LECTURES ON THE TWISTED HIGHER HARMONIC SIGNATURE FOR FOLIATIONS February 3, 2010

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Abstract. These lectures review the classical Hirzebruch signature theorem, and show how it extends to the signature defined on Riemannian foliations using leafwise differential forms with coefficients in a leafwise $U(p, q)$-flat complex bundle. We give background on all the concepts needed to make this extension.

1. Introduction

This is a write up of a mini-course given at the conference Géometrie Non Commutative et Physique, which was held at the Université de Kenitra in Morocco in April 2009. I would like to express here my warmest thanks for the invitation to speak at the conference, and for the wonderful hospitality of the organizers.

The purpose of the lectures was to introduce the rich circle of ideas surrounding the classical signature theorem of Hirzebruch, as well as their extension foliations, which is in the realm of non-commutative theory. In particular, they were to serve as a gentle introduction to the results obtained with Moulay-Tahar Benemeur, which appear in the paper [BH09], and whose complexity may be daunting to the uninitiated. In that paper, we prove that the higher harmonic signature, that is the signature defined using harmonic forms rather than cohomology, of a even dimensional oriented Riemannian foliation of a compact manifold M, twisted by a leafwise $U(p, q)$-flat complex bundle over M, is a leafwise homotopy invariant. Rather than being just an integer, this signature is a cohomology class, which may have non-trivial terms in dimensions greater than zero, hence the name higher signature. Its relation to the classical signature is akin to the relationship of the families index theorem to the index theorem on compact manifolds, or more correctly, the Connes index theorem for foliations, [C94], to the index theorem on compact manifolds. As such, it requires a good understanding of non-commutative theory, in particular the theory of foliations, and the non-commutative structures associated to it. In addition to the classical elements of foliation theory (Haefliger cohomology, homotopy groupoid, leafwise operators), we also introduce new concepts (transversely smooth idempotents, Chern-Weil theory and the Chern-Connes character for such idempotents), needed for the statement and proof of the theorem. This result also has important consequences for the classical Novikov conjecture for groups, and for the Baum-Connes Novikov conjecture for foliations. For the sake of simplicity, we give just two such applications here.

For the most part, results are only stated without proof, although we do give a brief outline of the proof of our foliation signature theorem. We refer the reader to [BH09] for the details of this theorem and its proof, and more background on the work of others on these questions. In addition, the reader should see [C94] and [HiS92] for another approach to the homotopy invariance of the higher signatures for foliations.

2. Notation and Review

Throughout this paper M denotes a smooth compact oriented manifold without boundary. The tangent bundle of M is denoted by TM, and its dual bundle by T^*M. If $E \to M$ is a vector bundle over M, we denote the space of smooth sections by $C^\infty(E)$ or by $C^\infty(M; E)$. The fiber of E over $x \in M$ is denoted E_x. The space of differential k forms on M is denoted $\Omega^k(M)$, and we set $\Omega^*(M) = \oplus_{k\geq 0}\Omega^k(M)$. The de Rham exterior derivative is denoted d (or d_M) and more specifically $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$. Recall that the de Rham cohomology of M is given by $\mathrm{H}^k(M; \mathbb{R}) = \mathrm{Ker}(d_k)/\,\mathrm{Im}(d_{k-1})$. Each $\mathrm{H}^k(M; \mathbb{R})$ is finite dimensional, since M is compact, and satisfies Poincaré duality, since M is oriented. That is, if the dimension of M is n, then $\mathrm{H}^k(M;\mathbb{R}) \simeq \mathrm{H}^{n-k}(M;\mathbb{R})$, and this duality may be realized as follows. If we choose a metric on T^*M, we can use it to construct an isomorphism $* : \wedge^k T^*M \to \wedge^{n-k}T^*M$, which extends to $* : \Omega^k(M) \to \Omega^{n-k}(M)$, as follows. Given $x \in M$, choose an oriented orthonormal basis $\omega_1, ..., \omega_n$ of T^*M_x. Then $\omega_1 \wedge \cdots \wedge \omega_n$ is a well defined n-form on M, which is denoted $dvol$. Now define $* : \wedge^k T^*M_x \to \wedge^{n-k}T^*M_x$ to be the unique *linear* map such that $*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}$ where $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \wedge \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}} = dvol$. Note that $*^2 \neq I$ in general, a defect that we will address below. Recall that each class $\alpha \in \mathrm{H}^k(M;\mathbb{R})$ has a unique representative ω so that $d\omega = 0$ and $d * \omega = 0$. These are the harmonic k forms. The mapping $\varphi(\alpha) = \varphi([\omega]) = [*\omega]$ then defines an isomorphism $\varphi : \mathrm{H}^k(M;\mathbb{R}) \simeq \mathrm{H}^{n-k}(M;\mathbb{R})$. It is not difficult to see that the pairing $S : \mathrm{H}^k(M;\mathbb{R}) \otimes \mathrm{H}^{n-k}(M;\mathbb{R}) \to \mathbb{R}$ given by

$$S([\omega_1],[\omega_2]) \;=\; <[M],[\omega_1] \cup [\omega_2]> \;=\; \int_M \omega_1 \wedge \omega_2$$

is non-degenerate, since $S(\alpha, \varphi(\alpha)) \neq 0$ whenever $\alpha \neq 0$. Here $[M] \in \mathrm{H}_n(M;\mathbb{Z})$ is the fundamental class determined by M.

We now briefly recall the construction of characteristic classes for real and complex bundles.

Definition 2.1. *A connection on a bundle $E \to M$ is a linear map $\nabla : C^\infty(E \otimes \wedge T^*M) \to C^\infty(E \otimes \wedge T^*M)$, of degree one, satisfying*

$$\nabla(\xi \otimes \omega) = \nabla\xi \wedge \omega + \xi \otimes d\omega.$$

Lemma 2.2. *Connections always exist.*

Proof. Let $\{U_i\}$ be a finite open cover of M so that $E\,|\,U_i$ is trivial. On each U_i, choose a local framing $\xi_1^i, ..., \xi_k^i$, and denote by ∇^i the local connection on $E\,|\,U_i$ determined by the requirement that $\nabla^i(\xi_j^i) = 0$ for all j. Choose a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{U_i\}$. For $\xi \in C^\infty(E)$, define $\nabla(\xi) = \sum_i \varphi_i \nabla^i(\xi\,|\,U_i)$, and extend to all of $C^\infty(E \otimes \wedge T^*M)$ by requiring that $\nabla(\xi \otimes \omega) = \nabla\xi \wedge \omega + \xi \otimes d\omega$. $\square$

The curvature of ∇ is the order two map $\theta = \nabla^2$.

Lemma 2.3. *For $\xi \in C^\infty(E)$ and $\omega \in \Omega^*(M)$, $\theta(\xi \otimes \omega) = (\theta\xi) \wedge \omega$.*

Proof. It is not difficult to show that ∇ is a local operator, and locally we may write $\nabla\xi = \xi_1 \otimes \omega_1$ (actually as a finite sum of such sections) where ξ_1 is a local section of E and ω_1 is a local one form. Then

$$\theta(\xi \otimes \omega) \;=\; \nabla^2(\xi \otimes \omega) \;=\; \nabla(\nabla(\xi \otimes \omega)) \;=\; \nabla(\nabla\xi \wedge \omega + \xi \otimes d\omega) \;=$$
$$\nabla(\xi_1 \otimes \omega_1 \wedge \omega + \xi \otimes d\omega) \;=\; \nabla\xi_1 \otimes \omega_1 \wedge \omega + \xi_1 \otimes d(\omega_1 \wedge \omega) + \nabla\xi \wedge d\omega \;=$$
$$\nabla\xi_1 \otimes \omega_1 \wedge \omega + \xi_1 \otimes d\omega_1 \wedge \omega - \xi_1 \otimes \omega_1 \wedge d\omega + \nabla\xi \wedge d\omega \;=\; (\nabla^2\xi) \wedge \omega \;=\; (\theta\xi) \wedge \omega.$$

$\square$

Thus the curvature operator θ is not just a local operator, but actually a pointwise operator. If $\xi_1, ..., \xi_k$ is a local framing of E, we may write

$$\theta(\xi_i) = \sum_j \xi_j \otimes \theta_j^i$$

where the θ_j^i are smooth local two forms on M. A simple calculation shows that the local matrix of two forms $\theta = [\theta_j^i]$ is well defined up to conjugation by elements of $Gl_k(\mathbb{R})$. So, for example, $\mathrm{tr}(\theta)$ is a globally well defined 2 form on M! To generalize this observation, we use the equation

$$\det(I - \frac{\lambda}{2i\pi}\theta) \;=\; 1 + \lambda c_1(\theta) + \lambda^2 c_2(\theta) + \cdots + \lambda^k c_k(\theta),$$

to define the Chern forms associated to the curvature θ. Each $c_j(\theta)$ is a globally well defined closed $2j$ form on M, whose cohomology class $[c_j(\theta)]$ depends only on E. The constant $1/2i\pi$ insures that these cohomology classes are in fact integral classes. For real bundles, the $c_{2j+1}(\theta)$ are exact, so $[c_{2j+1}(\theta)] = 0$ in $\mathrm{H}^{4j+2}(M;\mathbb{R})$.

Definition 2.4. *Suppose that E is a real bundle over M. The j th Pontrjagin class of E is $p_j(E) = [c_{2j}(\theta)] \in \mathrm{H}^{4j}(M;\mathbb{R})$.*

We want to construct certain linear combinations of the Pontrjagin classes of TM which are central to the Hirzebruch Signature Theorem. So consider the even series $\prod_{j=1}^{k} \frac{x_j}{\tanh(x_j)}$, which may be written as a series in the elementary symmetric functions σ_i of the variables $x_1^2, ..., x_k^2$. That is, we define the series $\mathrm{L}_k(y_1, ..., y_k)$ by the requirement that

$$\mathrm{L}_k(\sigma_1, ..., \sigma_k) \;=\; \prod_{j=1}^{k} \frac{x_j}{\tanh(x_j)}.$$

If the $\dim M = 4\ell$, we set

$$\mathrm{L}(TM) \;=\; \mathrm{L}_\ell(p_1(TM), ..., p_\ell(TM)) \quad \in \quad \mathrm{H}^{4\ell}(M; \mathbb{R}).$$

It turns out that $\int_M \mathrm{L}(TM)$, which, a priori, is only a real number, is in fact an integer, and it is a very important topological invariant of M. More on this in the next section.

If E is a complex bundle, we get the same results, except that now the $c_{2j+1}(\theta)$ are not necessarily exact.

Definition 2.5. *For a complex bundle E, the j th Chern class of E is $c_j(E) = [c_j(\theta)] \in \mathrm{H}^{2j}(M; \mathbb{R})$.*

An important invariant of complex bundles is their Chern character. In particular, define $C_k(y_1, ..., y_k)$ by the requirement that

$$C_k(\sigma_1, ..., \sigma_k) \;=\; \sum_{j=1}^{k} e^{x_j},$$

where σ_i is the i th elementary symmetric function in the variables $x_1, ..., x_k$.

Definition 2.6. *The Chern character of a complex bundle E is $\mathrm{ch}(E) = C_k(c_1(E), ..., c_k(E))$.*

Note that the Chern character of E can also be expressed as $\mathrm{ch}(E) = [\mathrm{tr}(\exp(-\theta/2i\pi))]$. This observation will be important for us later.

Denote by $\mathrm{K}(M)$ the complex K-theory of M, that is formal differences of equivalence classes of complex bundles. Given an element β in $\mathrm{K}(M)$, write it as $\beta = [E_1] - [E_2]$. Then $\mathrm{ch}(\beta) = \mathrm{ch}(E_1) - \mathrm{ch}(E_2)$ is well-defined. A fundamental theorem involving the Chern character is

Theorem 2.7. $\mathrm{ch} : \mathrm{K}(M) \otimes \mathbb{R} \to \mathrm{H}^{2*}(M; \mathbb{R})$ *is an isomorphism of algebras.*

Examples of $\mathbb{C}$ bundles over the torus $\mathbb{T}^2$ with non-trival Chern characters.

Realize $\mathbb{T}^2$ as the square $\mathbb{I}^2 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x, y \leq 1\}$, with opposite edges identified. Now consider the $\mathbb{C}$ bundle E_n over $\mathbb{T}^2$ defined as follows. On $\mathbb{I}^2 \times \mathbb{C}$ identify $(0, y, z)$ with $(1, y, z)$, and identify $(x, 0, z)$ with $(x, 1, e^{2\pi inx}z)$. Denote by U_0 the open subset of $\mathbb{T}^2$ determined by $\{(x, y) \in \mathbb{I}^2 \,|\, \frac{1}{4} < y < \frac{3}{4}\}$, and by U_1 the open subset determined by $\{(x, y) \in \mathbb{I}^2 \,|\, y < \frac{1}{3} \text{ or } y > \frac{2}{3}\}$. Choose a point $z_0 \in \mathbb{C}$. On U_0 define the local section of E_n by $\sigma(x, y) = z_0$, and define the local connection form by $\nabla^0\sigma = 0$. On U_1 define the local section by $\gamma(x, y) = z_0$ if $y < \frac{1}{3}$ and $\gamma(x, y) = e^{2\pi inx}z_0$ if $y > \frac{2}{3}$, and define the local connection form by $\nabla^1\gamma = 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with $\varphi = 1$ on $[1/3, 2/3]$, and $\varphi = 0$ on $[0, 1/4]$ and $[3/4, 1]$. Set $\nabla = \varphi(y)\nabla^0 + (1 - \varphi(y))\nabla^1$. On the set determined by $\{(x, y) \in \mathbb{I}^2 \,|\, 0 \leq y < \frac{2}{3}\}$, we have the framing of E_n given by the section $\sigma(x, y) = z_0$ and $\nabla(\sigma) = 0$ so $\theta = 0$ on this set. On the set determined by $\{(x, y) \in \mathbb{I}^2 \,|\, y \geq \frac{2}{3}\}$, we have the framing of E_n given by the restriction of the section γ. Now we compute

$$\nabla(\gamma) \;=\; \varphi\nabla^0(\gamma) + (1 - \varphi)\nabla^1(\gamma) \;=\; \varphi\nabla^0(\gamma) \;=$$
$$\varphi\nabla^0(e^{2\pi inx}\sigma) \;=\; \varphi\sigma \otimes d(e^{2\pi inx}) \;=\; \varphi\sigma \otimes 2\pi ine^{2\pi inx}dx \;=\; \gamma \otimes \varphi 2\pi indx,$$

so

$$\theta(\gamma) \;=\; \nabla^2(\gamma) \;=\; \nabla(\gamma) \wedge \varphi 2\pi indx + \gamma \otimes d(\varphi 2\pi indx) \;=\; 0 + (d\varphi/dy)2\pi indy \wedge dx.$$

So, on $0 \leq y < 2/3$, $\theta = 0$ and $c_1(\theta) = 0$, while on $y \geq 2/3$, $\theta = (d\varphi/dy)2\pi i n dy \wedge dx$, and

$$c_1(\theta) \;=\; \frac{-1}{2\pi i}(d\varphi/dy)2\pi i n dy \wedge dx \;=\; -n(d\varphi/dy)dy \wedge dx.$$

Thus

$$\int_{\mathbb{T}^2} c_1(E_n) \;=\; -\int_0^1 \Big[\int_{2/3}^1 n(d\varphi/dy)dy\Big]dx \;=\; n,$$

provided we choose the orientation $dy \wedge dx$ on $\mathbb{T}^2$. Thus, $\mathrm{ch}(E_n) = 1 + n[dy \wedge dx]$, in particular, the bundles E_n, $n \in \mathbb{Z}$ are all distinct.

An interesting extension involving these ideas is the following. Realize the two sphere $\mathbb{S}^2$ as two copies of the unit disc in the complex plane, $\mathbb{S}^2 = \mathbb{D}^2 \cup_{\mathbb{S}^1} \mathbb{D}^2$, which are identified on their boundaries ($= \mathbb{S}^1$) by the map $z \to 1/z$. Denote by E_n the $\mathbb{C}$ bundle over $\mathbb{S}^2$ which is given as follows.

$$E_n \;=\; (\mathbb{D}^2 \times \mathbb{C}) \cup_{\mathbb{S}^1} (\mathbb{D}^2 \times \mathbb{C}),$$

where the gluing map along $\mathbb{S}^1 \times \mathbb{C}$ is $(z, w) \to (1/z, z^n w)$. Then prove that $\displaystyle\int_{\mathbb{S}^2} c_1(E_n) = n$ where $\mathbb{S}^2$ is given its natural orientation as a complex manifold.

3. The Classical Signature

In this section, we recall the Hirzebruch Signature Theorem [H66], which is a special case of the Atiyah-Singer Index Theorem [AS68]. Associated to every compact oriented manifold M without boundary of dimension 4ℓ, there is a profound integer invariant $\sigma(M)$ defined as follows. The pairing $S : \mathrm{H}^{2\ell}(M;\mathbb{R}) \otimes \mathrm{H}^{2\ell}(M;\mathbb{R}) \to \mathbb{R}$,

$$S([\omega_1],[\omega_2]) \;= \;<[M],[\omega_1]\cup[\omega_2]> \;=\; \int_M \omega_1 \wedge \omega_2,$$

which is now a non-degenerate *symmetric* pairing, since the dimension of M is divisible by 4. It is a classical result that for any non-degenerate symmetric pairing of a finite dimensional vector space H, the space H may be written as $H = H_+ \oplus H_-$, where the pairing is $\pm$ definite on $H_\pm$. The spaces $H_\pm$ are not unique, but their dimensions are. The signature of M, $\sigma(M)$, is defined by

Definition 3.1. $$\sigma(M) \;=\; \dim \mathrm{H}^{2\ell}_+(M;\mathbb{R}) \;-\; \dim \mathrm{H}^{2\ell}_-(M;\mathbb{R}).$$

The Hirzebruch Signature Theorem states the following.

Theorem 3.2. *Suppose that M is a compact oriented manifold of dimension 4ℓ. Then the signature $\sigma(M)$ of M is an oriented homotopy invariant, and*

$$\sigma(M) \;=\; \int_M \mathrm{L}(TM).$$

Proof. If $f : M \to N$ is an oriented homotopy equivalence, it induces isomorphisms

$$f^* : \mathrm{H}^*(N;\mathbb{R}) \to \mathrm{H}^*(M;\mathbb{R}) \quad \text{and} \quad f_* : \mathrm{H}_*(M;\mathbb{Z}) \to \mathrm{H}_*(N;\mathbb{Z}),$$

where f^* is an **algebra** isomorphism, and $f_*([M]) \;=\; [N]$. Thus

$$S(f^*[\omega_1]), f^*[\omega_2]) \;=\; \int_M f^*(\omega_1) \wedge f^*(\omega_2) \;=\; \int_M f^*(\omega_1 \wedge \omega_2) \;=$$
$$\int_{f_*(M)} \omega_1 \wedge \omega_2 \;=\; \int_N \omega_1 \wedge \omega_2 \;=\; S([\omega_1],[\omega_2]).$$

So $\sigma(M)$ is an oriented homotopy invariant. Now for the formula.

First we fix the problem that $*^2 \neq I$. Denote by τ the involution of $\Omega^*(M)$, which on $\Omega^p(M)$ is given by $\tau = (-1)^p *$. As $\tau^2 = I$, we get the splitting

$$\Omega^*(M) \;=\; \Omega^*_+(M) \oplus \Omega^*_-(M)$$

where $\Omega^*_\pm = \pm 1$ eigenspaces of τ. The metric on M induces an inner product $(\cdot,\cdot)$ on $\Omega^*(M)$ which is given by $(\alpha,\beta) = \int_M \alpha \wedge *\beta$. Using this inner product, we may form the adjoint $d^* : \Omega^*(M) \to \Omega^*(M)$ of $d : \Omega^*(M) \to \Omega^*(M)$, and a simple computation shows that $d^* = - * d *$. Now consider the operator $d + d^* : \Omega^*(M) \to \Omega^*(M)$, which anti-commutes with τ, so it reverses the splitting $\Omega^*_+(M) \oplus \Omega^*_-(M)$. The operator $D^+ = d + d^* : \Omega^*_+(M) \to \Omega^*_-(M)$ is the signature operator of M. It is an elliptic operator, which implies that $\dim \mathrm{Ker}(D^+)$ and $\dim \mathrm{Coker}(D^+)$ are finite, so we may form the index of D^+, $\mathrm{Ind}(D^+) = \dim \mathrm{Ker}(D^+) - \dim \mathrm{Coker}(D^+) \in \mathbb{Z}$. The famous Atiyah-Singer Index Theorem [AS68] tells us that the index of any elliptic operator may be computed by integration over M of the product of certain characteristic classes of TM with the Chern character of an auxiliary bundle constructed out of the operator (the so-called symbol bundle of the operator). In the case of the signature operator, the formula reduces to

$$\mathrm{Ind}(D^+) \ = \ \int_M \mathrm{L}(TM).$$

Thus we need only show that $\sigma(M) = \mathrm{Ind}(D^+)$.

Recall that the harmonic forms on M, here denoted H, are given by $H = \mathrm{Ker}(\Delta) \subset \Omega^*(M)$, where $\Delta = (d + d^*)^2$. It is not difficult to show that $\omega \in H$ if and only if $d\omega = d^*\omega = 0$. The harmonic forms are isomorphic to the cohomology of M, that is each cohomology class on M has a unique harmonic representative. The operator $D^- = d + d^* : \Omega^*_-(M) \to \Omega^*_+(M)$ is also elliptic, and in fact is the adjoint of D^+. So $\dim \mathrm{Ker}(D^-) = \dim \mathrm{Coker}(D^+)$. Now $\mathrm{Ker}(D^\pm) = H \cap \Omega^*_\pm = H_\pm$. So we have $\mathrm{Ind}(D^+) = \dim H_+ - \dim H_-$. Since both $D^\pm$ anti-commute with τ, $\Delta\tau = \tau\Delta$, and τ restricts to the involution $\tau : H \to H$. Thus $H_\pm = \pm 1$ eigenspaces of $\tau \,|\, H$. For $0 \le k < 2\ell$, $\tau : V_k = H^k \oplus H^{4\ell-k} \to H^{4\ell-k} \oplus H^k = V_k$, and $H_+ \cap V_k = (1+\tau)H^k \simeq H^k \simeq (1-\tau)H^k = H_- \cap V_k$. So the V_k contribute nothing to $\mathrm{Ind}(D^+)$, that is $\dim(H_+ \cap V_k)$ and $\dim(H_- \cap V_k)$ cancel out when we compute $\mathrm{Ind}(D^+)$. Thus $\mathrm{Ind}(D^+) = \dim H^{2\ell}_+ - \dim H^{2\ell}_-$, where $H^{2\ell}_\pm = \pm 1$ eigenspaces of τ in $H^{2\ell} \ \simeq \ \mathrm{H}^{2\ell}(M;\mathbb{R})$. Now suppose $0 \ne \omega \in H^{2\ell}_+$. Then $\omega \ = \ \tau(\omega) \ = \ (-1)^{2\ell} * \omega \ = \ *\omega$, and

$$S(\omega,\omega) \ = \ \int_M \omega \wedge \omega \ = \ \int_M \omega \wedge *\omega \ > \ 0.$$

For $0 \ne \omega \in H^{2\ell}_-$, $\omega = -\tau(\omega) = - * \omega$, and

$$S(\omega,\omega) \ = \ \int_M \omega \wedge \omega \ = \ -\int_M \omega \wedge *\omega \ < \ 0.$$

Thus S is positive definite on $H^{2\ell}_+$, and negative definite on $H^{2\ell}_-$, so

$$\sigma(M) \ = \ \dim H^{2\ell}_+ - \dim H^{2\ell}_- \ = \ \mathrm{Ind}(D^+) \ = \ \int_M \mathrm{L}(TM).$$

□

The reader may wonder where the product $\prod_{j=1}^{\ell} x_j/\tanh(x_j)$ comes from in the definition of $\mathrm{L}(TM)$, that is what property does it have that makes it useful to us. The answer is that the coefficient of $x^{2\ell}$ in $(x/\tanh(x))^{2\ell+1}$ is 1. To see why we need this, consider the even dimensional complex projective space $\mathbb{C}P_{2\ell}$, which as a real manifold has dimension 4ℓ. We orient it by using the complex structure on $T\mathbb{C}P_{2\ell}$. The cohomology of $\mathbb{C}P_{2\ell}$ is the truncated polynomial algebra

$$\mathrm{H}^*(\mathbb{C}P_{2\ell};\mathbb{R}) \ = \ \mathbb{R}[x]/\{x^{2\ell+1} = 0\},$$

where x is a 2-dimensional class which satisfies $\int_{\mathbb{C}P_{2\ell}} x^{2\ell} = 1$. It follows immediately that $\sigma(\mathbb{C}P_{2\ell}) = 1$. Now the signature defines an algebra homomorphism from the oriented cobordism ring to the integers, and any such homomorphism is completely determined by the values it takes on the $\mathbb{C}P_{2\ell}$, since modulo torsion, these manifolds generate the oriented cobordism ring. In addition any such homomorphism must be of the form $[M^{4\ell}] \to \int_M K_\ell(p_1(TM),...,p_\ell(TM))$, where $K_\ell(y_1,...,y_\ell)$ is some polynomial. Because the homomorphism is multiplicative, these K_ℓ must have special properties. In particular, if we write formally $1 + p_1(TM) + \cdots + p_\ell(TM) = \prod_{i=1}^{k}(1 + x_i^2)$, where the x_i have degree 2, then there is some series $f(y)$ so that $\displaystyle\int_M K_\ell(p_1(TM), ..., p_\ell(TM)) = \int_M \prod_{i=1}^{k} f(x_i^2)$. Now, $1 + p_1(T\mathbb{C}P_{2\ell}) + \cdots + p_\ell(T\mathbb{C}P_{2\ell}) = (1 + x^2)^{2\ell+1}$, so we need a series $f(y)$ so that the coefficient of $x^{2\ell}$ in $[f(x^2)]^{2\ell+1}$ equals 1. The series $f(y) = \sqrt{y}/\tanh(\sqrt{y})$ is such a series.

The functions L_k are quite complicated in general, and their coefficients are non-integer rational numbers. For instance

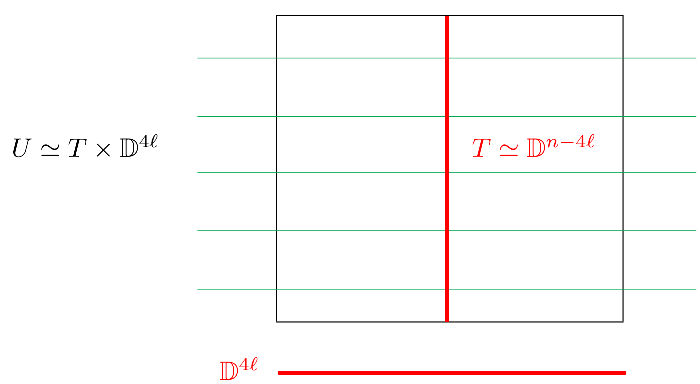
$$\mathrm{L}_4 \;=\; \frac{1}{14175}\Big(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4\Big).$$

This has very interesting applications. As a simple case consider $\mathrm{L}_1(p_1)$, which equals $\frac{1}{3}p_1$. We have immediately that for any compact, orientable, 4 dimensional manifold M^4, $\displaystyle\int_M p_1(TM)$ is an integer divisible by 3, since $\displaystyle\sigma(M) = \int_M \frac{1}{3}p_1(TM) \in \mathbb{Z}$.

For more on all of this, see [H66] and [M74].

4. Background on Foliations

The manifold M now may have arbitrary dimension, say n. A foliation F on M of dimension 4ℓ is a partition of M into disjoint submanifolds, each of dimension of 4ℓ, so that locally they are diffeomorphic to a product $T \times \mathbb{D}^{4\ell}$, where T (the transversal) is diffeomorphic to $\mathbb{D}^{n-4\ell}$, and $\mathbb{D}^k$ is the unit disk in $\mathbb{R}^k$. The connected submanifolds defining the foliation are called its leaves. We assume that F is Riemannian, which means that there is a metric on M so that the distance between leaves is constant. Since the foliation is locally trivial, M can be covered by a finite number of foliation charts, typically denoted U, which look as follows. The horizontal lines are the leaves of F. Note that they are a fixed distance apart. This does not happen in general foliations.



We extend the Hirzebruch Signature Theorem to such structures. In particular, we prove the following.

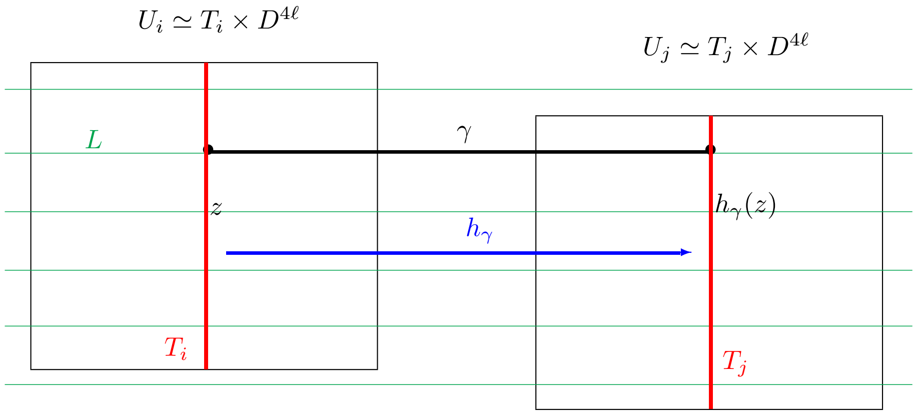
Theorem 4.1. *Suppose that M is a compact manifold with oriented Riemannian foliation F of dimension 4ℓ. Then the leafwise signature $\sigma(F)$ of F is an oriented leafwise homotopy invariant, and*

$$\sigma(F) = \int_F \mathrm{L}(TF).$$

Although the theorem looks very much like the classical signature theorem, this similarity is deceptive. There are a number of new concepts which need to be introduced and explained.

Holonomy and Local Integration

We will need the holonomy maps determined by F, as well as the local integration over F. Suppose we have two foliation charts U_i and U_j and a leaf L of F which passes through both of them. Choose a path γ in L, which starts on T_i and ends on T_j. Then we have the following picture.



The holonomy map h_γ has domain a subset of T_i and range a subset of T_j. Hueristically, it is obtained by sliding the transversal T_i along the path γ, keeping the points of T_i in their leaves, until we arrive at T_j. Then h_γ is a diffeomorphism, and the set of all such maps defines the holonomy pseudogroup $\mathcal{H}$. For each h_γ, we have the map $h_\gamma^* : \Omega_c^k(T_j \cap h_\gamma(T_i)) \to \Omega_c^k(T_i)$, where Ω_c^k denotes the smooth k forms with compact support.

Finally, given $\omega_i \in \Omega_c^{4\ell+k}(U_i)$, we get $\displaystyle\int \omega_i \in \Omega_c^k(T_i)$, which is just the intergration of the form ω_i over the fibers of the fibration $U_i \simeq T_i \times \mathbb{D}^{4\ell} \to T_i$.

Haefliger Cohomology

The invariant $\sigma(F)$ lives in the "cohomology" of the leaf space of F. The quotation marks appear since the leaf space of a foliation is usually a rather badly behaved space, so we can't use the usual cohomology of spaces. Instead we use the so-called Haefliger cohomology, [Ha80].

Choose a finite open cover of M by foliation charts $\{U_i\}$ for F, and choose transversals $T_i \subset U_i$ so that $T = \bigcup T_i$ is disjoint union. In $\Omega_c^k(T)$, consider the closed subspace $A^k = \overline{\operatorname{span}\{\alpha - h_\gamma^*\alpha\}}$, $h_\gamma \in \mathcal{H}$, the holonomy pseudogroup. Set

$$\Omega_c^k(M/F) = \Omega_c^k(T)/A^k.$$

The de Rham operator $d : \Omega_c^k(T) \to \Omega_c^{k+1}(T)$ induces a well defined operator $d_H : \Omega_c^k(M/F) \to \Omega_c^{k+1}(M/F)$. The Haefliger cohomology of F is the cohomology of this complex, and is denoted $\mathrm{H}_c^*(M/F)$. It is independent of all choices made in defining it. If F given by a fibration $M \to B$, then $\mathrm{H}_c^*(M/F) = \mathrm{H}^*(B;\mathbb{R})$.

Integration over F

We can now define the integration over the foliation F, which is a map $\displaystyle\int_F : \Omega^{4\ell+k}(M) \to \Omega_c^k(M/F)$, which commutes with the de Rham differentials. In particular, given $\omega \in \Omega^{4\ell+k}(M)$, write $\omega = \displaystyle\sum_i \omega_i$, where $\omega_i \in \Omega_c^{4\ell+k}(U_i)$. Then integrate ω_i along the fibers of $U_i \to T_i$ to get $\displaystyle\int \omega_i \in \Omega_c^k(T_i)$. The Haefliger differential form $\displaystyle\int_F \omega \in \Omega_c^k(M/F)$ is then defined to be the class of $\displaystyle\sum_i \int \omega_i$, which is well defined. As $d_H \circ \displaystyle\int_F = \int_F \circ\, d$, we get the well defined induced map in cohomology $\displaystyle\int_F : \mathrm{H}^{4\ell+k}(M) \to \mathrm{H}_c^k(M/F)$.

Homotopy groupoid $\mathcal{G}$ of F

In general, the foliation F does not define a fibration with M as total space (consider the one dimensional foliation of the two dimensional torus given by parallel lines with irrational slope). To overcome this defect, we work on another space, the homotopy groupoid $\mathcal{G}$ of F, where this problem disappears. The points of $\mathcal{G}$ are equivalence classes of leafwise paths in M, where two paths are equivalent if they are homotopic in their leaf, with the end points of the homotopy fixed. Then $\mathcal{G}$ is a fiber bundle over M, with the projection $s : \mathcal{G} \to M$ given by $s\big([\gamma]\big) = \gamma(0)$. Denote by F_s the foliation of $\mathcal{G}$ whose leaves are the fibers of s, that is $\widetilde{L}_x = s^{-1}(x)$. Denote by $r : \mathcal{G} \to M$ the map given by $r\big([\gamma]\big) = \gamma(1)$. Then $r : \widetilde{L}_x \to L_x$ is the simply connected cover of L_x, where L_x is the leaf of F containing the point $x \in M$. For each point $x \in M$ we have the element $\overline{x} \in \mathcal{G}$, which is the class of constant path at x. So, $x \to \overline{x}$ gives a diffeomorphism between M and the space of units $\mathcal{G}_0 \subset \mathcal{G}$. Thus we may consider M as a submanifold of $\mathcal{G}$.

Connections on Transversely Smooth Idempotents

Consider $\Omega^*_{(2)}(F_s) \to M$, which is the bundle of L^2 differential forms on leaves of F_s. Thus the fiber over $x \in M$ is the (infinite dimensional) space $(\Omega^*_{(2)}(F_s))_x = L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$.

An operator $A : \Omega^*_{(2)}(F_s) \to \Omega^*_{(2)}(F_s)$ assigns to each $x \in M$, an operator $A_x : L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x) \to L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$. If A_x is a sufficiently nice operator, it has a smooth Schwartz kernel K^A_x, which assigns to each pair of points $y, z \in \widetilde{L}_x$, an operator $K^A_x(y,z) \in \operatorname{Hom}((\wedge T^*\widetilde{L}_x)_z, (\wedge T^*\widetilde{L}_x)_y)$, so that for any $\xi \in L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$,

$$A(\xi)(y) \;=\; \int_{\widetilde{L}_x} K^A_x(y,z)\xi(z)\,dz.$$

We say that A is transversely smooth if all the derivatives of K^A_x with respect to x define operators on $L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$ which are smoothing and are bounded independently of x. Recall that an operator A_x on $L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$ is smoothing if it extends to an operator from any Sobolev space W^k associated to $L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$ to any other Sobolev space. Roughly speaking, for $k \in \mathbb{Z}^+$, W^k consists of all sections of $\wedge T^*\widetilde{L}_x$, with the property that all their derivatives up to order k are L^2 sections. For negative k, W^k is the dual of W^{-k}. Smoothing implies that A_x is bounded (but the bound might depend on x), and that all the derivarives of its Schwartz kernel $K^A_x(y,z)$ with respect to y and z define smoothing operators. A good example of such an operator is the orthogonal projection onto the harmonic forms in $L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$.

Any connection $\nabla : C^\infty(\wedge T^*F \otimes \wedge T^*M) \to C^\infty(\wedge T^*F \otimes \wedge T^*M)$ on $\wedge T^*F$, induces a connection ∇^{F_s} on $\wedge T^*F_s$, that is

$$\nabla^{F_s} : C^\infty(\wedge T^*F_s \otimes \wedge T^*\mathcal{G}) \to C^\infty(\wedge T^*F_s \otimes \wedge T^*\mathcal{G}).$$

Denote by $\nu^*_s \subset T^*\mathcal{G}$ the dual normal bundle of F_s, and note that $\nu^*_s = s^*(T^*M)$, so there is a natural inclusion and action of $\Omega^*(M)$ on $C^\infty(\wedge\nu^*_s)$. Denote by $p_\nu : \wedge T^*\mathcal{G} \to \wedge\nu^*_s$, the projection. Suppose that $\rho : \Omega^*_{(2)}(F_s) \to \Omega^*_{(2)}(F_s)$ is an idempotent, that is for all $x \in M$, $\rho^2_x = \rho_x$.

Definition 4.2. *A connection ∇ on ρ is a $\mathcal{G}$ invariant operator on $\Omega^*_{(2)}(F_s) \otimes C^\infty(\wedge\nu^*_s)$ of degree one, so that*

1. *for $\xi \in \Omega^*_{(2)}(F_s)$ and $\omega \in \Omega^k(M)$, $\nabla(\omega \otimes \xi) = \nabla\xi \wedge \omega + \xi \otimes d_M\omega$;*
2. *∇ can be written as*

$$\nabla \;=\; \rho\Big(p_\nu\nabla^{F_s} \;+\; A\Big)\rho,$$

where A a transversely smooth $\mathcal{G}$ invariant leafwise operator on $\Omega^*_{(2)}(F_s) \otimes C^\infty(\wedge\nu^*_s)$.

Connections always exist, since we may take $\nabla = \rho\Big(p_\nu\nabla^{F_s}\Big)\rho$. We need the inclusion of the operator A in the definition to insure that the pull-back of a connection is also connection.

The notion of $\mathcal{G}$ invariance for ∇ means that $\nabla \,|\, \widetilde{L}_{x_1} = \nabla \,|\, \widetilde{L}_{x_2}$, where $x_1, x_2 \in L$, and similarly for A.

As in the classical case, we call $\theta = \nabla^2$ the curvature of ∇, and we can use it to construct a Chern-Connes character of the idempotent ρ. All powers θ^k of θ are transversely smooth $\mathcal{G}$ invariant leafwise operators, and we denote their Schwartz kernels by $K_x^{\theta^k}(y,z)$. Recall that for each $x \in M$, $\overline{x}$ is the class of the constant path at x. Then for each k, and each $x \in M$, $K_x^{\theta^k}(\overline{x},\overline{x})$ is a linear operator on a finite dimensional space and so has a well-defined trace, $\mathrm{tr}(K_x^{\theta^k}(\overline{x},\overline{x}))$. In fact, property (1) of 4.2 and the proof of Lemma 2.3 show that θ^k is an $\Omega^*(M)$ equivariant operator, so we can take its equivariant trace, and interpret $\mathrm{tr}(K_x^{\theta^k}(\overline{x},\overline{x}))$ as a $2k$ form on M. With a little more work, we can see that $\mathrm{tr}(K_x^{\theta^k}(\overline{x},\overline{x}))$ is actually a normal form for F, that is a section of $C^\infty(\wedge^{2k}\nu^*)$. Denote by dx the volume form on the leaves of F. Thus $x \to \mathrm{tr}(K_x^{\theta^k}(\overline{x},\overline{x}))dx$ gives a well defined $4\ell + 2k$ differential form on M.

Definition 4.3. *Set* $\mathrm{Tr}(\theta^k) = \displaystyle\int_F \mathrm{tr}(K_x^{\theta^k}(\overline{x},\overline{x}))dx \quad \in \quad \Omega_c^{2k}(M/F)$.

In complete analogy with the classical case, we have

Proposition 4.4. $\mathrm{Tr}(\exp(-\theta/2i\pi))$ *is a closed Haefliger form, whose Haefliger class* $[\mathrm{Tr}(\exp(-\theta/2i\pi))]$ *is independent of* ∇.

Recall that the Chern character of a complex bundle E can be expressed as $\mathrm{ch}(E) = [\mathrm{tr}(\exp(-\theta/2i\pi))]$, which inspires the following definition. Note that we have used $\mathbb{R}$ valued forms here, but we could have just as well used $\mathbb{C}$ valued forms, and that for the twisted case (see below), the twisting bundle is in fact a $\mathbb{C}$ bundle.

Definition 4.5. $\mathrm{ch}_a(\rho) \;=\; [\mathrm{Tr}(\exp(-\theta/2i\pi))] \quad \in \quad \mathrm{H}_c^*(M/F)$.

5. The Higher Harmonic Signatures for Foliations

All the structures we considered in the classical case of the signature of a compact manifold extend to the leaves of the foliation F_s. In particular, on each leaf we have the involution τ, which gives an involution on $\Omega^*_{(2)}(F_s)$, so this bundle splits as $\Omega^*_{(2)}(F_s) = \Omega^*_+(F_s) \oplus \Omega^*_-(F_s)$. The leafwise operator $D = d + d^*$ reverses this splitting, and leafwise Laplacian $\Delta = D^2$ preserves the splitting. Denote by $\mathrm{Ker}(\Delta)$ the bundle over M whose fiber over x is $\mathrm{Ker}(\Delta_x) \subset L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$, and by $\rho_x : L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x) \to \mathrm{Ker}(\Delta_x)$ the orthogonal projection. The amalgamation of the ρ_x then defines the operator $\rho : \Omega^*_{(2)}(F_s) \to \Omega^*_{(2)}(F_s)$, the projection onto the leafwise harmonic forms. Denote by $\rho_\pm : \Omega^*_{(2)}(F_s) \to \mathrm{Ker}(\Delta^\pm_{2\ell}) = \mathrm{Ker}(\Delta) \cap \Omega^{2\ell}_\pm(F_s)$, the projection onto the plus and minus leafwise harmonic forms in the middle dimension. So for each $x \in M$, $(\rho_\pm)_x : L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x) \to (\mathrm{Ker}(\Delta^\pm_{2\ell}))_x = \mathrm{Ker}(\Delta_x) \cap L^2_\pm(\widetilde{L}_x, \wedge^{2\ell} T^*\widetilde{L}_x)$. Generalizing an unpublished result of Gong and Rothenberg, [GR97], we have

Theorem 5.1. *The projections* $\rho_\pm$ *are transversely smooth.*

Definition 5.2. *The Higher Harmonic Signature* $\sigma(F)$ *of* (M,F) *is the Haefliger class*

$$\sigma(F) = \mathrm{ch}_a(\rho_+) - \mathrm{ch}_a(\rho_-).$$

To see that this is an extension of the classical case, suppose that the foliation F consists of a single leaf, namely the entire manifold M, and for simplicity assume that M is simply connected. Then $\mathcal{G} = M \times M$, and for each $x \in M$, $\rho_{\pm,x}$ is just projection onto the $H^{2\ell}_\pm$, the $\pm$ harmonic forms on M in the middle dimension. Then, since the co-dimension of F is zero, we need only consider the $k=0$ term in $\mathrm{Tr}(\exp(-\nabla^2/2i\pi))$, so

$$\mathrm{ch}_a(\rho_\pm) \;=\; \int_F \mathrm{tr}(K_x^{\nabla^0}(\overline{x},\overline{x}))dx \;=\; \int_M \mathrm{tr}(K^\pm(x,x))dx,$$

where $K^\pm$ are the Schwartz kernels of the projections $\rho_\pm$. It is a fairly easy calculation to show that $\displaystyle\int_M \mathrm{tr}(K^\pm(x,x))dx = \dim(H^{2\ell}_\pm)$.

A leafwise homotopy equivalence between foliated manifolds (M,F) and (M',F') is a smooth map $f : M \to M'$ which takes leaves to leaves, so that there is a smooth map $g : M' \to M$, which takes leaves to leaves, and smooth maps $H : M \times [0,1] \to M'$ and $H' : M' \times [0,1] \to M$, which take a leaf $\times[0,1]$ to a leaf, so that $H(x,1) = x$, $H(x,0) = g \circ f(x)$, $H'(x',1) = x'$, and $H(x',0) = f \circ g(x')$. The equivalence is oriented if both foliations are oriented, and f and g preserve the orientations. If f is such a map, then f induces the isomorphism on Haefliger cohomology $f^* : \mathrm{H}^*_c(M'/F') \to \mathrm{H}^*_c(M/F)$.

Recall the First Main Theorem

Theorem 5.3. *Suppose that M is a compact manifold with oriented Riemannian foliation F of dimension 4ℓ. Then the leafwise signature $\sigma(F)$ of F_s is an oriented leafwise homotopy invariant, and*

$$\sigma(F) = \int_F \mathrm{L}(TF).$$

Now consider the space $C^\infty_c(\mathcal{G}; \mathrm{Hom}(\wedge T^*F_s))$, where the fiber of $\mathrm{Hom}(\wedge T^*F_s)$ over an element $[\gamma] \in \mathcal{G}$ consists of all linear homomorphisms from $\wedge T^*F_{s(\gamma)}$ to $\wedge T^*F_{r(\gamma)}$, and note that we may identify $(\wedge T^*F_s)_{[\gamma]}$ with $\wedge T^*F_{r(\gamma)}$. If $A \in C^\infty_c(\mathcal{G}; \mathrm{Hom}(\wedge T^*F_s))$, it defines a leafwise operator on $\wedge T^*F_s$, which is $\mathcal{G}$ invariant, bounded, and smoothing. For $\xi \in L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x)$, and $[\gamma] \in \widetilde{L}_x$,

$$A(\xi)([\gamma]) \;=\; \int_{\widetilde{L}_x} A(\gamma\gamma_1^{-1})\xi(\gamma_1).$$

As above, the operator D defines $D^+ : \Omega^*_+(F_s) \to \Omega^*_-(F_s)$, which we call the leafwise signature operator. On each leaf, D^+ is elliptic, and it is $\mathcal{G}$ invariant and invertible modulo $C^\infty_c(\mathcal{G}; \mathrm{Hom}(\wedge T^*F_s))$, so by a now classical procedure, [C81], it has an index class

$$\mathrm{Ind}^\infty_c(D^+) \quad \in \quad \mathrm{K}_0(C^\infty_c(\mathcal{G}; \mathrm{Hom}(\wedge T^*F_s))).$$

In [BH08], we constructed a Chern-Connes character also denoted $\mathrm{ch}_a : \mathrm{K}_0(C^\infty_c(\mathcal{G}; \mathrm{Hom}(\wedge T^*F_s))) \to \mathrm{H}^*_c(M/F)$, and we showed that

Theorem 5.4.

$$\mathrm{ch}_a(\mathrm{Ind}^\infty_c(D^+)) = \int_F \mathrm{L}(TF).$$

Recent results of Azzali, Goette, and Schick [AGS], improving results of [HL99] and [BH08] immediately give the following.

Theorem 5.5. *Suppose that M is a compact manifold with oriented Riemannian foliation F of dimension 4ℓ. Then*

$$\mathrm{ch}_a(\mathrm{Ind}^\infty_c(D^+)) = \sigma(F).$$

Thus we have that $\sigma(F) = \displaystyle\int_F \mathrm{L}(TF)$, and it only remains to show that $\sigma(F)$ is a leafwise homotopy invariant, that is, if $f : (M, F) \to (M', F')$ is an oriented leafwise homotopy equivalence, then $f^*(\sigma(F')) = \sigma(F)$. An outline of the proof of this is in Section 7.

Second Main Theorem

We can extend the First Main Theorem to the higher harmonic signature of an even dimensional oriented Riemannian foliation F of a compact manifold M with coefficients in a leafwise $U(p, q)$-flat complex bundle. In particular, suppose that the dimesnion of F is 2ℓ, and let E be a complex bundle over M, which restricted to any leaf of F is flat, i.e. a leafwise flat bundle. Assume that E admits a non-degenerate *possibly indefinite* Hermitian metric, i.e. a $U(p, q)$ structure, which is preserved by the leafwise flat structure. The bundle $E \,|\, L$ pulls back to a flat bundle (also denoted E) on each $\widetilde{L}_x$, and it determine leafwise Laplacians Δ^E and Hodge $*$ operators on the differential forms on $\widetilde{L}_x$ with coefficients in E. The Hodge operator determines an involution on forms with coefficients in E which commutes with Δ^E, so Δ^E splits as a sum $\Delta^E = \Delta^{E,+} + \Delta^{E,-}$, in particular in dimension ℓ, $\Delta^E_\ell = \Delta^{E,+}_\ell + \Delta^{E,-}_\ell$. We assume that the projection onto $\mathrm{Ker}(\Delta^E_\ell)$ is transversely smooth, which implies that the projections $\rho^E_\pm$ onto $\mathrm{Ker}(\Delta^{E,+}_\ell)$ and $\mathrm{Ker}(\Delta^{E,-}_\ell)$ are transversely smooth. This is true whenever the leafwise parallel translation on E defined by the flat structure is a bounded map, in particular whenever the preserved metric on E is positive definite. It is satisfied for important examples, e.g., the examples of Lusztig [Lu72] which proved the Novikov conjecture for free abelian groups, and it is always true whenever E is a bundle associated to the normal bundle of the foliation. Of course, the smoothness assumption is fulfilled for the (untwisted) leafwise signature operator, since this is Theorem 5.1.

Definition 5.6. *The higher harmonic signature of F twisted by E is*

$$\sigma(F,E) \;=\; \mathrm{ch}_a(\rho_+^E) - \mathrm{ch}_a(\rho_-^E).$$

Our Second Main Theorem is the following.

Theorem 5.7. Suppose that M is a compact manifold, with oriented Riemannian foliation F of dimension 2ℓ, and that E is a leafwise flat complex bundle over M with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the projection onto $\mathrm{Ker}(\Delta_\ell^E)$ for the associated foliation F_s of the homotopy groupoid of F is transversely smooth. Then $\sigma(F,E)$ is a leafwise homotopy invariant.

The proof of this theorem is essentially the same as for the First Main Theorem, with some complications introduced because of the auxiliary bundle E.

Note that Theorem 5.7 does not give a formula for $\sigma(F,E)$ in terms of characteristic classes. The bundle E splits as $E = E^+ \oplus E^-$, where the metric is $\pm$ definite on $E^\pm$. The splitting isn't unique, but the isomorphism classes of the bundles $E^\pm$ are. We have the following [BH08].

Conjecture 5.8.

$$\sigma(F,E) = \int_F \mathrm{L}(TF)\Big(\mathrm{ch}(E^+) - \mathrm{ch}(E^-)\Big).$$

In [HL99], and improved in [BH08], this conjecture was proven for certain foliations with nice spectra. In [AGS], Azzali, Goette, and Schick prove it for globally flat E. We expect that their proof will extend to leafwise flat E.

6. Applications

We briefly state two applications of Theorem 5.7. For more results of this form, see [BH09].

First recall the famous conjecture of Novikov.

Conjecture 6.1 (Novikov)**.** *Suppose that N is a compact manifold with universal cover $\widetilde{N}$, and that $f : N \to B\pi_1 N$ classifies the $\pi_1 N$ bundle $\widetilde{N} \to N$. Then for any $x \in \mathrm{H}^*(B\pi_1 N;\mathbb{Q})$, $\displaystyle\int_N \mathrm{L}(TN) f^*(x)$ is a homotopy invariant.*

Theorem 5.7 implies this conjecture for $\pi_1 N = \mathbb{Z}^n$ and for all surface groups. The original proofs were given by Lusztig [Lu72]. In addition, Theorem 5.7 implies this conjecture for any cohomology class in $\mathrm{H}^*(B\pi_1 N;\mathbb{Q})$ of the form $\mathrm{ch}(E^+) - \mathrm{ch}(E^-)$, where $E^+ \oplus E^-$ is a $U(p,q)$ flat bundle over $B\pi_1 N$.

Baum and Connes have extended the Novikov conjecture to the case of foliations.

Conjecture 6.2 (Baum-Connes)**.** *Suppose that F is a foliation of a compact manifold M and that $f : M \to B\mathcal{G}$ is a classifying map for F. Then for any $x \in \mathrm{H}^*(B\mathcal{G};\mathbb{Q})$, $\displaystyle\int_F \mathrm{L}(TF) f^* x$ is a leafwise homotopy invariant.*

Theorem 5.7 implies this for conjecture for any cohomology class in $\mathrm{H}^*(B\mathcal{G};\mathbb{Q})$ of the form $\mathrm{ch}(E^+) - \mathrm{ch}(E^-)$, where $E^+ \oplus E^-$ is a $U(p,q)$ flat bundle over $B\mathcal{G}$.

7. Outline of the Proof of the First Main Theorem

Suppose that $f : M, F \to M', F'$ is an oriented leafwise homotopy equivalence between oriented Riemannian foliations. We must show that $f^*(\sigma(F')) = \sigma(F)$. In fact we show more. Denote by $\rho'_\pm$ the idempotents used in the definition of $\sigma(F')$. Then we show that $f^*(\mathrm{ch}_a(\rho'_\pm)) = \mathrm{ch}_a(\rho_\pm)$. As the two proofs are identical, we concentrate on the case of ρ'_+.

Since we are working on the homotopy groupoids, we extend f to a leafwise map $\widetilde{f} : \mathcal{G}, F_s \to \mathcal{G}', F'_s$ by setting $\widetilde{f}([\gamma]) = [\widetilde{f} \circ \gamma]$. Our first step is to use $\widetilde{f}^*$ to pull back the idempotent ρ'_+ to an idempotents ρ^f_+. There are two problems with this simple definition of $\widetilde{f}$ which prevent us from doing this.

First, it might be the case that $\widetilde{f}^*$ does not induce a map on L^2 leafwise forms. To see what can go wrong, it is an instructive exercise to construct an oriented leafwise homotopy equivalence of an irrational constant slope foliation of the two torus to itself which does not take leafwise L^2 forms to L^2 forms. To solve this problem, we adapt the method of Hilsum-Skandalis [HiS92], which essentially says that we can "fatten up" $\widetilde{f}$ to a map, also denoted $\widetilde{f}$, which is a leafwise submersion. Once we have done this, a good deal of analysis shows that the new $\widetilde{f}$ induces bounded maps on all leafwise Sobolev spaces.

The second problem we encounter is that the action of the Hilsum-Skandalis $\widetilde{f}^*$ on the algebra of forms, or more correctly the leafwise cohomology algebra, is not so obvious. To solve this problem, we use the results of [HL91] (à la Dodziuk [D77]) to construct another $\widetilde{f}^*$, which passes through the leafwise simplicial cohomology of F_s and does have the algebraic properties we require. We then show that on leafwise cohomology the Heitsch-Lazarov $\widetilde{f}^*$ is the same as the Hilsum-Skandalis $\widetilde{f}^*$.

Now we are in a position to define the pull-back of the transversely smooth idempotent ρ'_+. Denote by $\rho_{2\ell}$ the projection to the leafwise harmonic forms in dimension 2ℓ, that is to $\mathrm{Ker}(\Delta_{2\ell})$.

Definition 7.1. *Let $g : M', F' \to M, F$ be a homotopy inverse for f. Set*

$$\rho^f_+ = \widetilde{f}^* \rho'_+ \widetilde{g}^* \rho_{2\ell}.$$

Proposition 7.2. *The map ρ^f_+ is a transversely smooth idempotent.*

Proof. To see that ρ^f_+ is an idempotent, note that

$$(\rho^f_+)^2 \;=\; \widetilde{f}^* \rho'_+ \widetilde{g}^* \rho_{2\ell} \widetilde{f}^* \rho'_+ \widetilde{g}^* \rho_{2\ell}.$$

Here we are actually working on cohomology, since $\rho_{2\ell}$ and ρ'_+ have image in the harmonic forms. But on cohomology (so also on the harmonic forms) $\widetilde{g}^*$ and $\widetilde{f}^*$ are inverses of each other, so $\widetilde{g}^* \rho_{2\ell} \widetilde{f}^* \rho'_+ = \rho'_+$, and

$$(\rho^f_+)^2 \;=\; \widetilde{f}^* \rho'_+ \rho'_+ \widetilde{g}^* \rho_{2\ell} \;=\; \widetilde{f}^* \rho'_+ \widetilde{g}^* \rho_{2\ell} \;=\; \rho^f_+.$$

Next, we need to show that ρ^f_+ and all its transverse derivatives take any leafwise Sobolev space to any other leafwise Sobolev space. As the idempotents $\rho_{2\ell}$ and ρ'_+ are transversely smooth, they do take any leafwise Sobolev space to any other leafwise Sobolev space. Now, $\widetilde{f}^*$ and $\widetilde{g}^*$ are bounded maps on all leafwise Sobolev k spaces, so we have that ρ^f_+ takes any leafwise Sobolev space to any other leafwise Sobolev space.

To see that the transverse derivatives of ρ^f_+ take any leafwise Sobolev space to any other leafwise Sobolev space, we need to relate the transverse derivatives on $\mathcal{G}'$ to those on $\mathcal{G}$. Recall the projections $p_\nu : \wedge T^*\mathcal{G} \to \wedge \nu^*_s$, and $p'_\nu : \wedge T^*\mathcal{G}' \to \wedge {\nu'}^*_s$. The transverse derivatives are computed by using the transverse de Rham operators $d_\nu = p_\nu d_{\mathcal{G}}$ and $d'_\nu = p'_\nu d_{\mathcal{G}'}$, coupled with interior product with transverse vector fields. Here we have denoted by $d_{\mathcal{G}}$ and $d_{\mathcal{G}'}$ the usual de Rham operators on $\mathcal{G}$ and $\mathcal{G}'$. Denote by d_s and d'_s the leafwise de Rham operators for F_s and F'_s. The crucial lemma is the following.

Lemma 7.3.

$$d_\nu \widetilde{f}^* - \widetilde{f}^* d'_\nu = \widetilde{f}^* d'_s - d_s \widetilde{f}^* \quad \text{and} \quad d'_\nu \widetilde{g}^* - \widetilde{g}^* d_\nu = \widetilde{g}^* d_s - d'_s \widetilde{g}^*.$$

This lemma allows us to transform questions about transverse derivatives into questions about tangential derivatives. Since the operators d_s and d'_s are the leafwise de Rham operators, they take any leafwise Sobolev k space to the leafwise Sobolev $k-1$ space. Then a good deal of functional analysis shows that since we are composing with transversely smooth operators, which take any leafwise Sobolev space to any other leafwise Sobolev space, the transverse derivatives of the ρ_+^f take any leafwise Sobolev space to any other leafwise Sobolev space, and so finishes the proof. $\square$

Proposition 7.4.
$$f^* \operatorname{ch}_a(\rho'_+) = \operatorname{ch}_a(\rho^f_+).$$

Proof. Given any connection ∇' on ρ'_+, we can define the pull-back connection $\nabla = \widetilde{f}^*(\nabla')$ on ρ^f_+ (more or less in the usual way, but of course with complications). This is where the use of the transversely smoothing operator A in the definition of a connection comes into play. It is not necessarily true that the pull-back of a connection on ρ'_+ is the compression of a connection on $\wedge T^* F_s$ to the pulled back bundle ρ^f_+. Then $\theta = \widetilde{f}^*(\theta')$ and $\operatorname{Tr}(\theta^k) = f^* \operatorname{Tr}({\theta'}^k)$ for all k, which gives the result. $\square$

The following is a standard result for Chern-Connes characters defined on idempotents.

Proposition 7.5. *If e_t, $0 \leq t \leq 1$, is a smooth family of $\mathcal{G}$ invariant transversely smooth idempotents, then* $\operatorname{ch}_a(e_0) = \operatorname{ch}_a(e_1)$.

Proposition 7.6.
$$\operatorname{ch}_a(\rho^f_+) = \operatorname{ch}_a(\rho_{2\ell}\rho^f_+).$$

Proof. A simple computation shows that $(1-t)\rho_{2\ell}\rho^f_+ + t\rho^f_+$ is a family of idempotents. As both $\rho_{2\ell}\rho^f_+$ and ρ^f_+ are transversely smooth, we are done. $\square$

The last major result we need is the following.

Proposition 7.7.
$$\operatorname{ch}_a(\rho_{2\ell}\rho^f_+) = \operatorname{ch}_a(\rho_+).$$

Proof. The proof of this proposition involves a good deal of heavy functional analysis, which is used to show the following two results. First, we show that the restriction of ρ_+ to $\operatorname{Im}(\rho_{2\ell}\rho^f_+)$ is an isomorphism onto $\operatorname{Im}(\rho_+)$ with uniformly bounded inverse, which we denote $\rho_+^{-1} : \operatorname{Im}(\rho_+) \to \operatorname{Im}(\rho_{2\ell}\rho^f_+)$. Second, we show that
$$\varphi_+ \;=\; \rho_+^{-1} \circ \rho_+ : \Omega^{2\ell}_{(2)}(F_s) \to \operatorname{Im}(\rho_{2\ell}\rho^f_+)$$
is a transversely smooth idempotent.

To finish the proof of the proposition, and so also the First Main Theorem, we need two easy results. Since the transversely smooth idempotents φ_+ and $\rho_{2\ell}\rho^f_+$ have the same image, $t\varphi_+ + (1-t)\rho_{2\ell}\rho^f_+$ is a smooth family of transversely smooth idempotents, and we have
$$\operatorname{ch}_a(\rho_{2\ell}\rho^f_+) = \operatorname{ch}_a(\varphi_+).$$

Finally, since φ_+ is projection onto $\operatorname{Im}(\rho_{2\ell}\rho^f_+)$ along $\operatorname{Ker}(\rho_+)$, we have $\varphi_+\rho_+ = \varphi_+$ and $\rho_+\varphi_+ = \rho_+$. Thus, $t\varphi_+ + (1-t)\rho_+$ is a smooth family of transversely smooth idempotents, and
$$\operatorname{ch}_a(\varphi_+) = \operatorname{ch}_a(\rho_+).$$

$\square$

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