Abstract

Higher relative index theorems for foliations

In this paper we solve the general case of the cohomological relative index problem for foliations of non-compact manifolds. In particular, we significantly generalize the groundbreaking results of Gromov and Lawson, [GL83], to Dirac operators defined along the leaves of foliations of non-compact complete Riemannian manifolds, by involving all the terms of the Connes-Chern character, especially the higher order terms in Haefliger cohomology. The zero-th order term corresponding to holonomy invariant measures was carried out in [BH21] and becomes a special case of our main results here. In particular, for two leafwise Dirac operators on two foliated manifolds which agree near infinity, we define a relative topological index and the Connes-Chern character of a relative analytic index, both being in relative Haefliger cohomology. We show that these are equal. This invariant can be paired with closed holonomy invariant currents (which agree near infinity) to produce higher relative scalar invariants. When we relate these invariants to the leafwise index bundles, we restrict to Riemannian foliations on manifolds of sub-exponential growth. This allows us to prove a higher relative index bundle theorem, extending the classical index bundle theorem of [BH08]. Finally, we construct examples of foliations and use these invariants to prove that their spaces of leafwise positive scalar curvature metrics have infinitely many path-connected components, completely new results which are not available from [BH21]. In particular, these results confirm the well-known idea that important geometric information of foliations is embodied in the higher terms of the $\widehat{A}$ genus.