beta(t)}} \mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2}))P_0)\wedge\omega_T \;=$$
$$\int_T \lim_{t\to 1}\ \mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2 + P_0\mathcal{O}(\beta(t)^{-1/2}))P_0)\wedge\omega_T \;=$$
$$\int_T \lim_{t\to 1}\ \mathrm{STr}(P_0\widetilde{\chi}^t((P_0\mathbb{B}_1P_0)^2)P_0)\wedge\omega_T \;=\; \int_T \lim_{t\to 1}\ \mathrm{STr}(P_0\chi^t(P_0\mathbb{B}_1P_0)P_0)\wedge\omega_T \;=$$
$$\int_T \mathrm{STr}(P_0e^{-(P_0\mathbb{B}_1P_0)^2/2}P_0)\wedge\omega_T = \langle \mathrm{ch}_a(P_0),\omega\rangle.$$

The last equality again follows from Section 5 of [BH08]. □

That finishes the proof of Theorem 3.3. To finish the proof of Theorem 3.5, we have the following.

Proposition 4.13. *Under the assumptions in Theorem 3.5,*

$$\big\langle \big(\mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}(\beta(t)))),\mathrm{S}\mathfrak{Tr}(\chi^t(\mathbb{B}'(\beta(t))))\big)\,,(\omega,\omega')\big\rangle \;=\; \langle([\mathrm{ch}_a(P_0)],[\mathrm{ch}_a(P_0')],(\omega,\omega')\rangle.$$

Proof. The proof follows [HL99], pp. 193-199. So, we no longer assume that there are spectral gaps at zero.

Now $\epsilon = t^{-1/a}$ where $a \in (6, 2NS(D)/q) \neq \varnothing$, and $a > 6$ is needed in the expression for $\bar{B}^2_{\epsilon,t}$ on p. 192. In particular, now

$$\bar{B}_{\epsilon,t} \;=\; (P_0+Q_\epsilon)\mathbb{B}(t)(P_0+Q_\epsilon) + P_{(0,\epsilon)}\mathbb{B}(t)P_{(0,\epsilon)} \;\text{ and }\; \bar{B}^2_{\epsilon,t} = \psi_t g_\epsilon^{-1}\psi_t^{-1}A\psi_t g_\epsilon\psi_t^{-1},$$

where, for some $\beta > 0$,

$$A \;=\; \left[\begin{array}{ccc} (P_0\mathbb{B}_1P_0)^2+\mathcal{O}(t^{-\beta}) & 0 & 0 \\ 0 & (Q_\epsilon\mathbb{B}(t)Q_\epsilon)^2+\mathcal{O}(t^{2/a}) & 0 \\ 0 & 0 & (P_{(0,\epsilon)}\mathbb{B}_tP_{(0,\epsilon)})^2 \end{array}\right].$$

We may again assume that $\psi_t g_\epsilon^{-1}\psi_t^{-1} = \psi_t g_\epsilon \psi_t^{-1} = \mathrm{I}$, so we are reduced to considering just A.

The proof in [HL99] shows that if $\mathbb{B}_t = A$, then $\lim_{t\to\infty}\mathrm{STr}(e^{-\mathbb{B}_t^2}) = \mathrm{STr}(e^{-(P_0\mathbb{B}_1P_0)^2}$ in Haefliger cohomology, which by [BH08] equals $\mathrm{ch}_a(P_0)$. It then shows that the terms in $B_t - A$ contribute nothing in the limit. As above, to prove Proposition 4.13, we need to show that that proof can be adapted to our case here.

With this in mind, note that the proof of Proposition 11 of [HL99] uses the Volterra series and estimates from the Spectral Mapping Theorem to show that, for any $\delta > 0$, as $t \to \infty$,

$$\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon B_tQ_\epsilon)^2+\mathcal{O}(t^{2/a}))}Q_\epsilon) \;=\; \mathcal{O}(t^{-\delta}),$$

that is, $\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon B_tQ_\epsilon)^2+\mathcal{O}(t^{2/a}))}Q_\epsilon)$ has pointwise uniform super polynomial decay as $t\to\infty$. Thus the corresponding terms, $||\,\mathrm{STr}(Q_\epsilon\widetilde{\chi}^t((Q_\epsilon B_{\beta(t)}Q_\epsilon)^2 + \mathcal{O}(\beta(t)^{2/a}))Q_\epsilon)$ and its twin, also have pointwise uniform super polynomial decay as $t \to 1$, and may be disregarded as above because of the sub-super polynomial growth assumption.

The proof of Proposition 12 of [HL99], again using the Volterra series and estimates from the Spectral Mapping Theorem, shows that for t large,

$$||\,\mathrm{STr}(P_{(0,\epsilon)}e^{-(P_{(0,\epsilon)}B_tP_{(0,\epsilon)})^2}P_{(0,\epsilon)})||\;\leqslant\;||C\,\mathrm{Tr}(P_{(0,\epsilon)})||,$$

where $||t^{-q/2}C||$ is bounded. Now,

$$||C\,\mathrm{Tr}(P_{(0,\epsilon)})|| \;=\; ||t^{-q/2}Ct^{q/2}\,\mathrm{Tr}(P_{(0,\epsilon)})|| \;\leqslant\; ||t^{-q/2}C||t^{q/2}\epsilon^{NS(D)} \;=\; ||t^{-q/2}C||t^{\frac{q}{2}-\frac{NS(D)}{a}} \to 0,$$

uniformly pointwise as $t\to\infty$, since $\epsilon = t^{-1/a}$ where $a \in (6, 2NS(D)/q)$ and $NS(D) > 3q$. This combined with the assumption that $\displaystyle\int_M \mathrm{tr}(P_{(0,\epsilon)})d\mu < \infty$, and the Bounded Convergence Theorem once again give that

$$\langle \mathrm{STr}(P_{(0,\epsilon)}\widetilde{\chi}^t(P_{(0,\epsilon)}B_{\beta(t)}P_{(0,\epsilon)})^2)P_{(0,\epsilon)}),\omega_T\rangle \to 0,$$

as $t \to 1$, and so it and its twin may be ignored.

Thus we have the Proposition for $\bar{\mathbb{B}}_{\epsilon,t}$ and $\bar{\mathbb{B}}'_{\epsilon,t}$, that is,

$$\lim_{t\to 1}\langle\left[\mathrm{STr}(\chi^t(\bar{\mathbb{B}}_{\epsilon,t}(\beta(t)))), \mathrm{STr}(\chi^t(\bar{\mathbb{B}}'_{\epsilon,t}(\beta(t))))\right], (\omega,\omega')\rangle \ = \ \langle([\mathrm{ch}_a(P_0)], [\mathrm{ch}_a(P_0')], (\omega,\omega')\rangle.$$

Finally, consider the remaining missing terms in [HL99], namely,

$$\mathbb{B}_t - \bar{\mathbb{B}}_{\epsilon,t} \ = \ (P_0 + Q_\epsilon)\mathbb{B}_t P_{(0,\epsilon)} + P_{(0,\epsilon)}\mathbb{B}_t(P_0+Q_\epsilon) \ =$$
$$(P_0+Q_\epsilon)(\mathbb{B}_t - \sqrt{t}\mathbb{B}_0)P_{(0,\epsilon)} + P_{(0,\epsilon)}(\mathbb{B}_t - \sqrt{t}\mathbb{B}_0)(P_0+Q_\epsilon),$$

since $(P_0+Q_\epsilon)\mathbb{B}_0 P_{(0,\epsilon)} = (P_0+Q_\epsilon)P_{(0,\epsilon)}\mathbb{B}_0 = 0$, as $\mathbb{B}_0 = D$ so commutes with $P_{(0,\epsilon)}$, and similarly for the other term. The proof of Proposition 14 of [HL99] shows that

$$\mathrm{STr}(e^{-\mathbb{B}_t^2}) - \mathrm{STr}(e^{-\bar{\mathbb{B}}_t^2}) \ = \ \mathrm{Tr}(CP_{(0,\epsilon)}) \ = \ \mathrm{Tr}(P_{(0,\epsilon)}CP_{(0,\epsilon)}),$$

where C is a bounded leafwise smoothing operator, with a bound depending on t, and $t^{-q/2}||C||$ is bounded independently of t for t large. Just as above we get,

$$||\,\mathrm{Tr}(P_{(0,\epsilon)}CP_{(0,\epsilon)})|| \ \leqslant \ ||\beta(t)^{-q/2}C||\beta(t)^{q/2}\epsilon^{NS(D)} \ = \ ||\beta(t)^{-q/2}C||\beta(t)^{\frac{q}{2}-\frac{NS(D)}{a}} \to 0,$$

uniformly pointwise as $t \to 1$.

Next, consider the corresponding term

$$\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))) - \mathrm{STr}(\chi^t(\bar{\mathbb{B}}(\beta(t)))) \ =$$
$$(\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))) - \mathrm{STr}(e^{-\mathbb{B}(\beta(t)))^2})) - (\mathrm{STr}(\chi^t(\bar{\mathbb{B}}(\beta(t)))) - \mathrm{STr}(e^{-\bar{\mathbb{B}}(\beta(t))^2}) +$$
$$(\mathrm{STr}(e^{-\mathbb{B}(\beta(t))^2}) - \mathrm{STr}(e^{-\bar{\mathbb{B}}(\beta(t))^2})).$$

The term

$$(\mathrm{STr}(e^{-\mathbb{B}(\beta(t))^2}) - \mathrm{STr}(e^{-\bar{\mathbb{B}}(\beta(t))^2})) \ = \ \mathrm{Tr}(P_{(0,\epsilon)}CP_{(0,\epsilon)}) \to 0,$$

uniformly pointwise as $t \to 1$ by the result quoted above. Using this, the assumption that $\displaystyle\int_M \mathrm{tr}(P_{(0,\epsilon)})d\mu < \infty$, and the Bounded Convergence Theorem shows that we may ignore this term.

Because STr is a trace, $P_0P_{(0,\epsilon)} = P_0Q_\epsilon = P_{(0,\epsilon)}Q_\epsilon = 0$, and we can factor $\chi^t(\mathbb{B}(\beta(t))) = \widetilde{\chi}^t(\mathbb{B}(\beta(t)))^2$ and $e^{-\mathbb{B}(\beta(t))^2} = e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}e^{-\frac{1}{2}\mathbb{B}(\beta(t))^2}$, a standard argument gives,

$$\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t)))) - \mathrm{STr}(e^{-\mathbb{B}(\beta(t))^2}) \ = \ \mathrm{STr}(Q_\epsilon(\chi^t(\mathbb{B}(\beta(t))) - e^{-\mathbb{B}(\beta(t))^2})Q_\epsilon) +$$
$$\mathrm{STr}(P_{(0,\epsilon)}(\chi^t(\mathbb{B}(\beta(t))) - e^{-\mathbb{B}(\beta(t))^2})P_{(0,\epsilon)}) \ + \ \mathrm{STr}(P_0(\chi^t(\mathbb{B}(\beta(t))) - e^{-\mathbb{B}(\beta(t))^2})P_0).$$

The first term on the right can be treated as in the proof of Proposition 11 of [HL99] to show that it goes to zero uniformly pointwise super-polynomially, so may be ignored thanks to the growth condition. The second term may be handled using the assumption that $\displaystyle\int_M \mathrm{tr}(P_{(0,\epsilon)})d\mu < \infty$, and the Bounded Convergence Theorem, so it too contributes zero. The third term also goes to zero using the facts that its two parts have the same limit as $t \to 1$, that $\displaystyle\int_M \mathrm{tr}(P_0)d\mu \leqslant \int_M \mathrm{tr}(P_{(0,\epsilon)})d\mu < \infty$, and the Bounded Convergence Theorem.

Finally, the term $\mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t)))) - \mathrm{STr}(\chi^t(\bar{\mathbb{B}}'(\beta(t))))$ may be treated the same way. □

Remark 4.14. *The reader may wonder why we can't just use the operators $e^{-\mathbb{B}_t^2}$ and $e^{-\mathbb{B}_t'^2}$. That is, why do we need operators with finite propagations? The answer is two fold. Even though we know that for any $\delta > 0$, as $t \to \infty$,*

$$\mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon B_t Q_\epsilon)^2 + \mathcal{O}(t^{2/a}))}Q_\epsilon) \ = \ \mathcal{O}(t^{-\delta}),$$

there in no growth condition (save zero, i.e. compactness) which will insure that

$$\langle \mathrm{STr}(Q_\epsilon e^{-((Q_\epsilon B_t Q_\epsilon)^2 + \mathcal{O}(t^{2/a}))}Q_\epsilon), \omega_T\rangle$$

is well defined.

A way around this problem is to define

$$\langle(\mathrm{STr}(e^{-\mathbb{B}_t^2}), \mathrm{STr}(e^{-\mathbb{B}_t'^2})), (\omega_T, \omega'_{T'})\rangle \;=\; \lim_{s\to\infty}\left[\int_{T\smallsetminus T_s} \mathrm{STr}(e^{-\mathbb{B}_t^2})\wedge\omega_T \;-\; \int_{T'\smallsetminus T'_s} \mathrm{STr}(e^{-\mathbb{B}_t'^2})\wedge\omega_{T'}\right],$$

and then show that this is well defined and independent of t. We do not know how to prove these. Note that our proof of these for $\mathrm{STr}(\chi^t(\mathbb{B}(\beta(t))))$ and $\mathrm{STr}(\chi^t(\mathbb{B}'(\beta(t))))$ uses the fact that they agree near infinity, which, as pointed out in the Introduction, is not necessarily true for $\mathrm{STr}(e^{-\mathbb{B}_t^2})$ and $\mathrm{STr}(e^{-\mathbb{B}_t'^2})$.

5. Examples

We first give the properties needed for general examples of the type of 5.1 below.

- N is a compact n dimensional Riemannian manifold whose fundamental group Γ acts on a compact manifold S.
- Denote by $\widetilde{N}$ the universal cover of N, and set $M = \widetilde{N}\times_\Gamma S$, and denote by F the foliation given by the fibration $S \to M \xrightarrow{\pi} N$. M also has the natural flat bundle foliation which is transverse to F.
- Assume that the tangent bundle of F, namely the tangent bundle along the fiber $TF = \widetilde{N}\times_\Gamma TS$, is spin, has a nowhere zero cross section (e.g. $\partial/\partial\theta$ as in 5.1 below), and $\displaystyle\int_F \widehat{A}(TF) \neq 0$ in $H^*_c(M/F) = H^*(N;\mathbb{R})$.
- Denote by X a vector field on N with at least two zeros and a trajectory which starts at one zero and ends at another. These always exist.
- Since M is compact, TF admits metrics of *bounded* scalar curvature.

Then, as we show below, the manifold $M\times\mathbb{S}^{4k-1}$, $k>1$, admits a non-Riemannian spin foliation F_X with Hausdorff holonomy groupoid, and the space of PSC metrics on F_X has infinitely many path connected components.

Recall the following example in [BH24], which is an adaptation of Example 1 of [H78].

Example 5.1. *Let $G = SL_2\mathbb{R}\times\cdots\times SL_2\mathbb{R}$ (q copies) and $K = SO_2\times\cdots\times SO_2$ (q copies). G acts naturally on $\mathbb{R}^{2q}\smallsetminus\{0\}$ and is well known to contain subgroups Γ with $N = \Gamma\backslash G/K$ compact, (in fact a product of q surfaces of higher genus). Set*

$$M = \Gamma\backslash G\times_K ((\mathbb{R}^{2q}\smallsetminus\{0\})/\mathbb{Z}) \simeq \Gamma\backslash G\times_K (\mathbb{S}^{2q-1}\times\mathbb{S}^1),$$

where $n\in\mathbb{Z}$ acts on $\mathbb{R}^{2q}\smallsetminus\{0\}$ by $n\cdot z = e^n z$.

M has two transverse foliations, F which is given by the fibers $\mathbb{S}^{2q-1}\times\mathbb{S}^1$ of the fibration $M\to N$, and a transverse foliation coming from the foliation τ of Example 1 of [H78]. τ is the natural foliation on the flat vector bundle $K\backslash G\times_\Gamma\mathbb{R}^{2q}$, and the zero section is a leaf of it. In addition, $K\backslash G\times_\Gamma\mathbb{R}^{2q}$ is diffeomorphic to $\Gamma\backslash G\times_K R^{2q}$, and the action of $\mathbb{Z}$ preserves τ, fixing the zero section, so it descends to a foliation on M, also denoted τ.

The following modification of F preserves the needed properties, but it is not Riemannian. In particular, the facts that TF is orientable, spin, has Hausdorff holonomy groupoid, and admits a metric with bounded (actually positive-but this is non-essential) scalar curvature are preserved. We alter F as follows. The base space $N = \prod_{i=1}^q \Sigma_i$, where each Σ_i is a surface of higher genus. On each Σ_i choose a smooth vector field X_i with isolated simple singularities so having indexes ±1. We may assume that there are integral curves of X_i starting at one singularity and ending at another. This insures that the resulting foliation is not Riemannian. The vector field $\prod_{i=1}^q X_i$ on N determines the vector field X on M which is tangent to the foliation τ. We also have the vector field $\partial/\partial\theta$ which is tangent to the fiber $\mathbb{S}^1$ of M. Denote by F_X the foliation determined by the fibers $\mathbb{S}^{2q-1}$ and the vector field $\partial/\partial\theta + X$. First note that the tangent bundle TF_X is a equivalent to the tangent bundle TF, so it is also spin and has the same characteristic classes. However, F is Riemannian but F_X is not. This is because there are families of leaves of F_X which become arbitrarily close to two different leaves of F, and this cannot happen in a Riemannian foliation. Note that the holonomy groupoid of F_X is Hausdorff, and that F_X admits a metric of positive scalar curvature.

As in [BH24], we have

Proposition 5.2. *There is a non-zero constant C_q so that* $\displaystyle\int_N \int_{F_X} \widehat{A}(TF_X) = C_q \operatorname{vol}(N)$.

Thus $\displaystyle\int_{F_X} \widehat{A}(TF_X) \neq 0$ in $H^*_c(M/F_X)$.

In [Ca88], Carr constructs examples of "exotic" PSC metrics $\mathfrak{g}_i$, $i \in \mathbb{Z}_+$ on $\mathbb{S}^{4k-1}$, for $k > 1$, and compact Riemannian $4k$ dimensional spin manifolds X_i with boundary $\mathbb{S}^{4k-1}$, so that the metric $\widehat{\mathfrak{g}}_i$ on X_i is $\mathfrak{g}_i \times dt^2$ in a neighborhood of $\mathbb{S}^{4k-1}$, and $\widehat{\mathfrak{g}}_i$ also has PSC. Set

$$X_{(i,j)} \;=\; X_i \cup (\mathbb{S}^{4k-1} \times [0,1]) \cup X_j,$$

where the metric on $\mathbb{S}^{4k-1} \times [0,1]$ is $\mathfrak{g}_t \times dt^2$, and $\mathfrak{g}_t$ is a path of metrics from $\mathfrak{g}_i$ to $\mathfrak{g}_j$. These examples have the property that the integer valued Gromov-Lawson invariant $i_{GL}(\mathfrak{g}_i, \mathfrak{g}_j)$, [GL83], is

$$i_{GL}(\mathfrak{g}_i, \mathfrak{g}_j) \;=\; \int_{X_{(i,j)}} \widehat{A}(TX_{(i,j)}) \;=\; C_k(i-j),$$

where $C_k \neq 0$.

Consider the manifold $M \times \mathbb{S}^{4k-1}$ with the foliation $TF_X \times T\mathbb{S}^{4k-1}$, and the metric g_i, which is the product of the metric on TF_X and $\mathfrak{g}_i$ on $T\mathbb{S}^{4k-1}$. These metrics have PSC and we claim that for $i \neq j$ they are not in the same path component of the space of PSC metrics on $TF_X \times T\mathbb{S}^{4k-1}$. To see this, set

$$\widehat{M} \;=\; M \times X_{(i,j)},$$

with the foliation $F_X \times X_{(i,j)}$. In [BH24], we define the invariant

$$i(g_i, g_j) \;=\; \int_{F_X \times X_{(i,j)}} \widehat{A}(TF_X \times TX_{(i,j)}) \;\in\; H^*_c(\widehat{M}/F_X \times X_{(i,j)}),$$

and show that if g_i and g_j are in the same path component of the space of PSC metrics on $TF_X \times T\mathbb{S}^{4k-1}$, then $i(g_i, g_j) = 0$. However, if $i \neq j$, then

$$\int_{F_X \times X_{(i,j)}} \widehat{A}(TF_X \times TX_{(i,j)}) \;=\; \int_{F_X \times X_{(i,j)}} \widehat{A}(TF_X)\widehat{A}(TX_{(i,j)}) \;=\; \int_{F_X} \widehat{A}(TF_X) \int_{X_{(i,j)}} \widehat{A}(TX_{(i,j)}) \;=$$

$$\widehat{A}(X_{(i,j)}) \int_{F_X} \widehat{A}(TF_X) \;=\; C_k(i-j) \int_{F_X} \widehat{A}(TF_X) \neq 0 \text{ in } \; H^*_c(\widehat{M}/F_X \times X_{(i,j)}),$$

so $i(g_i, g_j) \neq 0$.

Further examples can be constructed using other examples in [H78], as well as those in [KS93].

Note that in Example 5.1, the fact that F admits a metric of positive scalar curvature is non-essential. What is essential is that it admits metrics of bounded scalar curvature. Then we may multiply the metrics $\mathfrak{g}_i$ by constants so that their scalar curvatures overwhelm the scalar curvature on the leaves of F_X, so the resulting metric on $TF_X \times T\mathbb{S}^{4k-1}$ has positive scalar curvature.

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Institut Montpellierain Alexander Grothendieck, UMR 5149 du CNRS, Université de Montpellier

Mathematics, Statistics, and Computer Science, University of Illinois at Chicago
E-mail address: heitsch@uic.edu