THE TWISTED HIGHER HARMONIC SIGNATURE FOR FOLIATIONS

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Abstract

We prove that the higher harmonic signature of an even dimensional oriented Riemannian foliation F of a compact Riemannian manifold M with coefficients in a leafwise $U(p,q)$-flat complex bundle is a leafwise homotopy invariant. We also prove the leafwise homotopy invariance of the twisted higher Betti classes. Consequences for the Novikov conjecture for foliations and for groups are investigated.

CONTENTS

1. Introduction

In this paper, we prove that the higher harmonic signature, $\sigma(F, E)$, of a 2ℓ dimensional oriented Riemannian foliation F of a compact Riemannian manifold M, twisted by a leafwise flat complex bundle E over M, is a leafwise homotopy invariant. We also derive important consequences for the Novikov conjecture for foliations and for groups. We assume that E admits a non-degenerate *possibly indefinite* Hermitian metric which is preserved by the leafwise flat structure. As explained in [**G96**], this includes the leafwise $O(p, q)$-flat and the leafwise symplectic-flat cases. We assume that the projection onto the twisted leafwise harmonic forms in dimension ℓ is transversely smooth. This is true whenever the leafwise parallel translation on E defined by the flat structure is a bounded map, in particular whenever the preserved metric on E is positive definite. It is satisfied for important examples, e.g. the examples of Lusztig [**Lu72**] which proved the Novikov conjecture for free abelian groups, and it is always true whenever E is a bundle associated to the normal bundle of the foliation. In particular, the smoothness assumption is fulfilled for the (untwisted) leafwise signature operator.

Any metric on M determines a metric on each leaf L of F, so also on all covers of L. The bundle $E \,|\, L$ can be pulled back to a flat bundle (also denoted E) on any cover of L. These leafwise metrics and the leafwise flat bundle E determine leafwise Laplacians Δ^E and Hodge $*$ operators on the differential forms on L with coefficients in $E \,|\, L$, as well as on all covers of L. The Hodge operator determines an involution which commutes with Δ^E, so Δ^E splits as a sum $\Delta^E = \Delta^{E,+} + \Delta^{E,-}$, in particular in dimension ℓ, $\Delta_\ell^E = \Delta_\ell^{E,+} + \Delta_\ell^{E,-}$. To each leaf L of F we associate the formal difference of the (in general, infinite dimensional) spaces $\mathrm{Ker}(\Delta_\ell^{E,+})$ and $\mathrm{Ker}(\Delta_\ell^{E,-})$ on $\widetilde{L}$, the simply connected cover of L. We assume that the Schwartz kernel of the projection onto $\mathrm{Ker}(\Delta_\ell^E) = \mathrm{Ker}(\Delta_\ell^{E,+}) \oplus \mathrm{Ker}(\Delta_\ell^{E,-})$ varies smoothly transversely. Roughly speaking, transverse smoothness means that the $\mathrm{Ker}(\Delta_\ell^{E,\pm})$ are "smooth bundles over the leaf space of F". There is a Chern-Connes character ch_a for such bundles which takes values in the Haefliger cohomology of F, [**BH08**]. The higher harmonic signature of F is defined as

$$\sigma(F, E) = \mathrm{ch}_a(\mathrm{Ker}(\Delta_\ell^{E,+})) - \mathrm{ch}_a(\mathrm{Ker}(\Delta_\ell^{E,-})).$$

Our main theorem is the following.

Theorem 9.1. *Suppose that M is a compact Riemannian manifold, with an oriented Riemannian foliation F of dimension 2ℓ, and that E is a leafwise flat complex bundle over M with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the projection onto $\mathrm{Ker}(\Delta_\ell^E)$ for the associated*

foliation F_s of the homotopy groupoid of F is transversely smooth. Then $\sigma(F, E)$ is a leafwise homotopy invariant.

In particular, suppose that M', F', and E' satisfy the hypothesis of Theorem 9.1, and that $f : M \to M'$ is a leafwise homotopy equivalence, which is leafwise oriented. Set $E = f^*(E')$ with the induced leafwise flat structure and preserved metric. Then f induces an isomorphism f^* from the Haefliger cohomology of F' to that of F, and

$$f^*(\sigma(F', E')) = \sigma(F, E).$$

A priori, $\sigma(F, E)$ depends on the metric on M. However, it is an immediate corollary of Theorem 9.1 that it is independent of this metric since the identity map is a leafwise homotopy equivalence between $(M, F; g_0)$ and $(M, F; g_1)$. In general, $\sigma(F, E)$ depends on the flat structure and the metric on E, in particular on the splitting of $E = E^+ \oplus E^-$ into positive (resp. negative) definite sub bundles.

Our techniques also give the leafwise homotopy invariance of the twisted higher Betti classes. When the twisting bundle E is trivial, this extends (in the Riemannian case) the main theorem of [**HL91**].

Theorem 10.6 *Suppose that M is a compact Riemannian manifold, with an oriented Riemannian foliation F of dimension p. Let E be a leafwise flat complex bundle over M with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the projection onto $\mathrm{Ker}(\Delta^E)$ for the associated foliation F_s of the homotopy groupoid of F is transversely smooth. Then the twisted higher Betti classes $\beta_j(F, E)$, $0 \le j \le p$, are leafwise homotopy invariants.*

We now give some background to place the results of this paper in context.

Let M and M' be closed oriented manifolds with oriented foliations F and F'. Let $\varphi : (M', F') \to (M, F)$ be an oriented, leafwise oriented, leafwise homotopy equivalence. Denote the homotopy groupoid of F by $\mathcal{G}$, and let $f : M \to B\mathcal{G}$ be a classifying map for F. The BC (Baum-Connes) Novikov conjecture predicts that for every $x \in H^*(B\mathcal{G}; \mathbb{R})$,

$$\int_M \mathbb{L}(TF) \cup f^*x = \int_{M'} \mathbb{L}(TF') \cup (f \circ \varphi)^*x.$$

It is easy to check that this conjecture reduces to the case where the leaves have even dimension. In the case of a foliation with a single closed leaf with fundamental group Γ and denoting by $f : M \to B\Gamma$ a classifying map for the universal cover of M, the BC Novikov conjecture reduces to the classical Novikov conjecture

$$\int_M \mathbb{L}(TM) \cup f^*x = \int_{M'} \mathbb{L}(TM') \cup (f \circ \varphi)^*x, \quad \forall x \in H^*(B\Gamma; \mathbb{R}).$$

A powerful approach to the Novikov conjecture was initiated by Kasparov in [**K88**]. He actually proves a stronger version of the Novikov conjecture, namely the injectivity of the famous Baum-Connes map [**KS03**, **HgK01**, **La02**]. See [**T99**] for a proof of this injectivity for a large class of foliations, including hyperbolic foliations. Note that it is still an open question whether the Baum-Connes map is rationally injective for Riemannian foliations.

A second approach to the Novikov conjecture was initiated by Connes and his collaborators [**CM90**] and uses cyclic cohomology and the homotopy invariance of the Miscenko symmetric signature in the K-theory of the reduced group C^*-algebra [**K88**, **M78**]. This method proved successful, [**CGM93**], for the largest known class of groups, including Gromov-hyperbolic groups. For foliations, the homotopy invariance of the corresponding Miscenko class in the K-theory of the C^*-algebra of $\mathcal{G}$ was explained in [**BC00**, **BC85**] and proved independently in [**KaM85**] and [**HiS92**]. It reduces the BC Novikov conjecture to an extension problem in the K-theory of foliations, together with a cohomological longitudinal index formula. The extension problem was first solved by Connes for certain cocycles in [**C86**] by using a highly non trivial analytic breakthrough. For general cocycles, the extension problem is a serious obstacle and many efforts have been made in this direction [**Cu04**, **CuQ97**, **LMN05**, **N97**, **P95**, **Me**]. See also the recent [**Ca**] for an alternative approach.

The present paper was inspired by a third method mainly due to Lusztig [**Lu72**], and to ideas of Gromov [**G96**]. It relies on the fact that for discrete groups having *enough finite dimensional $U(p,q)$ representations*, the even cohomology of the classifying space $B\Gamma$ is generated by $U(p,q)$ flat K-theory classes. The main theorem needed in this approach is the oriented homotopy invariance of the twisted signature by such K-theory classes. This approach has been extended in [**CGM90**, **CGM93**] to cover all the known cases, using the concept of groups having *enough almost representations* and almost flat K-theory classes.

Recall that in non-commutative geometry, the index of an elliptic operator is usually defined as a certain C^*-algebra K theory class constructed out of the operator itself, without reference to its kernel or cokernel. In the special (commutative) sub-case of a fibration, the Chern character of this operator K theory class coincides with the Chern character of the index bundle determined by the operator. In the (non-commutative) case of foliations, this equality is not known in general. See [**BH08**], where conditions are given for it to hold, as well as [**N97**] and the recent [**AGS**]. For the signature operator, and its twists by leafwise almost flat K-theory classes, the C^*-algebra K-theory index is well known to be a leafwise homotopy invariant of the foliation [**HiS92**].

However, in order to deduce explicit results on the BC Novikov conjecture for foliations, one needs to define a Chern-Connes character of this C^*-algebra K-theory class and to compute it. Our approach to this problem is to use the index bundle of the twisted leafwise signature operator, whose Chern-Connes character in Haefliger cohomology is well defined as soon as the bundle is. It is therefore a natural problem to prove directly the homotopy invariance of the Chern-Connes character of the leafwise signature index bundle and its twists by leafwise (almost) flat K-theory classes.

Our program to attack the BC Novikov conjecture for foliations consists of three steps.

- Given a K-theory class $y = [E^+] - [E^-]$ over $B\mathcal{G}$, prove that the characteristic number $\displaystyle\int_M \mathbb{L}(TF) \cup f^* \operatorname{ch}(y)$ equals the higher leafwise harmonic signature twisted by f^*y.
- Prove that the higher leafwise harmonic signature twisted by leafwise almost flat K-theory classes of the ambient manifold is a leafwise oriented, leafwise homotopy invariant.
- Prove that complex bundles $E = E^+ \oplus E^-$, such that $[f^*E^+] - [f^*E^-]$ is a leafwise almost flat K-theory class, generate the K-theory of $B\mathcal{G}$.

It is clear that solving these three problems for a class of foliations implies the BC Novikov conjecture for that class. The first step was partially completed in our previous papers [**BH04**, **BH08**], where we proved this equality under certain assumptions, which were subsequently removed in [**AGS**], provided the bundle $E^+ \oplus E^-$ is *globally* flat. We conjecture that the result is still true under the far less restrictive assumption that $E^+ \oplus E^-$ is only *leafwise* flat. The second step is the goal of the present paper, when the coefficient bundle E has a leafwise flat structure and the foliation is Riemannian. See [**BH**] for further results on this question.

Our results so far on the third step rely on deep but now classical results of Gromov [**G96**], and allow us to prove, for instance, the BC Novikov conjecture, without extra assumptions, for the subring of $H^*(B\mathcal{G};\mathbb{R})$ generated by $H^1(B\mathcal{G};\mathbb{R})$ and $H^2(B\mathcal{G};\mathbb{R})$. Again see the forthcoming paper [**BH**].

We now briefly describe the contents of each section. Section 2 contains notation and some review. In Section 3, we recall the Chern-Connes character for transversely smooth idempotents, which takes values in the Haefliger cohomology of the foliation. In Section 4, we define the twisted higher harmonic signature, and prove that if the parallel translation using the flat structure on E is bounded, then the projection to the twisted harmonic forms is transversely smooth. Section 5

contains two important concepts essential to the proof of our main theorem, namely the notion of a "smooth bundle over the space of leaves of F", and the extension to such bundles of the classical Chern-Weil theory of characteristic classes. This allows us to compare the characteristic classes of such bundles on different manifolds. Section 6 contains results on leafwise homotopy equivalences. Outlines of the proofs are given in the Appendix. In general, leafwise homotopy equivalences do not behave well on Sobolev spaces. To overcome this problem, we use a construction, due to Hilsum-Skandalis [**HiS92**], which produces smooth bounded maps between Sobolev spaces. We then use the Whitney isomorphism between simplicial and smooth cohomology to get control of the leafwise cohomologies. In Section 7, we prove that the pull-backs under leafwise homotopy equivalences of certain smooth bundles over the space of leaves are still smooth bundles. Section 8 extends the notion of pulled-back connections. Section 9 contains the proof of the main theorem. In Section 10, we prove the equality between the twisted higher harmonic signature and the Chern-Connes character of the index bundle of the twisted leafwise signature operator. We explain how our methods extend to prove Theorem 10.6. We also conjecture a cohomological formula for the twisted higher harmonic signature, which is already known to be true in some cases. See [**H95**, **HL99**, **BH08**] and the forthcoming [**AGS**]. Finally, in Section 11 we show how our results lead to important consequences for the Novikov conjecture for foliations and for groups.

2. Notation and review

Throughout this paper M denotes a smooth compact Riemannian manifold of dimension n, and F denotes an oriented Riemannian foliation of M of dimension $p = 2\ell$ and codimension q. So $n = p + q$. The tangent bundle of F is denoted by TF, its normal bundle by ν, and its dual normal bundle by ν^*. We assume that the metric on M, when restricted to ν, is bundle like, so the holonomy maps of ν and ν^* are isometries. A leaf of F is denoted L. We denote by $\mathcal{U}$ a finite good cover of M by foliation charts as defined in [**HL90**].

If $V \to N$ is a vector bundle over a manifold N, we denote the space of smooth sections by $C^\infty(V)$ or by $C^\infty(N; V)$ if we want to emphasize the base space of the bundle. The compactly supported sections are denoted by $C_c^\infty(V)$ or $C_c^\infty(N; V)$. The space of differential k forms on N

is denoted $\mathcal{A}^k(N)$, and we set $\mathcal{A}^*(N) = \oplus_{k \geq 0} \mathcal{A}^k(N)$. The space of compactly supported k forms is denoted $\mathcal{A}_c^k(N)$, and $\mathcal{A}_c^*(N) = \oplus_{k \geq 0} \mathcal{A}_c^k(N)$. The de Rham exterior derivative is denoted d or d_N. The tangent and cotangent bundles of N will be denoted TN and T^*N.

We will freely use the notation and constructions of [**BH08**] in this paper (very briefly recalled below), with two notable differences. In constructing the (reduced) Haefliger cohomology of F, [**Ha80**], we use the quotient of $\mathcal{A}_c^k(T)$, where T is a complete transversal for F, by the closure $\overline{L^k}$ of L^k taken in the following sense. (The reader should note that in previous papers, we said that we used the C^∞ topology to take this closure, but in fact we used the one given here.) $\overline{L^k}$ consists of all elements in $\omega \in \mathcal{A}_c^k(T)$, so that there are sequences $\{\omega_n\}, \{\widehat{\omega}_n\} \subset L^k$ with $||\omega - \omega_n|| \to 0$ and $||d_T(\omega) - \widehat{\omega}_n|| \to 0$. The norm $|| \cdot ||$ is the sup norm, that is $||\omega|| = \sup_{x \in T} ||\omega(x)||_x$, where $|| \cdot ||_x$ is the norm on $(\wedge^k T^*T)_x$.

The reader should note that this cohomology appears as a quotient in the general computation of cyclic homology for foliations carried out in [**BN94**]. It is worth pointing out that since F is Riemannian and transversely oriented, $\mathrm{H}_c^*(M/F)$ is well behaved and is easily computed in specific examples. In particular, it is finite dimensional, Hausdorff, has twisted duality, and is invariant by foliation homotopies (which are weaker than leafwise homotopies). This is because $\mathrm{H}_c^*(M/F)$ is related to $\mathrm{H}_b^*(M, F)$, the basic cohomology of F, that is the cohomology of the transverse forms which are invariant under holonomy. For transversely oriented Riemannian foliations of codimension q, $\mathrm{H}_c^k(M/F) \simeq \mathrm{H}_b^{q-k}(M, F)$, and the latter has all these properties. See [**KT83**, **EHS85**, **EH86**].

The second exception is that we will be working on the homotopy groupoids (also called the monodromy groupoids) of our foliations, but our results extend to the holonomy groupoid, as well as any groupoids between these two extremes. Recall that the homotopy groupoid $\mathcal{G}$ of F consists of equivalence classes of paths $\gamma : [0, 1] \to M$ such that the image of γ is contained in a leaf of F. Two such paths γ_1 and γ_2 are equivalent if they are in the same leaf and homotopy equivalent (with endpoints fixed) in that leaf.

For Riemannian foliations, $\mathcal{G}$ is a Hausdorff dimension $2p+q$ manifold, in fact a fibration, and we have the usual basic open sets defining its manifold structure denoted (U, γ, V), where $U, V \in \mathcal{U}$ and γ is a leafwise path starting in U and ending in V.

As the bundle TF is oriented, just as in the original case [**Ha80**], there is a continuous open surjective linear map, called integration over the leaves,

$$\int_F : \mathcal{A}^{p+k}(M) \longrightarrow \mathcal{A}_c^k(M/F)$$

which commutes with the exterior derivatives d_M and d_H, and so induces the map $\int_F : \mathrm{H}^{p+k}(M;\mathbb{R}) \to \mathrm{H}_c^k(M/F)$.

We also have the source and range maps s, $r : \mathcal{G} \to M$ and the two natural transverse foliations F_s and F_r whose leaves are respectively $\widetilde{L}_x = s^{-1}(x)$, and $\widetilde{L}^x = r^{-1}(x)$, for each $x \in M$. Note that $r : \widetilde{L}_x \to L$ is the simply connected covering of L. We will work with the foliation F_s. Note that the intersection of any leaf $\widetilde{L}_x$ and any basic open set (U, γ, V) consists of at most one plaque of the foliation F_s in (U, γ, V), i.e. each $\widetilde{L}_x$ passes through any (U, γ, V) at most once.

There is a canonical lift of the normal bundle ν of F to a bundle $\nu_\mathcal{G} \subset T\mathcal{G}$ so that $T\mathcal{G} = TF_s \oplus TF_r \oplus \nu_\mathcal{G}$, and $r_* \nu_\mathcal{G} = \nu$ and $s_* \nu_\mathcal{G} = \nu$. The metric g_0 on M induces a canonical metric g_0 on $\mathcal{G}$ as follows. The bundles TF_s, TF_r, and $\nu_\mathcal{G}$ and are mutually orthogonal. So the normal bundle ν_s of TF_s is $\nu_s = TF_r \oplus \nu_\mathcal{G}$. On TF_r, g_0 is $s^*\big(g_0|TF\big)$, on TF_s it is $r^*\big(g_0|TF\big)$, and on $\nu_\mathcal{G}$ it is $r^*\big(g_0|\nu\big)$, which, since F is Riemannian and the metric on ν is bundle-like, is the same as $s^*\big(g_0|\nu\big)$.

We denote by E a leafwise flat complex bundle over M. This means that there is a connection ∇_E on E over M which, when restricted to any leaf L of F, is a flat connection, i.e. its curvature $(\nabla_E)^2 \,|\, L = (\nabla_E \,|\, L)^2 = 0$. This is equivalent to the condition that the parallel translation defined by $\nabla_E \,|\, L$, when restricted to contractible loops in L, is the identity. We assume that E admits a (possibly indefinite) non-degenerate Hermitian metric, denoted $\{\cdot, \cdot\}$, which is preserved by the leafwise flat structure. This means that if ϕ_1 and ϕ_2 are local leafwise flat sections of E, then their inner product $\{\phi_1, \phi_2\}$ is a locally constant function on each leaf. More generally, it is characterized by the fact that for general sections ϕ_1 and ϕ_2, and for any vector field X tangent to F,
$$X\{\phi_1, \phi_2\} \;=\; \{\nabla_{E,X}\phi_1, \phi_2\} + \{\phi_1, \nabla_{E,X}\phi_2\}.$$
We denote also by E its pull back by r to a leafwise (for the foliation F_s) flat bundle on $\mathcal{G}$ along with its invariant metric and leafwise flat connection. The context should make it clear which bundle we are using. A splitting of E is a decomposition $E = E^+ \oplus E^-$ (of E on M) into an orthogonal sum of two sub-bundles so that the metric is $\pm$ definite on $E^\pm$. Splittings always exist and any two are homotopic. The splitting defines an involution γ of E. If ϕ is a local section of E with $\phi = \phi^+ + \phi^-$ where $\phi^\pm$ is a local section of $E^\pm$, then $\gamma\phi = \phi^+ - \phi^-$. If we change the sign of the metric on E^-, we obtain a positive definite Hermitian metric on E^- and so also on E over both M and $\mathcal{G}$. In general, this new metric on E, denoted $(\cdot, \cdot)$, is not preserved by the flat structure.

Example 2.1. Assume that the codimension of F is even, say $q = 2k$. Set $E = \wedge^k \nu^* \otimes \mathbb{C}$. The bundles ν and ν^* have natural flat structures

along the leaves given by the holonomy maps (which define flat local sections). Since the metric on ν is bundle-like, the induced volume form on ν^* is invariant under the holonomy of F. Denote by $*_\nu$ the Hodge $*$ operator on $\wedge^*\nu^*$, and also its extension to E. Given two elements ϕ_1 and ϕ_2 of E_x, set

$$\{\phi_1, \phi_2\} \;=\; \sqrt{-1}^{k^2} *_\nu (\phi_1 \wedge_\nu \overline{\phi_2}),$$

where $\wedge_\nu : E \otimes E \to \wedge^{2k}\nu^* \otimes \mathbb{C}$. We leave it to the reader to check that E and $\{\cdot,\cdot\}$ satisfy the hypothesis of Theorem 9.1.

3. Chern-Connes character for transversely smooth idempotents

We will need the "transverse differential" ∂_ν and the graded trace Tr used in [**BH08**].

Consider the connection ∇ on $\wedge T^*F_s \otimes E$ given by $\nabla = r^*(\nabla_F \otimes \nabla_E)$ where ∇_F is a connection on $\wedge T^*F$ defined by a connection on T^*F. Then ∇ is an operator of degree one on $C^\infty(\wedge T^*\mathcal{G} \otimes \wedge T^*F_s \otimes E)$, where on decomposable sections $\omega \otimes \phi$, with $\omega \in C^\infty(\wedge^k T^*\mathcal{G})$, $\nabla(\omega \otimes \phi) = d\omega \otimes \phi + (-1)^k \omega \wedge \nabla\phi$. The foliation F_s has dual normal bundle $\nu_s^* = s^*(T^*M)$, and ∇ defines a *quasi-connection* ∇^ν acting on $C^\infty(\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E)$ by the composition

$$C^\infty(\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E) \xrightarrow{\;i\;} C^\infty(\wedge T^*\mathcal{G} \otimes \wedge T^*F_s \otimes E) \xrightarrow{\;\nabla\;}$$

$$C^\infty(\wedge T^*\mathcal{G} \otimes \wedge T^*F_s \otimes E) \xrightarrow{\;p_\nu\;} C^\infty(\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E),$$

where i is the inclusion and p_ν is induced by the projection $p_\nu : T^*\mathcal{G} \to \nu_s^*$ determined by the decomposition $T\mathcal{G} = TF_s \oplus \nu_s$.

Denote by $\partial_\nu : \mathrm{End}(C^\infty(\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E)) \to \mathrm{End}(C^\infty(\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E))$ the linear operator given by the graded commutator

$$\partial_\nu(H) = [\nabla^\nu, H].$$

Recall: that $(\partial_\nu)^2$ is given by the commutator with the curvature $\theta^\nu = (\nabla^\nu)^2$ of ∇^ν; that θ^ν is a leafwise differential operator which is at worst order one; and that the derivatives of all orders of its coefficients are uniformly bounded, with the bound possibly depending on the order of the derivative. See [**BH08**].

Now, suppose that H is an $\mathcal{A}^*(M)$-equivariant bounded leafwise smoothing operator on $\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E$. If H is such an operator, we can write it as

$$H = H_{[0]} + H_{[1]} + \cdots + H_{[n]},$$

where $H_{[k]}$ is homogeneous of degree k, that is, for all j,

$$H_{[k]} : C^\infty(\wedge^j\nu_s^* \otimes \wedge T^*F_s \otimes E) \to C^\infty(\wedge^{j+k}\nu_s^* \otimes \wedge T^*F_s \otimes E).$$

Recall that any $X \in C^\infty(\wedge^k TM)$ defines a section, denoted $\widehat{X}$, of $\wedge^k \nu_s$. For such X, $i_{\widehat{X}} H_{[k]}$ is a bounded leafwise smoothing operator on $\wedge T^* F_s \otimes E$. For any vector field Y on M, set

$$\partial_\nu^Y (i_X H_{[k]}) = i_{\widehat{Y}} (\partial_\nu (i_{\widehat{X}} H_{[k]})),$$

which (if it exists) is an operator on $\wedge T^* F_s \otimes E$.

Definition 3.1. An $\mathcal{A}^*(M)$ equivariant bounded leafwise smoothing operator H on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$ is transversely smooth provided that for any $X \in C^\infty(\wedge^k TM)$ and any vector fields $Y_1, ..., Y_m$ on M, the operator

$$\partial_\nu^{Y_1} ... \partial_\nu^{Y_m} (i_X H_{[k]})$$

is a bounded leafwise smoothing operator on $\wedge T^* F_s \otimes E$.

If the leafwise parallel translation along E is a bounded map, then the projection onto the leafwise harmonic forms with coefficients in E (for the foliation F_s) is transversely smooth. See Theorem 4.4 below. Since ∂_ν is a derivation, it is immediate that the composition of transversely smooth operators is transversely smooth. It is also easy to prove that the Schwartz kernel of any transversely smooth operator is a smooth section in all variables, see [**BH08**].

The trace of H is the Haefliger form $\mathrm{Tr}(H)$ given by

$$\mathrm{Tr}(H) = \int_F \mathrm{tr}(H(\overline{x})) dx = \int_F i^* (\mathrm{tr}(H \,|\, i(M))) dx,$$

where $\overline{x}$ is the class of the constant path at x, $\mathrm{tr}(H(\overline{x}))$ is the $\mathcal{A}^*(M)$-equivariant trace of the Schwartz kernel of H at $\overline{x}$ and so belongs to $\wedge T^* M_x$, and dx is the leafwise volume form associated with the fixed orientation of the foliation F. Tr is a graded trace which satisfies $\mathrm{Tr} \circ \partial_\nu = d_H \circ \mathrm{Tr}$. See [**BH04**] and [**BH08**].

If K is a bounded leafwise smoothing operator on $\wedge T^* F_s \otimes E$, we may extend it to an $\mathcal{A}^*(M)$ equivariant bounded leafwise smoothing operator on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$ by using the natural $\mathcal{A}^*(M)$ module structure of $C^\infty(\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E)$.

The proof of Lemma 4.5 of [**BH08**] extends easily to give the following.

Lemma 3.2. *Suppose that A is an $\mathcal{A}^*(M)$-equivariant leafwise differential operator of finite order on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$, and that the derivatives of all orders of its coefficients are uniformly bounded, with the bound possibly depending on the order of the derivative. Suppose that K is a bounded leafwise smoothing operator on $\wedge T^* F_s \otimes E$, and extend it to an $\mathcal{A}^*(M)$-equivariant bounded leafwise smoothing operator on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$. Then AK and KA are $\mathcal{A}^*(M)$-equivariant bounded leafwise smoothing operators on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$. If K is transversely smooth, so are AK and KA.*

Note that operators A which are the pull backs of operators on M, such as θ^ν and τ, satisfy the hypothesis of Lemma 3.2. Using Lemma 3.2, it is easy to show that being transversely smooth is independent of the choice of ∇^ν.

Finally, we need the concept of $\mathcal{G}$ invariant $\mathcal{A}^*(M)$-equivariant operators. Suppose that $H = H_{[0]} + H_{[1]} + \cdots + H_{[n]}$ is an $\mathcal{A}^*(M)$-equivariant bounded leafwise smoothing operator acting on the sections of $\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E$. Then H is $\mathcal{G}$ invariant provided it satisfies two requirements.

(1) For any $X = X_1 \wedge \cdots \wedge X_k \in C^\infty(\wedge^k TM)$ with some $X_j \in C^\infty(TF)$, $i_{\widehat{X}} H_{[k]} = 0$.

This means that $H_{[k]}$ defines an operator $H_{[k]} : C^\infty(\wedge^j \nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E) \to C^\infty(\wedge^{j+k}\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E)$, and that for $k > q$, $H_{[k]} = 0$.

Each $\gamma \in \widetilde{L}_x^y \equiv \widetilde{L}_x \cap \widetilde{L}^y$, defines an action $W_\gamma : C^\infty(\widetilde{L}_x, \wedge^*\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E) \to C^\infty(\widetilde{L}_y, \wedge^*\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E)$ given by

$$[W_\gamma \xi](\gamma') = \xi(\gamma'\gamma), \quad \gamma' \in \widetilde{L}_y.$$

Let $y' = r(\gamma')$, and note that $[W_\gamma \xi](\gamma') \in (\wedge^*\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E)_{\gamma'}$, which we identify with $\wedge^*\nu_y^* \otimes (\wedge T^*F \otimes E)_{y'}$, while $\xi(\gamma'\gamma) \in (\wedge^*\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E)_{\gamma'\gamma}$, which we identify with $\wedge^*\nu_x^* \otimes (\wedge T^*F \otimes E)_{y'}$. To effect this action, we identify $\wedge^*\nu_x^*$ with $\wedge^*\nu_y^*$ using the holonomy along γ. The second requirement of H is:

(2) For any $\gamma \in \widetilde{L}_x^y$,

$$(\gamma \cdot H)_y \equiv W_\gamma \circ H_x \circ W_\gamma^{-1} = H_y,$$

where H_x is the action of H on $\wedge^*\nu_{\mathcal{G}}^* \otimes \wedge T^*F_s \otimes E \,|\, \widetilde{L}_x$.

Essentially then, H is $\mathcal{G}$ invariant means that it defines the same operator on each $\widetilde{L} \subset s^{-1}(L)$ for each leaf L of F. Note that ∂_ν preserves $\mathcal{G}$ invariant $\mathcal{A}^*(M)$-equivariant transversely smooth operators.

In [**BH08**], we defined a Chern-Connes character $\mathrm{ch}_a(e) \in \mathrm{H}_c^k(M/F)$ for any $\mathcal{G}$ invariant transversely smooth idempotent e. The essential result needed was Lemma 4.13 of that paper, which we state for further reference.

Lemma 3.3. *Let H and K be $\mathcal{G}$ invariant $\mathcal{A}^*(M)$-equivariant transversely smooth operators acting on the sections of $\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E$. Then*

$$\mathrm{Tr}([H, K]) = 0.$$

Lemma 3.3 and the proof of Theorem 4.1 of [**BH04**] immediately imply

Theorem 3.4. *Let e be a $\mathcal{G}$ invariant transversely smooth idempotent. Then $\mathrm{ch}_a(e)$ depends only on e. In addition, if e_t, $0 \le t \le 1$, is a smooth family of such idempotents, then*

$$\mathrm{ch}_a(e_0) = \mathrm{ch}_a(e_1).$$

Lemma 3.5. *Two $\mathcal{G}$ invariant transversely smooth idempotents which have the same image, have the same Chern-Connes character.*

Proof. Suppose that e^0 and e^1 are two such idempotents. Then $e^0 \circ e^1 = e^1$ and $e^1 \circ e^0 = e^0$, and the family $e^t = te^1 + (1-t)e^0$ is a smooth family of $\mathcal{G}$ invariant transversely smooth idempotents connecting e^0 to e^1. Theorem 3.4 then gives the result.   q.e.d.

4. The twisted higher harmonic signature

We now define the twisted higher harmonic signature $\sigma(F, E)$.

Denote by $\mathcal{A}_c^*(F_s, E)$ the graded algebra of leafwise (for F_s) differential forms on $\mathcal{G}$ with coefficients in E which have compact support when restricted to any leaf of F_s. A Riemannian structure on F induces one on F_s. As usual there is the leafwise Riemannian Hodge operator $*$, which gives an inner product on each $\mathcal{A}_c^k(F_s, E)$. In particular, if α_1 and α_2 are leafwise $\mathbb{R}$ valued k forms and ϕ_1 and ϕ_2 are sections of E, then

$$\langle \alpha_1 \otimes \phi_1, \alpha_2 \otimes \phi_2 \rangle(x) = \int_{\widetilde{L}_x} (\phi_1, \phi_2)\alpha_1 \wedge *\alpha_2 = \int_{\widetilde{L}_x} \{\phi_1, \gamma\phi_2\}\alpha_1 \wedge *\alpha_2.$$

We denote by $\mathcal{A}_{(2)}^*(F_s, E)$ the field of Hilbert spaces over M which is the leafwise L^2 completion of these differential forms under this inner product, i.e.

$$\mathcal{A}_{(2)}^*(F_s, E)_x = L^2(\widetilde{L}_x; \wedge T^*F_s \otimes E).$$

This is a continuous field of Hilbert spaces, see [**C79**]. Because M is compact, the spaces $L^2(\widetilde{L}_x; \wedge^k T^*F_s \otimes E)$ do not depend on our choice of metrics. However, the inner products on these spaces do depend on the metrics, as do the Hilbert norms, denoted $\|\cdot\|_0$.

If E is the one dimensional trivial bundle with the trivial flat structure, then $\mathcal{A}_{(2)}^*(F_s, E)$ is just the leafwise L^2 forms (now with coefficients in $\mathbb{C}$) for the foliation F_s and is denoted $\mathcal{A}_{(2)}^*(F_s, \mathbb{C})$.

The leafwise de Rham differential on $\mathcal{G}$ extends to a closed operator on $\mathcal{A}_{(2)}^*(F_s, \mathbb{C})$ which coincides with the lifted one from the foliation (M, F) and it is denoted by d_s. The leafwise formal adjoint of d_s with respect to the Hilbert structure is well defined and is denoted by δ_s, and $\delta_s = -*d_s*$. Denote by Δ the Laplacian given by $\Delta = (d_s + \delta_s)^2 = d_s\delta_s + \delta_s d_s$, and denote by Δ_k its action on $\mathcal{A}_{(2)}^k(F_s, \mathbb{C})$. The leafwise $*$ operator also gives the leafwise involution τ on $\mathcal{A}_{(2)}^*(F_s, \mathbb{C})$, where as usual,

$$\tau = \sqrt{-1}^{k(k-1)+\ell} *$$

on $\mathcal{A}_{(2)}^k(F_s, \mathbb{C})$, and it is easy to check that $\delta_s = -\tau d_s \tau$, so $\tau(d_s + \delta_s) = -(d_s + \delta_s)\tau$, and $\Delta\tau = \tau\Delta$.

These operators extend to $\mathcal{A}^*_{(2)}(F_s, E)$ as follows. Since the operators are all leafwise, local and linear, we need only define them for local sections of the form $\alpha \otimes \phi$, where α is a local k form on $\widetilde{L}$ and ϕ is a local section of $E \,|\, \widetilde{L}$. Then

$$d_s(\alpha \otimes \phi) = d_s\alpha \otimes \phi + (-1)^k \alpha \wedge \nabla^{\widetilde{L}}_E \phi, \quad \widehat{*}(\alpha \otimes \phi) = *\alpha \otimes \gamma\phi, \quad \text{and}$$
$$\widehat{\tau}(\alpha \otimes \phi) = \tau\alpha \otimes \gamma\phi,$$

where $\nabla^{\widetilde{L}}_E$ is ∇_E restricted to $\widetilde{L}$, so $\nabla^{\widetilde{L}}_E\phi$ is a local section of $T^*\widetilde{L} \otimes E$. We define the wedge product $\alpha \wedge \nabla^{\widetilde{L}}_E\phi$ (as a local section of $\wedge^{k+1}T^*\widetilde{L}\otimes E$) in the obvious way.

Lemma 4.1. *We have*

$$\delta_s \;=\; -\widehat{*}\, d_s\, \widehat{*} \;=\; -\widehat{\tau}\, d_s\, \widehat{\tau}.$$

Note that $d_s^2 = 0$, so also $\delta_s^2 = 0$ since $\widehat{*}^2 = \pm 1$.

Proof. Consider two sections $\alpha_1 \otimes \phi_1$ and $\alpha_2 \otimes \phi_2$, and set

$$Q(\alpha_1 \otimes \phi_1, \alpha_2 \otimes \phi_2)(x) \;=\; \int_{\widetilde{L}_x} \{\phi_1, \phi_2\}\alpha_1 \wedge \alpha_2,$$

(and extend to all of $\mathcal{A}^*_{(2)}(F_s, E)$ by linearity). Then

$$\langle \alpha_1 \otimes \phi_1, \alpha_2 \otimes \phi_2 \rangle \;=\; Q(\alpha_1 \otimes \phi_1, \widehat{*}(\alpha_2 \otimes \phi_2)).$$

Now suppose that α_1 is a local $k-1$ form on $\widetilde{L}$, α_2 is a local k form on $\widetilde{L}$, and ϕ_1 is flat. If ϕ_2 is an arbitrary section of E, set $\{\phi_1, \alpha_2 \otimes \phi_2\} = \alpha_2\{\phi_1, \phi_2\}$. (Note that α_2 is $\mathbb{C}$ valued). As $\{\cdot, \cdot\}$ is preserved by the flat structure and ϕ_1 is flat, it follows that on $\widetilde{L}$, $\nabla^{\widetilde{L}}_E\{\phi_1, \phi_2\} = \{\phi_1, \nabla^{\widetilde{L}}_E\phi_2\}$. Acting on functions on $\widetilde{L}$, $d_s = \nabla^{\widetilde{L}}_E$, so

$$d_s\{\phi_1, \phi_2\} \;=\; \{\phi_1, \nabla^{\widetilde{L}}_E\phi_2\}.$$

Then

$$\langle d_s(\alpha_1 \otimes \phi_1), \alpha_2 \otimes \phi_2 \rangle \;=\; \int_{\widetilde{L}_x} \{\phi_1, \gamma\phi_2\}d_s\alpha_1 \wedge *\alpha_2 \;=\;$$
$$(-1)^k \int_{\widetilde{L}_x} \alpha_1 \wedge d_s(\{\phi_1, \gamma\phi_2\} * \alpha_2),$$

while

$$\langle \alpha_1 \otimes \phi_1, -\widehat{*}\, d_s\, \widehat{*}(\alpha_2 \otimes \phi_2) \rangle \;=\;$$
$$(-1)Q(\alpha_1 \otimes \phi_1, \widehat{*}^2 d_s\, \widehat{*}(\alpha_2 \otimes \phi_2)) \;=\; (-1)^k Q(\alpha_1 \otimes \phi_1, d_s\, \widehat{*}(\alpha_2 \otimes \phi_2)) \;=\;$$
$$(-1)^k Q(\alpha_1 \otimes \phi_1, (d_s * \alpha_2) \otimes \gamma\phi_2 + (-1)^k * \alpha_2 \wedge \nabla^{\widetilde{L}}_E\gamma\phi_2) \;=\;$$
$$(-1)^k \int_{\widetilde{L}_x} \alpha_1 \wedge (d_s * \alpha_2)\{\phi_1, \gamma\phi_2\} + (-1)^k \alpha_1 \wedge *\alpha_2 \wedge \{\phi_1, \nabla^{\widetilde{L}}_E\gamma\phi_2\} \;=\;$$
$$(-1)^k \int_{\widetilde{L}_x} \alpha_1 \wedge d_s(*\alpha_2\{\phi_1, \gamma\phi_2\}) \;=\; (-1)^k \int_{\widetilde{L}_x} \alpha_1 \wedge d_s(\{\phi_1, \gamma\phi_2\} * \alpha_2).$$

<div align="right">q.e.d.</div>

Denote by Δ^E the Laplacian given by $\Delta^E = (d_s + \delta_s)^2 = d_s\,\delta_s + \delta_s\,d_s$, and denote by Δ_k^E its action on $\mathcal{A}_{(2)}^k(F_s, E)$. Note that $\widehat{\tau}$ is still an involution even at the bundle level, and that $\widehat{\tau}(d_s + \delta_s) = -(d_s + \delta_s)\widehat{\tau}$ and $\Delta^E \widehat{\tau} = \widehat{\tau}\Delta^E$ still hold.

As usual, the space of twisted harmonic forms $\mathrm{Ker}(\Delta^E)$ is related to the leafwise cohomology of the twisted forms. The space of closed L^2 forms in $\mathcal{A}_{(2)}^*(F_s, E)$ is denoted by $Z_{(2)}^*(F_s, E)$ and it is a Hilbert subspace. The space of exact L^2 forms in $\mathcal{A}_{(2)}^*(F_s, E)$ is denoted by $B_{(2)}^*(F_s, E)$, and we denote its closure by $\overline{B}_{(2)}^*(F_s, E)$. We denote by $\mathrm{H}_{(2)}^*(F_s, E)$ the leafwise reduced twisted L^2 cohomology of the foliation, that is

$$\mathrm{H}_{(2)}^*(F_s, E) = Z_{(2)}^*(F_s, E)/\overline{B}_{(2)}^*(F_s, E).$$

Here is a well known Hodge result that we state for further use. See the Appendix of [**HL90**]

Lemma 4.2. *The field* $\mathrm{Ker}(\Delta_k^E)$ *is a subfield of* $Z_{(2)}^k(F_s, E)$, *and* $Z_{(2)}^k(F_s, E) = \mathrm{Ker}(\Delta_k^E) \oplus \overline{B}_{(2)}^k(F_s, E)$. *Thus the restriction of the natural projection* $Z_{(2)}^k(F_s, E) \to \mathrm{H}_{(2)}^k(F_s, E)$ *induces an isomorphism*

$$\mathrm{Ker}(\Delta_k^E) \simeq \mathrm{H}_{(2)}^k(F_s, E).$$

In addition

$$\mathcal{A}_{(2)}^*(F_s, E) = \mathrm{Ker}(d_s + (d_s)^*) \oplus \overline{\mathrm{Im}(d_s)} \oplus \overline{\mathrm{Im}(\delta_s)}.$$

That is, for each $x \in M$,

$$L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x \otimes (E\,|\,\widetilde{L}_x)) = \mathrm{Ker}(d_s^x + \delta_s^x) \oplus \overline{\mathrm{Im}(d_s^x)} \oplus \overline{\mathrm{Im}(\delta_s^x)}.$$

We assume the projection P_ℓ onto $\mathrm{Ker}(\Delta_\ell^E)$ is transversely smooth. It is a classical result that this projection is a bounded leafwise smoothing operator, so what we are really assuming is a form of smoothness in transverse directions. This condition holds in many important cases, see the comments below after the statement of Theorem 4.4. Denote by $\mathcal{A}_\pm^*(F_s, E)$ the ± 1 eigenspaces of $\widehat{\tau}$, and by $\mathrm{Ker}(\Delta_\ell^{E\pm})$ the intersections $\mathcal{A}_\pm^*(F_s, E) \cap \mathrm{Ker}(\Delta_\ell^E)$. Denote by $\pi_\pm = \frac{1}{2}(P_\ell \pm \widehat{\tau} \circ P_\ell)$, and note that these are the projections onto $\mathrm{Ker}(\Delta_\ell^{E\pm})$, respectively. Since the operator $\widehat{\tau}$ satisfies the hypothesis of Lemma 3.2, both $\pi_\pm$ are transversely smooth, and their Chern-Connes characters $\mathrm{ch}_a(\pi_\pm)$ are well defined Haefliger cohomology classes.

Definition 4.3. Suppose that the projection P_ℓ onto $\mathrm{Ker}(\Delta_\ell^E)$ is transversely smooth. The higher twisted harmonic signature $\sigma(F, E)$ is the difference

$$\sigma(F, E) = \mathrm{ch}_a(\pi_+) - \mathrm{ch}_a(\pi_-).$$

To justify our claim that our assumption of transverse smoothness for P_ℓ holds in important cases, we have the following which is an extension of a result due to Gong and Rothenberg, [**GR97**].

Theorem 4.4. *Suppose that the leafwise parallel translation along E is a bounded map. Then the projection P onto $\mathrm{Ker}(\Delta^E)$ is transversely smooth.*

The conclusion of Gong-Rothenberg is that the Schwartz kernel of P is smooth in all its variables.

For Riemannian foliations, P is always transversely smooth for the classical signature operator (that is, with coefficients in the trivial one dimensional bundle) for both the holonomy and the homotopy groupoid. P is transversely smooth whenever the preserved metric on E is positive definite. It is smooth in important examples, e.g. Lusztig [**Lu72**]. If the leafwise parallel translation along E is a bounded map, P is also transversely smooth using the holonomy groupoid, provided that the flat structure on E over each holonomy covering has no holonomy (so using the flat structure to translate a frame of a single fiber of $E \,|\, \widetilde{L}$ to all of $\widetilde{L}$ trivializes $E \,|\, \widetilde{L}$).

It is an open question whether the projection to the leafwise harmonic forms has transversely smooth Schwartz kernel when F is not Riemannian. It is satisfied for all foliations with compact leaves and Hausdorff groupoid [**EMS76**, **Ep76**].

Since the paper [**GR97**] has not been published, we give their proof here that P depends smoothly on $x \in M$, and then show how to get transverse smoothness from it.

Proof. Let $U \subset M$ be a foliation chart and choose $x_0 \in U$. Then there is a diffeomorphism $\varphi_U : U \times \widetilde{L}_{x_0} \simeq s^{-1}(U)$, and a bundle isomorphism $\psi_U : U \times (E \,|\, \widetilde{L}_{x_0}) \simeq E \,|\, s^{-1}(U)$, covering φ_U and preserving the leafwise flat structure. They are constructed as follows. The normal bundle $\nu_s = TF_r \oplus \nu_{\mathcal{G}} \simeq s^*(TM)$ defines a local transverse translation for the leaves of the foliation F_s. See [**Hu93**, **W83**]. We may assume that U is the diffeomorphic image under $\exp_{x_0}$ of a neighborhood $\widehat{U}$ of $0 \in TM_{x_0}$. Then for all $x \in U$, there is a unique $X \in \widehat{U}$ so that $x = \exp_{x_0}(X)$. Define $\gamma_x : [0,1] \to M$ to be $\gamma_x(t) = \exp_{x_0}(tX)$. Given x sufficiently close to x_0, for any $z \in \widetilde{L}_{x_0}$ there is a unique path $\widehat{\gamma}_z(t)$ in $\mathcal{G}$ so that $\widehat{\gamma}_z(0) = z$, $\widehat{\gamma}_z(t) \in \widetilde{L}_{\gamma_x(t)}$, and $\widehat{\gamma}'_z(t) \in (\nu_s)_{\widehat{\gamma}_z(t)}$. The transverse translate $\Phi_x(z)$ of z to $\widetilde{L}_x$ is just $\widehat{\gamma}_z(1)$. Φ_x is a smooth diffeomorphism from $\widetilde{L}_{x_0}$ to $\widetilde{L}_x$, and we set $\varphi_U(x, z) = \Phi_x(z)$, which is a smooth diffeomorphism from $U \times \widetilde{L}_{x_0}$ to $s^{-1}(U)$.

Since we are using the homotopy groupoid, each $\widetilde{L}$ is simply connected, so $E \,|\, \widetilde{L}_x$ is a trivial bundle for each $x \in M$, and using the flat

structure to translate a frame of a single fiber of $E \,|\, \widetilde{L}_x$ to all of $\widetilde{L}_x$ trivializes $E \,|\, \widetilde{L}_x$. Choose a local orthonormal framing $e_1, ..., e_k$ of $E \,|\, U$ (on M). This framing is also a local framing of $E \,|\, i(U)$ (on $\mathcal{G}$). Using the leafwise flat structure of E to translate it along the $\widetilde{L}$, we get a leafwise flat framing $e_1^s, ..., e_k^s$ of $E \,|\, s^{-1}(U)$. For $(x, \sum_j a_j e_j^s(z)) \in U \times (E \,|\, \widetilde{L}_{x_0})$, set

$$\psi_U\Big(x, \sum_j a_j e_j^s(z)\Big) = \sum_j a_j e_j^s(\varphi_U(x, z)).$$

That is, the image of $\phi \in E_z$ (where $z \in \widetilde{L}_{x_0}$) under $\psi_U(x, \cdot)$ is obtained by first parallel translating ϕ along $\widetilde{L}_{x_0}$ to $E_{i(x_0)}$, obtaining $\sum_j a_j e_j(i(x_0))$, and then parallel translating $\sum_j a_j e_j(i(x))$ along $\widetilde{L}_x$ to $E_{\varphi_U(x,z)}$. It is clear that ψ_U covers φ_U and preserves the leafwise flat structure.

There is a naturally defined bundle map

$$\Psi_U(x) : \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}) \to \wedge T^* \widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)$$

for each $x \in U$, which on a decomposable element $\alpha \otimes \phi \in (\wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))_z$ is given by

$$\Psi_U(x)(\alpha \otimes \phi) = ((\Phi_x^{-1})^* \alpha) \otimes \psi_U(x, \phi).$$

We also denote by Ψ_U the induced map

$$\Psi_U(x) : C_c^\infty(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) \to C_c^\infty(\widetilde{L}_x; \wedge T^* \widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)).$$

$\Psi_U(x)$ is invertible, commutes with the extended de Rham operators, and depends smoothly on x. Note that Φ_x^{-1} is a diffeomorphism of uniformly bounded dilation (as is Φ_x). If the leafwise parallel translation along E is a bounded map, then the map ψ_U is a bounded map, and Ψ_U extends to the following commutative diagram,

$$
\begin{array}{ccc}
L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) & \xrightarrow{\ d_s^{x_0}\ } & L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) \\
\Psi_U(x) \downarrow & & \downarrow \Psi_U(x) \\
L^2(\widetilde{L}_x; \wedge T^* \widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)) & \xrightarrow{\ d_s^x\ } & L^2(\widetilde{L}_x; \wedge T^* \widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)).
\end{array}
$$

4.5.

So $\Psi_U(x)(\mathrm{Ker}(d_s^{x_0})) \subset \mathrm{Ker}(d_s^x)$ and $\Psi_U(x)(\overline{\mathrm{Im}(d_s^{x_0})}) \subset \overline{\mathrm{Im}(d_s^x)}$. By Lemma 4.2, we have

$$L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) = \mathrm{Ker}(d_s^{x_0} + \delta_s^{x_0}) \oplus \overline{\mathrm{Im}(d_s^{x_0})} \oplus \overline{\mathrm{Im}(\delta_s^{x_0})},$$

and

$$L^2(\widetilde{L}_x; \wedge T^* \widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)) = \mathrm{Ker}(d_s^x + \delta_s^x) \oplus \overline{\mathrm{Im}(d_s^x)} \oplus \overline{\mathrm{Im}(\delta_s^x)}.$$

With respect to these decompositions, we may write

$$\Psi_U(x) = \begin{bmatrix} \Psi_{11}(x) & 0 & \Psi_{13}(x) \\ \Psi_{21}(x) & \Psi_{22}(x) & \Psi_{23}(x) \\ 0 & 0 & \Psi_{33}(x) \end{bmatrix}.$$

It follows immediately that $\Psi_{22}(x) : \overline{\mathrm{Im}(d_s^{x_0})} \to \overline{\mathrm{Im}(d_s^x)}$ is an invertible map which depends smoothly on x. Let $R_{x_0} : L^2(\widetilde{L}_{x_0}; \wedge T^*\widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) \to \overline{\mathrm{Im}(d_s^{x_0})}$ be the orthogonal projection. Define

$$\widetilde{R}_x \;=\; \Psi_{22}(x)R_{x_0}\Psi_U^{-1}(x), \quad \text{which equals} \quad \Psi_U(x)R_{x_0}\Psi_U^{-1}(x),$$

since $\Psi_{22}(x)R_{x_0} = \Psi_U(x)R_{x_0}$. Then $\widetilde{R}_x$ is an idempotent which varies smoothly in x, and has image $\overline{\mathrm{Im}(d_s^x)}$. However, it might not be an orthogonal projection. Set

$$Q_x = I + (\widetilde{R}_x - \widetilde{R}_x^*)(\widetilde{R}_x^* - \widetilde{R}_x).$$

Then Q_x is an invertible self adjoint operator which depends smoothly on x, and the orthogonal projection $R_x : L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)) \to \overline{\mathrm{Im}(d_s^x)}$ is just

$$R_x \;=\; \widetilde{R}_x \widetilde{R}_x^* Q_x^{-1},$$

so R_x depends smoothly on x.

Let τ_x be the Hodge type operator such that $\delta_s^x = \pm\tau_x^{-1}\widehat{d}_s^x\tau_x$, where $\widehat{d}_s^x$ is the differential associated with the antidual bundle $\overline{E}^*$ of E. The operator τ_x^{-1} maps $\overline{\mathrm{Im}(\widehat{d}_s^x)}$ onto $\overline{\mathrm{Im}(\delta_s^x)}$. Set $\widehat{S}_x = \tau_x^{-1}\widetilde{S}_x\tau_x$, where $\widetilde{S}_x$ is the operator for $\widehat{d}_s^x$ corresponding to the operator $\widetilde{R}_x$ for d_s^x. The argument above, with E replaced by its antidual $\overline{E}^*$, shows that $\widetilde{S}_x$, so also $\widehat{S}_x$, is an idempotent depending smoothly on x. Note that $\widehat{S}_x$ has image $\overline{\mathrm{Im}(\delta_s^x)}$. As above, we get that the orthogonal projection $S_x : L^2(\widetilde{L}_x; \wedge T^*\widetilde{L}_x \otimes (E \,|\, \widetilde{L}_x)) \to \overline{\mathrm{Im}(\delta_s^x)}$ depends smoothly on x. Thus the orthogonal projection $P = I - (R_x + S_x)$ depends smoothly on x.

We now show that P is transversely smooth. To do this, we view everything on $U \times \widetilde{L}_{x_0}$, using φ_U, and ψ_U. Thanks to Diagram 4.5, we are reduced to considering the operator $d_s^{x_0} : L^2(\widetilde{L}_{x_0}; \wedge T^*\widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) \to L^2(\widetilde{L}_{x_0}; \wedge T^*\widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$ acting over each point $x \in U$, that is the twisted leafwise de Rham operator on the foliation $U \times \widetilde{L}_{x_0}$. We use ϕ_U and ψ_U to pull back the structures on $s^{-1}(U)$ and we use the same notation to denote these pull backs. In particular, we have the connection ∇ and the normal bundle ν_s used to define ∂_ν. The leafwise projection P_x onto the twisted leafwise harmonics depends on the leafwise metrics on $\widetilde{L}_{x_0}$ and $E \,|\, \widetilde{L}_{x_0}$, which vary with $x \in U$.

First we prove that we may assume that the normal bundle ν_s is the bundle $TU \subset T(U \times \widetilde{L}_{x_0})$. Denote the operator corresponding to ∂_ν constructed using TU by ∂_U. Given a (bounded) vector field Y on U, we have two lifts, $\widehat{Y}$ to ν_s and $\widehat{Y}_0$ to TU. The difference $\widehat{Y} - \widehat{Y}_0$ is tangent to the fibers $\widetilde{L}_{x_0}$, so the difference of the operators $\partial_\nu^Y - \partial_U^Y = [\nabla_{\widehat{Y}} - \nabla_{\widehat{Y}_0}, \cdot] = [\nabla_{\widehat{Y}-\widehat{Y}_0}, \cdot]$ is the commutator with a leafwise differential operator of order one, whose coefficients and all their derivatives are uniformly bounded, with the bound possibly depending on the order of the derivative. For $s \in \mathbb{Z}$, we denote by $W_s = W_s^*(\widetilde{L}_{x_0}, E)$ the usual s-th Sobolev space which is the completion of $C_c^\infty(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \mid \widetilde{L}_{x_0}))$ under the usual s-th Sobolev norm. Then $\Upsilon(Y) := \nabla_{\widehat{Y}} - \nabla_{\widehat{Y}_0}$ defines a bounded leafwise operator from any W_s^* to W_{s-1}^*, and both $\Upsilon(Y)P_x$ and $P_x \Upsilon(Y)$ are bounded leafwise smoothing operators since P_x is leafwise smoothing. As

$$\partial_\nu^Y P_x = \partial_U^Y P_x + [\Upsilon(Y), P_x],$$

$\partial_\nu^Y P_x$ is a bounded leafwise smoothing operator if and only $\partial_U^Y P_x$ is.

By a simple induction argument using the fact that $\partial_U^{Y_2} \Upsilon(Y_1)$ has the same properties as $\Upsilon(Y_1)$, we have that $\partial_\nu^{Y_1} P_x$, $\partial_\nu^{Y_2} \partial_\nu^{Y_1} P_x, \dots$ are bounded leafwise smoothing operators if and only if $\partial_U^{Y_1} P_x$, $\partial_U^{Y_2} \partial_U^{Y_1} P_x, \dots$ are. Thus we may assume that $\nu_s = TU$.

Next we show that we may use any connection we please, provided it is in the same bounded geometry class as ∇. Suppose that ∂_0 is another derivation constructed from the connection ∇^0 in the same bounded geometry class as ∇. Then $\partial_\nu^Y - \partial_0^Y = [\nabla_Y^\nu - \nabla_Y^{0,\nu}, \cdot]$, and $\nabla_Y^\nu - \nabla_Y^{0,\nu}$ is a leafwise differential operator of order zero, whose coefficients and all their derivatives are uniformly bounded, with the bound possibly depending on the order of the derivative. So $\nabla_Y^\nu - \nabla_Y^{0,\nu}$ defines a bounded operator from any Sobolev space W_s to itself. Proceeding just as we did above, we have that $\partial_\nu^{Y_1} P_x$, $\partial_\nu^{Y_2} \partial_\nu^{Y_1} P_x, \dots$ are bounded leafwise smoothing operators if and only if $\partial_0^{Y_1} P_x$, $\partial_0^{Y_2} \partial_0^{Y_1} P_x, \dots$ are. Thus, we are reduced to showing that $\partial_0^{Y_m} ... \partial_0^{Y_1}(P)$ is a bounded leafwise smoothing operator.

The connection we choose is that pulled back from L_{x_0} under the obvious map $U \times \widetilde{L}_{x_0} \to L_{x_0}$. We leave it to the reader to show that this is in the same bounded geometry class as ∇. Now we can choose coordinates on U so we may think of $U = \mathbb{D}^n$ with coordinates, $x_1, ..., x_n$, and $x_0 = (0, ..., 0)$. When we do,

$$\partial_0^{\partial/\partial x_{i_m}} ... \partial_0^{\partial/\partial x_{i_1}} P_x = \partial^m P_x / \partial x_{i_m} ... \partial x_{i_1}.$$

Thus we are reduced to considering a smooth family of smoothing operators P_x acting on the space of sections of $\wedge T^* \widetilde{L}_{x_0} \otimes (E \mid \widetilde{L}_{x_0})$. The parameter x determines the metric g_x we use on this space, and P_x is the associated projection onto the twisted harmonic sections. Note that

the associated Sobolev spaces W_s^* are the same for all the g_x since these metrics are all in the same bounded geometry class. The norms on W_s^* do depend on the parameter x. However they are all comparable, so we may assume that we have a single norm $|| \cdot ||_s$ on each W_s^*, which is independent of x.

Denote $\partial^m / \partial x_{i_m} ... \partial x_{i_1}$ by $\partial_{i_m ... i_1}^m$. We need to prove that for all s and $k \geq 0$, $\partial_{i_m ... i_1}^m P_x$ defines a bounded map from W_s^* to W_{s+k}^*. Given $K : W_s^* \to W_{s+k}^*$, denote the $s, s+k$ norm of K by $||K||_{s+k,s}$. Then

$$||K||_{s+k,s} = ||(1+\Delta)^{(s+k)/2} K (1+\Delta)^{-s/2}||_{0,0},$$

where Δ is the Laplacian associated to the metric on $\wedge T^* \widetilde{L}_{x_0} \otimes (E \mid \widetilde{L}_{x_0})$. Since the norms associated to different metrics are comparable, we may use any metric g_x with associated Laplacian Δ_x we like. Now $P_x = (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2}$, so

$$\partial_i P_x = \partial_i ((1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2}) =$$

$$\partial_i (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2} + (1+\Delta_x)^{(s+k)/2} \partial_i P_x (1+\Delta_x)^{-s/2} +$$

$$(1+\Delta_x)^{(s+k)/2} P_x \partial_i (1+\Delta_x)^{-s/2},$$

which gives

$$(1+\Delta_x)^{(s+k)/2} \partial_i P_x (1+\Delta_x)^{-s/2} =$$

$$\partial_i P_x - \partial_i (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2} - (1+\Delta_x)^{(s+k)/2} P_x \partial_i (1+\Delta_x)^{-s/2}.$$

So,

$$||\partial_i P_x||_{s+k,s} = ||(1+\Delta_x)^{(s+k)/2} \partial_i P_x (1+\Delta_x)^{-s/2}||_{0,0} =$$

$$||\partial_i P_x - \partial_i (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2} -$$

$$(1+\Delta_x)^{(s+k)/2} P_x \partial_i (1+\Delta_x)^{-s/2}||_{0,0} \leq$$

$$||\partial_i P_x||_{0,0} + ||\partial_i (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2}||_{0,0} +$$

$$||(1+\Delta_x)^{(s+k)/2} P_x \partial_i (1+\Delta_x)^{-s/2}||_{0,0}.$$

Now for any r, $(1+\Delta_x)^{r/2}$ and $\partial_i (1+\Delta_x)^{r/2}$ are leafwise differential operators of order r, whose coefficients and all their derivatives are uniformly bounded, with the bound possibly depending on the order of the derivative, *but independent of x*. So they define bounded operators from W_s^* to W_{s-r}^* for any s, with bound independent of x. Since P_x is leafwise smoothing, it defines a bounded operator from any W_r^* to any W_s^* whose bound is also independent of x, since

$$||P_x||_{s,r} = ||(1+\Delta_x)^{-s/2} P_x (1+\Delta_x)^{-r/2}||_{0,0} = ||P_x||_{0,0} \leq 1,$$

Thus we have

$$||\partial_i (1+\Delta_x)^{(s+k)/2} P_x (1+\Delta_x)^{-s/2}||_{0,0} \leq$$

$$||\partial_i (1+\Delta_x)^{(s+k)/2}||_{0,s+k} \ ||P_x||_{s+k,-s} \ ||(1+\Delta_x)^{-s/2}||_{-s,0}$$

is bounded independently of x. Similarly $||(1 + \Delta_x)^{(s+k)/2} P_x \partial_i (1 + \Delta_x)^{-s/2}||_{0,0}$ is bounded independently of x. Thus $\partial_i P_x : W_s^* \to W_{s+k}^*$ is bounded if and only if $\partial_i P_x : W_0^* \to W_0^*$ is.

Now for any m and any r, $\partial_{i_m...i_1}^m (1 + \Delta_x)^{r/2}$ is also a leafwise differential operator of order r, whose coefficients and all their derivatives are uniformly bounded, with the bound possibly depending on the order of the derivative, but independent of x. Using this fact, a straightforward induction argument shows that $\partial_{i_m...i_1}^m P_x : W_s^* \to W_{s+k}^*$ is bounded if and only if $\partial_{i_m...i_1}^m P_x : W_0^* \to W_0^*$ is.

Now we have (working on $W_0^* = L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$) that $P_x = I - (R_x + S_x)$, where R_x is the orthogonal projection $R_x :$ $L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0})) \to \overline{\mathrm{Im}(d_s^{x_0})} \subset L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$ obtained using the metric g_x. At the point x, R_{x_0} also has image $\overline{\mathrm{Im}(d_s^{x_0})}$, but R_{x_0} might not be an orthogonal projection using the metric g_x. As above R_x is given by

$$R_x = R_{x_0} R_{x_0}^{*x} Q_x^{-1},$$

where

$$Q_x = I + (R_{x_0} - R_{x_0}^{*x})(R_{x_0}^{*x} - R_{x_0}),$$

and $R_{x_0}^{*x}$ is the adjoint of R_{x_0} constructed using the metric g_x. Since $I = Q_x Q_x^{-1}$ and $\partial_i I = 0$, we have that

$$\partial_i Q_x^{-1} = -Q_x^{-1}(\partial_i Q_x) Q_x^{-1},$$

and a boot-strapping argument shows that $\partial_{i_m...i_1}^m (Q_x^{-1})$ is bounded if $\partial_{i_m...i_1}^m Q_x$ is. It follows that $\partial_{i_m...i_1}^m R_x$ is bounded if $\partial_{i_m...i_1}^m R_{x_0}$ and $\partial_{i_m...i_1}^m R_{x_0}^{*x}$ are bounded. As $\partial_i R_{x_0} = 0$ for all i, we are reduced to considering $R_{x_0}^{*x}$.

We may write the metric g_x as $g_x(u, v) = g_{x_0}(G_x u, v)$ where G_x is a nonnegative self-adjoint (invertible) operator with respect to g_{x_0}, as is its inverse. Since g_x is the pull back of a family of metrics defined on the compact manifold M, G_x is smooth in all its variables, and it and all its derivatives are uniformly bounded, and the same is true for the inverse G_x^{-1}. Thus for all m, both $\partial_{i_m...i_1}^m G_x$ and $\partial_{i_m...i_1}^m G_x^{-1}$ define bounded operators on $L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$ (since they are order zero differential operators). For any bounded operator A on $L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$, the adjoint of A with respect to g_x is

$$A^{*x} = G_x^{-1} A^* G_x,$$

where A^* is the adjoint with respect to g_{x_0}. It follows immediately that for all m, $\partial_{i_m...i_1}^m R_{x_0}^{*x} = \partial_{i_m...i_1}^m (G_x^{-1} R_{x_0}^* G_x)$ is a bounded operator on $W_0^* = L^2(\widetilde{L}_{x_0}; \wedge T^* \widetilde{L}_{x_0} \otimes (E \,|\, \widetilde{L}_{x_0}))$, since $\partial_i R_{x_0}^* = 0$ for all i. Thus for all m, $\partial_{i_m...i_1}^m R_x$ is a bounded operator on W_0^*.

It remains to show that for all m, $\partial_{i_m...i_1}^m S_x$ is a bounded operator on W_0^*. To do this we may proceed as we did above, using the operators

$\widetilde{S}_x$ and $\widehat{S}_x$. We need only observe that the Hodge type operator τ_x has the same properties that G_x does. Thus for all m, $\partial^m_{i_m \dots i_1} P_x = -(\partial^m_{i_m \dots i_1} R_x + \partial^m_{i_m \dots i_1} S_x)$ is a bounded operator on W_0^*, and we conclude that P_x is transversely smooth.

<div align="right">q.e.d.</div>

Proposition 4.6. *If P is transversely smooth, then the projections onto $\mathcal{A}^*_\pm(F_s, E) \cap (\mathrm{Ker}(\Delta^E_k) \oplus \mathrm{Ker}(\Delta^E_{p-k}))$, $k \neq \ell$, and $\mathrm{Ker}(\Delta^{E\pm}_\ell)$ are transversely smooth.*

Proof. Denote by P_k the projection onto $\mathrm{Ker}(\Delta^E_k)$. It is immediate that P is transversely smooth if and only if all the P_k are transversely smooth. For $k \neq \ell$, the projection onto $\mathcal{A}^*_\pm(F_s, E) \cap (\mathrm{Ker}(\Delta^E_k) \oplus \mathrm{Ker}(\Delta^E_{p-k}))$ is given by $\pi^\pm_k = P_k \pm \tau \circ P_k$, (since $P_k \circ \tau \circ P_k = 0$ in those cases), and the projection onto $\mathrm{Ker}(\Delta^{E\pm}_\ell)$ is given by $\pi_\pm = \frac{1}{2}(P_\ell \pm \tau \circ P_\ell)$. As the operator τ satisfies the hypothesis of Lemma 3.2, and each P_k is transversely smooth, so is each $\tau \circ P_k$, so all of the projections are also transversely smooth. q.e.d.

5. Connections, curvature, and the Chern-Connes character

We now give an alternate construction of the Chern-Connes characters $\mathrm{ch}_a(\pi_+)$ and $\mathrm{ch}_a(\pi_-)$ using "connections" and "curvatures" defined on "smooth sub-bundles" of $\mathcal{A}^*_{(2)}(F_s, E)$.

Definition 5.1. A smooth subbundle of $\mathcal{A}^*_{(2)}(F_s, E)$ over M/F is a $\mathcal{G}$ invariant transversely smooth idempotent π_0 acting on $\mathcal{A}^*_{(2)}(F_s, E)$.

Example 5.2. 1) Any idempotent in the algebra of superexponentially decaying operators on $\wedge T^* F_s \otimes E$, defined in [**BH08**], is a smooth subbundle of $\mathcal{A}^*_{(2)}(F_s, E)$ over M/F. So any smooth compactly supported idempotent is a smooth subbundle of $\mathcal{A}^*_{(2)}(F_s, E)$ over M/F.

2) The Wassermann idempotent of the leafwise signature operator, as defined for instance in [**BH08**], is a very important special case of (1) above. In this case we take $E = M \times \mathbb{C}$.

3) A paradigm for such a smooth subbundle is given by projection onto the kernel of a leafwise elliptic operator acting on $\mathcal{A}^*_{(2)}(F_s, E)$ (induced from a leafwise elliptic operator on F). In particular, the projections π_+ and π_-.

Definition 5.3. The space $C^\infty_2(\wedge T^* F_s \otimes E)$ consists of all elements $\xi \in C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E) \cap \mathcal{A}^*_{(2)}(F_s, E)$ such that for any quasi-connection ∇^ν and any vector fields $Y_1, \dots, Y_m$ on M,

$$\nabla^\nu_{Y_1} \dots \nabla^\nu_{Y_m}(\xi) \in C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E) \cap \mathcal{A}^*_{(2)}(F_s, E),$$

where $\nabla^\nu_{Y_i} = i_{\widehat{Y}_i} \nabla^\nu$.

Note that if $\xi \in C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E)$, $\nabla^\nu_{Y_1} ... \nabla^\nu_{Y_m}(\xi)$ is automatically in $C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E)$, and that if $\xi \in C^\infty_2(\wedge T^* F_s \otimes E)$, then $\nabla^\nu_{Y_1} ... \nabla^\nu_{Y_m}(\xi) \in C^\infty_2(\wedge T^* F_s \otimes E)$. Note also that $C^\infty_c(\mathcal{G}; \wedge T^* F_s \otimes E) \subset C^\infty_2(\wedge T^* F_s \otimes E)$.

Proposition 5.4. *If H is a transversely smooth operator on $\wedge T^* F_s \otimes E$, then H maps $C^\infty_2(\wedge T^* F_s \otimes E)$ to itself.*

Proof. Let $\xi \in C^\infty_2(\wedge T^* F_s \otimes E)$. As H is transversely smooth, it follows easily that $H\xi \in C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E) \cap \mathcal{A}^*_{(2)}(F_s, E)$. Fix a quasi-connection ∇^ν and let Y be a vector field on M. Then

$$\nabla^\nu_Y(H\xi) = \nabla^\nu_Y H\xi - H\nabla^\nu_Y \xi + H\nabla^\nu_Y \xi = (\partial^Y_\nu H)\xi + H(\nabla^\nu_Y \xi),$$

which is in $C^\infty(\mathcal{G}; \wedge T^* F_s \otimes E) \cap \mathcal{A}^*_{(2)}(F_s, E)$, since H and $\partial^Y_\nu H$ are transversely smooth and ξ and $\nabla^\nu_Y \xi$ are in $C^\infty_2(\wedge T^* F_s \otimes E)$. An obvious induction argument now shows that $H\xi \in C^\infty_2(\wedge T^* F_s \otimes E)$.   q.e.d.

Let π_0 be a smooth subbundle of $\mathcal{A}^*_{(2)}(F_s, E)$ over M/F.

Definition 5.5. A C^∞ section of π_0 is an element $\xi \in C^\infty_2(\wedge T^* F_s \otimes E)$ which satisfies $\pi_0\xi = \xi$. The set of all smooth sections is denoted $C^\infty(\pi_0)$.

$C^\infty(\pi_0)$ is a $C^\infty(M)$ module, with $(f \cdot \xi)([\gamma]) = f(s(\gamma))\xi([\gamma])$. In addition, $C^\infty(\pi_0) = \pi_0(C^\infty_2(\wedge T^* F_s \otimes E)) \supset \pi_0(C^\infty_c(\mathcal{G}; \wedge T^* F_s \otimes E))$.

Definition 5.6. $C^\infty(\wedge T^* M; \pi_0)$ is the set of all smooth sections of $\wedge T^* M$ with coefficients in $C^\infty(\pi_0)$, and $C^\infty_c(\wedge T^* M; \wedge T^* F_s \otimes E)$ is the set of all smooth sections of $\wedge T^* M$ with coefficients in $C^\infty_c(\mathcal{G}; \wedge T^* F_s \otimes E)$.

There are natural actions of $\mathcal{A}^*(M)$ on both $C^\infty(\wedge T^* M; \pi_0)$ and $C^\infty_c(\wedge T^* M; \wedge T^* F_s \otimes E)$, and under these actions

$$C^\infty(\wedge T^* M; \pi_0) \simeq \mathcal{A}^*(M) \widehat{\otimes}_{C^\infty(M)} C^\infty(\pi_0),$$

and

$$C^\infty_c(\wedge T^* M; \wedge T^* F_s \otimes E) \simeq \mathcal{A}^*(M) \widehat{\otimes}_{C^\infty(M)} C^\infty_c(\mathcal{G}; \wedge T^* F_s \otimes E),$$

with the right completions. Thus $\pi_0 : C^\infty_c(\mathcal{G}; \wedge T^* F_s \otimes E) \to C^\infty(\pi_0)$ extends to the $\mathcal{A}^*(M)$ equivariant map

$$\pi_0 : C^\infty_c(\wedge T^* M; \wedge T^* F_s \otimes E) \to C^\infty(\wedge T^* M; \pi_0).$$

A *local invariant* element is a local section ξ of $\mathcal{A}^*_{(2)}(F_s, E)$ defined on an open subset $U \subset M$ so that for any leafwise path γ_1 in U, $\xi([\gamma]) = \xi([\gamma\gamma_1])$ for all γ with $s(\gamma) = r(\gamma_1)$. Local invariant elements are common. In particular, any locally defined element $\xi \in \mathcal{A}^*_{(2)}(F_s, E)$ defines local invariant elements. Suppose that ξ is defined on a foliation chart $U \subset M$ for F, and let P_x be the plaque in U containing the point x. Given $y \in P_x$, let γ_y be a path in P_x starting at x and ending at y.

Define $\widetilde{\xi}_y \in L^2(\widetilde{L}_y; \wedge T^* F_s \otimes E)$ by $\widetilde{\xi}_y([\gamma]) = \xi_x([\gamma\gamma_y])$. Then $\widetilde{\xi}$ is a local invariant element of $\mathcal{A}^*_{(2)}(F_s, E)$ defined along P_x. By restricting ξ to a transversal T in a foliation chart U and then extending invariantly to $\widetilde{\widetilde{\xi}}$ we obtain local invariant elements of $\mathcal{A}^*_{(2)}(F_s, E)$ defined over U. One can of course extend this construction from chart to chart as far as one likes, for example along any path $\gamma : [0, 1] \to L$ in a leaf L. If γ is a closed loop, the section at 1 will not agree in general with the section at 0, so one does not in general obtain global invariant sections this way.

Definition 5.7. A connection ∇ on π_0 is a linear map

$$\nabla : C^\infty(\wedge T^* M; \pi_0) \to C^\infty(\wedge T^* M; \pi_0)$$

of degree one, so that

1) for $\omega \in \mathcal{A}^k(M)$ and $\xi \in C^\infty(\pi_0)$,

$$\nabla(\omega \otimes \xi) = d_M \omega \otimes \xi + (-1)^k \omega \wedge \nabla\xi;$$

2) for local invariant $\xi \in C^\infty(\pi_0)$ and $X \in C^\infty(TF)$, $\nabla_X \xi = 0$, i. e. ∇ is flat along F;

3) ∇ is invariant under the right action of $\mathcal{G}$;

4) the leafwise operator $\nabla\pi_0 - \pi_0\nabla^\nu\pi_0 : C_c^\infty(\wedge T^* M; \wedge T^* F_s \otimes E) \to C^\infty(\wedge T^* M; \pi_0)$ is transversely smooth.

The usual proof shows that since ∇ satisfies (1), it is local in the sense that $\nabla\xi(x)$ depends only on $\xi | U$ where U is any open set in M with $x \in U$.

For ∇ to be invariant under the right action of $\mathcal{G}$ means the following. Let γ be a leafwise path in M from $x = s(\gamma)$ to $y = r(\gamma)$. Let ξ be a local invariant section of π_0 defined on a neighborhood of the path γ. For $X \in \nu_x$, we may use the natural flat structure on ν to parallel translate X to $\gamma_*(X) \in \nu_y$. Then we require,

$$\nabla_X \xi = (R_\gamma)^{-1} \nabla_{\gamma_*(X)} \xi = R_{\gamma^{-1}} \nabla_{\gamma_*(X)} \xi,$$

where the isomorphism $R_\gamma : L^2(\widetilde{L}_{s(\gamma)}; \wedge T^* F_s \otimes E) \to L^2(\widetilde{L}_{r(\gamma)}; \wedge T^* F_s \otimes E)$ is given by $R_\gamma(\xi)([\gamma_1]) = \xi([\gamma_1\gamma])$. Note that this condition does not depend on the choice of normal bundle ν because the ambiguity involves things of the form $\nabla_Y \xi$ where $Y \in TF$. But this is zero because ∇ is flat along F.

To see that $\nabla\pi_0 - \pi_0\nabla^\nu\pi_0$ is a leafwise operator, let $\omega \in \mathcal{A}^k(M)$ and $\xi \in C_c^\infty(\mathcal{G}; \wedge T^* F_s \otimes E)$. Then

$$(\nabla\pi_0 - \pi_0\nabla^\nu\pi_0)(\omega \otimes \xi) = \pi_0(\nabla - \nabla^\nu)\pi_0(\omega \otimes \xi) = \pi_0(\nabla - \nabla^\nu)(\omega \otimes \pi_0(\xi)) =$$

$$\pi_0\Big(d_M\omega \otimes \pi_0(\xi) + (-1)^k \omega \wedge \nabla\pi_0(\xi) - d_M\omega \otimes \pi_0(\xi) - (-1)^k \omega \wedge \nabla^\nu\pi_0(\xi)\Big) =$$

$$(-1)^k \pi_0\Big(\omega \wedge (\nabla - \nabla^\nu)\pi_0(\xi)\Big) = (-1)^k \omega \wedge (\nabla\pi_0 - \pi_0\nabla^\nu\pi_0)\xi,$$

so $\nabla\pi_0 - \pi_0\nabla^\nu\pi_0$ is a leafwise operator.

Next we show that $C_c^\infty(\wedge T^*M; \wedge T^*F_s \otimes E)$ is in the domain of $\pi_0 \nabla^\nu \pi_0$. Identify $C_c^\infty(\wedge T^*M; \wedge T^*F_s \otimes E)$ with the subspace $C_c^\infty(\wedge \nu_s^* \otimes \wedge T^*F_s \otimes E))$ of $C^\infty(\wedge \nu_s^* \otimes \wedge T^*F_s \otimes E))$. Now $\partial_\nu(\pi_0) = [\nabla^\nu, \pi_0]$, and (by assumption) it is transversely smooth. Thus we have

$$\nabla^\nu \pi_0 = \pi_0 \nabla^\nu + \partial_\nu(\pi_0),$$

so

$$\pi_0 \nabla^\nu \pi_0 = \pi_0 \nabla^\nu + \pi_0 \partial_\nu(\pi_0).$$

The domain of the operator on the right contains $C_c^\infty(\wedge \nu_s^* \otimes \wedge T^*F_s \otimes E)$.

Lemma 5.8. $\pi_0 \nabla^\nu$ *is a connection on* π_0.

Proof. As π_0 commutes with the action of $\mathcal{A}^*(M)$ on $C^\infty(\wedge T^*M; \pi_0)$, to show that $\pi_0 \nabla^\nu$ maps the space $C^\infty(\wedge T^*M; \pi_0)$ to itself, we need only show that for any local section $\xi \in C^\infty(\pi_0)$ and any local vector field X on M,

$$(\pi_0 \nabla^\nu \xi)(X) = \pi_0(i_{\widehat{X}} \nabla^\nu \xi) = \pi_0(\nabla_X^\nu \xi)$$

is in $C^\infty(\pi_0)$, where $\widehat{X}$ is the lift of X to ν_s. As $\xi \in C^\infty(\pi_0)$, it is in $C_2^\infty(\wedge T^*F_s \otimes E)$ and $\pi_0(\xi) = \xi$, so $\nabla_X^\nu \pi_0 \xi = \nabla_X^\nu \xi \in C_2^\infty(\wedge T^*F_s \otimes E)$. As π_0 is transversely smooth, $i_{\widehat{X}} \partial_\nu(\pi_0)(\xi) \in C_2^\infty(\wedge T^*F_s \otimes E)$. Since $\pi_0 \nabla^\nu = \nabla^\nu \pi_0 + \partial_\nu(\pi_0)$, we have $(\pi_0 \nabla^\nu \xi)(X) \in C_2^\infty(\wedge T^*F_s \otimes E)$. Finally, as $\pi_0^2 = \pi_0$, $\pi_0(\pi_0(\nabla_X^\nu \xi)) = \pi_0(\nabla_X^\nu \xi)$. Thus $(\pi_0 \nabla^\nu \xi)(X) \in C^\infty(\pi_0)$, and $\pi_0 \nabla^\nu$ maps $C^\infty(\wedge T^*M; \pi_0)$ to itself.

The operator $\pi_0 \nabla^\nu$ satisfies (1) because π_0 commutes with the action of $\mathcal{A}^*(M)$ on $C^\infty(\wedge T^*M; \pi_0)$. In particular, for $\omega \in \mathcal{A}^k(M)$ and $\xi \in C^\infty(\pi_0)$, we have

$$\pi_0 \nabla^\nu(s^*\omega \otimes \xi) = \pi_0 \rho_\nu(r^*(\nabla_F \otimes \nabla_E)(s^*\omega \otimes \xi)) =$$

$$\pi_0 \rho_\nu \Big(d_{\mathcal{G}}(s^*\omega) \otimes \xi + (-1)^k s^*\omega \wedge r^*(\nabla_F \otimes \nabla_E)\xi\Big) =$$

$$\pi_0 \rho_\nu(s^* d_M \omega \otimes \xi) + (-1)^k \pi_0 \rho_\nu(s^*\omega \wedge r^*(\nabla_F \otimes \nabla_E)\xi) =$$

$$s^* d_M \omega \otimes \pi_0 \xi + (-1)^k s^*\omega \wedge \pi_0 \rho_\nu r^*(\nabla_F \otimes \nabla_E)\xi = d_M \omega \otimes \xi + (-1)^k \omega \wedge \pi_0 \nabla^\nu \xi.$$

To show that $\pi_0 \nabla^\nu$ satisfies (2), let $X \in TF_x$, ξ be a local invariant section of π_0 defined near x, and $[\gamma] \in \widetilde{L}_x$. The fact that ξ is invariant means that there is a section $\widehat{\xi}$ of $\wedge T^*F \otimes E$ defined in a neighborhood of $r(\gamma)$ so that $\xi = r^*\widehat{\xi}$ in a neighborhood of $[\gamma]$. Recall that $\nu_s = TF_r \oplus \nu_{\mathcal{G}}$. Since $X \in TF_x$, $\widehat{X} \in TF_r$ and $r_*(\widehat{X}) = 0$. Now

$$\pi_0 \nabla_X^\nu \xi = \pi_0(r^*(\nabla_F \otimes \nabla_E)_{\widehat{X}}(\xi)),$$

but at $[\gamma]$,

$$\Big(r^*(\nabla_F \otimes \nabla_E)_{\widehat{X}}(\xi)\Big)[\gamma] = (\nabla_F \otimes \nabla_E)_{r_*(\widehat{X}[\gamma])}\widehat{\xi} = (\nabla_F \otimes \nabla_E)_0 \widehat{\xi} = 0,$$

so $\pi_0 \nabla_X^\nu \xi = 0$.

We leave it to the reader to check that $\pi_0 \nabla^\nu$ satisfies (3) of Definition 5.7, which is a straight forward computation, using the fact that for

$X \in \nu_x$ and $[\gamma] \in \widetilde{L}_x$, $r_*(\widehat{X}_{[\gamma]}) = \gamma_*(X)$, the parallel translate of X along γ to $\nu_{r(\gamma)}$.

$\pi_0 \nabla^\nu$ obviously satisfies (4).   q.e.d.

Remark 5.9. If $\widehat{\nabla}^\nu$ is another partial connection, then the difference $\widehat{\nabla}^\nu - \nabla^\nu$ is a leafwise operator which satisfies the hypothesis of Lemma 3.2, so $\pi_0 \widehat{\nabla}^\nu \pi_0 - \pi_0 \nabla^\nu \pi_0 = \pi_0(\widehat{\nabla}^\nu - \nabla^\nu)\pi_0$ is transversely smooth and $\pi_0 \widehat{\nabla}^\nu$ is also a connection on π_0. So, as in the classical case, the space of connections is an affine space whose linear part is composed of transversely smooth operators.

Now suppose that ∇ is any connection on π_0. Define the curvature θ of ∇ to be

$$\theta = \nabla^2.$$

The usual computation shows that θ is a pointwise operator, that is

Lemma 5.10. *For any $\omega \in \mathcal{A}^*(M)$ and any $\xi \in C^\infty(\pi_0)$,*

$$\nabla^2(\omega \otimes \xi) = \omega \wedge \nabla^2(\xi).$$

Denote by $C^\infty(\wedge T^*M; \mathcal{A}^*_{(2)}(F_s \otimes E))$ the space of all smooth sections of $\wedge T^*M$ with coefficients in $\mathcal{A}^*_{(2)}(F_s \otimes E)$. Smoothness means that the section is smooth when viewed as a section of $\wedge \nu_s^* \otimes \wedge T^*F_s \otimes E$ over $\mathcal{G}$. Extend ∇ to an operator on $C^\infty(\wedge T^*M; \mathcal{A}^*_{(2)}(F_s \otimes E))$ by composing it with the obvious extension of π_0 to $C^\infty(\wedge T^*M; \mathcal{A}^*_{(2)}(F_s \otimes E))$. The curvature of $\nabla \circ \pi_0$ is given by $(\nabla \circ \pi_0)^2 = \nabla \circ \pi_0 \circ \nabla \circ \pi_0 = \nabla \circ \nabla \circ \pi_0 = \theta \circ \pi_0$, since $\pi_0 \circ \nabla = \nabla$. We will also denote these new operators by ∇ and θ. Note that although ∇ is an operator which differentiates transversely to the foliation F_s, the operator θ is a purely leafwise operator thanks to Lemma 5.10. Also note that

$$\theta = \pi_0 \theta = \theta \pi_0.$$

Lemma 5.11. θ *is transversely smooth.*

Proof. Set $A = \pi_0 \nabla \pi_0 - \pi_0 \nabla^\nu \pi_0$, a transversely smooth operator. Then

$$\theta = (\pi_0 \nabla \pi_0)^2 = \pi_0 \nabla^\nu \pi_0 \nabla^\nu \pi_0 + \pi_0 \nabla^\nu \pi_0 A \pi_0 + \pi_0 A \pi_0 \nabla^\nu \pi_0 + A^2.$$

As A is transversely smooth, so is A^2. Since $\pi_0 A = A \pi_0 = A$, the terms

$$\pi_0 \nabla^\nu \pi_0 A \pi_0 + \pi_0 A \pi_0 \nabla^\nu \pi_0 = \pi_0 \nabla^\nu A \pi_0 + \pi_0 A \nabla^\nu \pi_0 =$$

$$\pi_0[\nabla^\nu, A]\pi_0 = \pi_0 \partial_\nu(A)\pi_0,$$

which is transversely smooth. Now $\nabla^\nu \pi_0 = \pi_0 \nabla^\nu + \partial_\nu(\pi_0)$, so

$$\pi_0 \nabla^\nu \pi_0 \nabla^\nu \pi_0 = \pi_0(\nabla^\nu)^2 \pi_0 + \pi_0 \partial_\nu(\pi_0)\nabla^\nu \pi_0 =$$

$$\pi_0 \theta^\nu \pi_0 + \pi_0 \partial_\nu(\pi_0)\pi_0 \nabla^\nu + \pi_0 \partial_\nu(\pi_0)\partial_\nu(\pi_0).$$

The curvature $\theta^\nu = (\nabla^\nu)^2$ satisfies the hypothesis of Lemma 3.2. As π_0 is transversely smooth, it follows from Lemma 3.2 that $\pi_0\theta^\nu\pi_0$ is transversely smooth. Using the facts that ∂_ν is a derivation and π_0 is an idempotent, it is a simple exercise to show that $\pi_0\partial_\nu(\pi_0)\pi_0 = 0$. Finally, $\pi_0\partial_\nu(\pi_0)\partial_\nu(\pi_0)$ is the composition of transversely smooth operators, so transversely smooth. Thus θ is transversely smooth.   q.e.d.

Set

$$\pi_0 e^{-\theta/2i\pi} = \pi_0 + \sum_{k=1}^{[n/2]} \frac{(-1)^k\theta^k}{(2i\pi)^k k!},$$

and consider the Haefliger form $\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})$. (Note that $2i\pi$ is the complex number.)

Theorem 5.12. *The Haefliger form* $\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})$ *is closed and its cohomology class does not depend on the connection used to define it.*

Proof. The zero-th order term of $\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})$ is $\mathrm{Tr}(\pi_0)$, and since π_0 is a uniformly bounded leafwise smoothing operator, we have (see [**BH08**])

$$d_H \mathrm{Tr}(\pi_0) = \mathrm{Tr}(\partial_\nu(\pi_0)) = \mathrm{Tr}(\partial_\nu(\pi_0^2)) =$$
$$2\,\mathrm{Tr}(\pi_0\partial_\nu(\pi_0)) = 2\,\mathrm{Tr}(\pi_0\partial_\nu(\pi_0)\pi_0) = 0,$$

since π_0 is a ($\mathcal{G}$ invariant transversely smooth) idempotent.

Lemma 5.13.   *For* $k > 0$, $d_H \mathrm{Tr}(\theta^k) = 0$.

Proof. First note that for $k > 0$,

$$[\nabla, \theta^k] = [\nabla, \nabla^{2k}] = \nabla \circ \nabla^{2k} - \nabla^{2k} \circ \nabla = 0.$$

Also note that $\nabla = \pi_0\nabla^\nu\pi_0 + A$, where A satisfies the hypothesis of Lemma 3.3, as does θ^k. Thus

$$0 = \mathrm{Tr}([\nabla, \theta^k]) = \mathrm{Tr}([\pi_0\nabla^\nu\pi_0 + A, \theta^k]) =$$
$$\mathrm{Tr}([\pi_0\nabla^\nu\pi_0, \theta^k]) = \mathrm{Tr}(\pi_0\nabla^\nu\theta^k - \theta^k\nabla^\nu\pi_0) =$$
$$\mathrm{Tr}((\pi_0 - 1)\nabla^\nu\theta^k + \nabla^\nu\theta^k - \theta^k\nabla^\nu - \theta^k\nabla^\nu(\pi_0 - 1)) =$$
$$\mathrm{Tr}((\pi_0 - 1)\nabla^\nu\theta^k) - \mathrm{Tr}(\theta^k\nabla^\nu(\pi_0 - 1)) + \mathrm{Tr}([\nabla^\nu, \theta^k]).$$

Note that the three terms are well defined since the three operators are $\mathcal{A}^*(M)$-equivariant. As $\theta = \pi_0\theta = \theta\pi_0$, $\theta^k = \pi_0\theta^k\pi_0$, and we have

$$\mathrm{Tr}((\pi_0 - 1)\nabla^\nu\theta^k) = \mathrm{Tr}((\pi_0 - 1)\nabla^\nu\pi_0\theta^k\pi_0) =$$
$$\mathrm{Tr}((\pi_0 - 1)\pi_0\nabla^\nu\theta^k\pi_0) + \mathrm{Tr}((\pi_0 - 1)\partial_\nu(\pi_0)\,\theta^k\pi_0) = 0,$$

since both terms are zero. The first term is zero because $(\pi_0 - 1)\pi_0 = 0$. The second term is zero because both $(\pi_0 - 1)\partial_\nu(\pi_0)\,\theta^k$ and π_0 are $\mathcal{G}$ invariant and transversely smooth, so by Lemma 3.3

$$\mathrm{Tr}((\pi_0 - 1)\partial_\nu(\pi_0)\,\theta^k\pi_0) = \mathrm{Tr}(\pi_0(\pi_0 - 1)\partial_\nu(\pi_0)\,\theta^k) = 0.$$

Similarly,

$$\mathrm{Tr}(\theta^k \nabla^\nu (\pi_0 - 1)) = 0.$$

Thus

$$0 = \mathrm{Tr}([\nabla^\nu, \theta^k]) = \mathrm{Tr}(\partial_\nu(\theta^k)).$$

It follows easily from Lemma 6.3 of [**BH08**] that $d_H \mathrm{Tr}(\theta^k) = \mathrm{Tr}(\partial_\nu(\theta^k))$, so we have the Lemma.   q.e.d.

To complete the proof of Theorem 5.12, we note that a standard argument in the theory of characteristic classes shows that

Lemma 5.14. *The Haefliger class of* $\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})$ *does not depend on the choice of connection* ∇ *on* π_0.

q.e.d.

Definition 5.15. The Chern-Connes character $\mathrm{ch}_a(\pi_0)$ of the transversely smooth idempotent π_0 is the cohomology class of the Haefliger form $\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})$, that is

$$\mathrm{ch}_a(\pi_0) = [\mathrm{Tr}(\pi_0 e^{-\theta/2i\pi})].$$

Remark 5.16. In [**H95**], [**BH04**], and [**BH08**] we defined Chern-Connes characters for various objects. It is clear from the results of those papers that the definition given here is consistent with those definitions. In particular, if $\nabla = \pi_0 \nabla^\nu$ is a connection on π_0 constructed from a connection $\nabla_F \otimes \nabla_E$ on $\wedge T^* F \otimes E$, then the material in Section 5 of [**BH08**] (which shows that the definitions of [**H95**] and [**BH04**] coincide) along with the comment after Definition 3.11 of [**BH08**] shows that the Chern-Connes character given here for π_0 and the Chern-Connes character for π_0 given in [**BH08**] are the same. Thus all three constructions of $\mathrm{ch}_a(\pi_0)$ yield the same Haefliger class.

Remark 5.17. Note that in Sections 3 and 5 we may replace the bundle $\wedge T^* F_s \otimes E$ by any bundle on $\mathcal{G}$ induced by r from a bundle on M, and the results are still valid.

Before leaving this section, we record some facts we will need later. In particular, we show that any connection ∇ is local in the sense that for X transverse to F and any local invariant section ξ of π_0, $\nabla_X \xi$ depends only on ξ restricted to any transversal T with X tangent to T. See Corollary 5.21 below.

Lemma 5.18. *Let* U *be a coordinate chart for* F. *There is a countable collection of smooth local invariant sections of* π_0 *on* U *which spans* $C^\infty(\pi_0)|U$ *as a module over* $C^\infty(U)$.

Proof. Let T be a transversal in U. The set $s^{-1}(T)$ is covered by a countable collection of coordinate charts of the form (U, γ, V). In each chart, choose a countable collection of smooth sections $\{\xi_i^{V,\gamma}\}$ of

$\wedge T^* F_s \otimes E$ with support in $(U, \gamma, V) \cap s^{-1}(T)$ so that for any section ξ of $\mathcal{A}^*_{(2)}(F_s, E)$, $\xi \,|\, (U, \gamma, V) \cap s^{-1}(T)$ may be written as a linear combination (over the functions on $s(U, \gamma, V) \cap T$) of the $\{\xi_i^{V,\gamma}\}$. Now extend the elements of this set to local invariant sections over U, also denoted $\{\xi_i^{V,\gamma}\}$. The collection of sections of $C^\infty(\pi_0) \,|\, U$

$$\mathcal{S} = \bigcup_{V,\gamma,i} \pi_0(\xi_i^{V,\gamma}),$$

then spans $C^\infty(\pi_0) \,|\, U$ as a module over $C^\infty(U)$, and the $\pi_0(\xi_i^{V,\gamma})$, are locally invariant sections over U.

<div align="right">q.e.d.</div>

As a consequence, we deduce the following.

Corollary 5.19. *If two connections ∇ and $\widehat{\nabla}$ on π_0 agree on local invariant sections, then they are the same.*

Note that the bundle $E = r^* E$ is flat (in fact trivial) along the leaves of the other foliation F_r of $\mathcal{G}$, since its leaves are just $r^{-1}(x)$ for $x \in M$. Denote by d_r the obvious differential associated to $\wedge T^* F_r \otimes E$. Given a local section $\xi \in C_2^\infty(\wedge T^* F_s \otimes E)$, we may view $d_r \xi$ as a local element of $C^\infty(\wedge T^* M; \wedge T^* F_s \otimes E)$. Note that $d_r^2 \xi = 0$, and ξ is locally invariant if and only if $d_r \xi = 0$. Note that for $\xi \in C^\infty(\pi_0)$ and $X \in C^\infty(TF)$, $\nabla_X \xi = d_r \xi(X)$. To see this, write $\xi = \sum_j g_j \xi_j$, where $\xi_j \in \mathcal{A}^k_{(2)}(F_s, E)$ are local invariant elements and the g_j are smooth local functions on M. Then Conditions (1) and (2) of Definition 5.7 give

$$\nabla_X \xi = \sum_j d_M g_j(X) \xi_j = \sum_j d_F g_j(X) \xi_j =$$

$$\sum_j d_F g_j(X) \xi_j + g_j d_r \xi_j(X) = d_r \xi(X).$$

Let U be a foliation chart for F with transversal T, and ∇ a connection on π_0. Then on U, ∇ is the pull back of ∇ restricted to $\pi_0 \,|\, T$. More specifically, for X tangent to T and $\xi \in C^\infty(\pi_0 \,|\, T)$, with local invariant extension $\widetilde{\xi}$ to $C^\infty(\pi_0 \,|\, U)$, define

$$\nabla_X^T \xi \equiv \nabla_X \widetilde{\xi}.$$

We may assume that $U \simeq \mathbb{R}^p \times T$ with coordinates (x, t) and plaques $\mathbb{R}^p \times t$. Denote by $\rho : U \to T$ the projection. Let $x \in U$ and $X \in TM_x$ and set $T_x = x \times T$. Write $X = X_F + \rho_*(X)$ where $X_F \in TF_x$ and $\rho_*(X)$ is tangent to T_x. Let $\xi \in C^\infty(\pi_0 \,|\, U)$ and define the pull back connection $\rho^*(\nabla^T)$ by

$$\rho^*(\nabla^T)_X \xi = d_r \xi(X_F) + \nabla_{\rho_*(X)}^T(\xi \,|\, T_x) = d_r \xi(X_F) + \nabla_{\rho_*(X)} \widetilde{(\xi \,|\, T_x)},$$

and extend to $C^\infty(\wedge T^*U; \pi_0)$ by using (1) of Definition 5.7 and the fact that $C^\infty(\wedge T^*U; \pi_0) \simeq \mathcal{A}^*(U) \otimes_{C^\infty(U)} C^\infty(\pi_0 \,|\, U)$.

Denote the curvature $(\nabla^T)^2$ of ∇^T by θ_T.

Proposition 5.20. $\nabla\,|\,U = \rho^*(\nabla^T)$ *and* $\theta\,|\,U = \rho^*(\theta_T)$.

Proof. Let $\xi \in C^\infty(\pi_0 \,|\, U)$ and suppose that $X \in TF$, so $X_F = X$ and $\rho_*(X) = 0$. Then

$$\rho^*(\nabla^T)_X \xi = d_r \xi(X) = \nabla_X \xi.$$

Next suppose that ξ is local invariant and X is tangent to T_x, so $X_F = 0$ and $\rho_*(X) = X$. Then

$$\rho^*(\nabla^T)_X \xi = \nabla_{\rho_*(X)} \widetilde{(\xi \,|\, T_x)} = \nabla_X \xi,$$

since $\widetilde{(\xi \,|\, T_x)} = \xi$ as ξ is local invariant. Thus $\nabla\,|\,U$ and $\rho^*(\nabla^T)$ agree on local invariant sections, so they are equal.

For the second equation, writing $\rho^*(\nabla^T) = d_r + \nabla^T$, we have

$$\theta\xi = d_r^2 \xi + \nabla^T d_r \xi + d_r \nabla^T \xi + (\nabla^T)^2 \xi = (\nabla^T)^2 \xi,$$

since $d_r^2 = 0$ and $\nabla^T \circ d_r = -d_r \circ \nabla^T$. But, with the notation $\rho^*(\nabla^T) = d_r + \nabla^T$, $(\nabla^T)^2 \xi = \rho^*(\theta_T)\xi$.   q.e.d.

The following is immediate.

Corollary 5.21. ∇ *is local in the sense that for X transverse to F and any local invariant section ξ of π_0, $\nabla_X \xi$ depends only on $\xi\,|\,T$ where T is any transversal with X tangent to it.*

6. Leafwise maps

In this section we collect some results for leafwise homotopy equivalences. Outlines of the proofs are in the Appendix.

Let M and M' be compact Riemannian manifolds with oriented foliations F and F'. The results of this section do not require F or F' to be Riemannian. Let $f : M \to M'$ be a smooth leafwise homotopy equivalence which preserves the leafwise orientations. (We need only assume transverse smoothness, and leafwise continuity. A standard argument then allows f to be approximated by a smooth map.) Suppose that $E' \to M'$ is a leafwise flat complex bundle over M' which satisfies the hypothesis of Theorem 9.1, and set $E = f^*(E')$. Let $g : M' \to M$ be a homotopy inverse of f.

Proposition 6.1. f *induces an isomorphism* $f^* : \mathrm{H}_c^*(M'/F') \to \mathrm{H}_c^*(M/F)$ *on Haefliger cohomology with inverse g^*.*

Define $\check{f} : \mathcal{G} \to \mathcal{G}'$ by $\check{f}([\gamma]) = [f \circ \gamma]$. Recall that $\check{f}$ is leafwise uniformly proper if for any C_0, there is C_1 so that if the leafwise distance

from $\check{f}(z_0)$ to $\check{f}(z_1)$ is less than C_0, then the leafwise distance from z_0 to z_1 is less than C_1.

Proposition 6.2. $\check{f}$ *is smooth and leafwise uniformly proper.*

Thus $\check{f}$ induces a well defined map on compactly supported leafwise forms, but in general it does not induce a well defined map on leafwise L^2 forms. To deal with this problem we adapt the construction of Hilsum-Skandalis, [**HiS92**], to our setting.

Let $i : M' \hookrightarrow \mathbb{R}^k$ be an imbedding of the compact manifold M' in some Euclidean space $\mathbb{R}^k$, and identify M' with its image. For $x' \in M'$ and $t \in \mathbb{R}^k$, define $p(x', t)$ to be the projection of the tangent vector $X^t = \dfrac{d}{ds}\big|_{s=0}(x' + st)$ at x' determined by t, to the leaf $L'_{x'}$ in $(M', F') \subset \mathbb{R}^k$. In particular, first project X^t to $TF'_{x'}$ and then exponentiate it to $L'_{x'}$, thinking of $L'_{x'}$ as a Riemannian manifold in its own right. Since M' is compact, we may choose a ball $B^k \subset \mathbb{R}^k$ so small that the restriction of the smooth map $p_f = p \circ (f, \mathrm{id}) : M \times B^k \to M'$ to any $p_f : L_x \times B^k \to L'_{f(x)}$ is a submersion. Lifting this map to the groupoids, we get

$$p_f : \mathcal{G} \times B^k \longrightarrow \mathcal{G}',$$

which is a leafwise map if $\mathcal{G} \times B^k$ is endowed with the foliation $F_s \times B^k$. Note that $p_f : \widetilde{L}_x \times B^k \to \widetilde{L}'_{f(x)}$ is the map induced on the coverings by $p_f : L_x \times B^k \to L'_{f(x)}$. In particular, $p_f([\gamma], t)$ is the composition of leafwise paths $p_f([\gamma], t) = [P_f(\gamma, t) \cdot (f \circ \gamma)]$, where $P_f(\gamma, t)(s) = p_f(r(\gamma), st)$. It is easy to see that p_f is a smooth map.

Denote by $\pi_2 : \mathcal{G} \times B^k \to B^k$ the projection, and choose a smooth compactly supported k-form ω on B^k whose integral is 1. We shall refer to such a form as a Bott form on B^k. Denote by e_ω the exterior multiplication by the differential k−form $\pi_2^*\omega$ on $\mathcal{G} \times B^k$. For $\xi \in \mathcal{A}_c^*(F'_s, E')$, we define $f^{(i,\omega)}(\xi) \in \mathcal{A}_c^*(F_s, E)$ as

$$f^{(i,\omega)}(\xi) = (\pi_{1,*} \circ e_\omega \circ p_f^*)(\xi).$$

The map $p_f : \mathcal{G} \times B^k \longrightarrow \mathcal{G}'$ is a leafwise (for $F_s \times B^k$) submersion extending $\check{f}$, so $p_f^*(\xi)$ is a leafwise form on $\mathcal{G} \times B^k$ with coefficients in the bundle $p_f^* E'$. The map $\pi_{1,*}$ is integration over the fiber of the projection $\pi_1 : \mathcal{G} \times B^k \to \mathcal{G}$ of such forms. In general, the fiber of $p_f^* E'$ is not constant on fibers of the fibration $\pi_1 : \mathcal{G} \times B^k \to \mathcal{G}$. To correct for this, we use the parallel translation given by the flat structure of $p_f^* E'$ to identify all the fibers of $p_f^* E' \,|\, z \times B^k$ with $(p_f^* E')_{(z,0)} = (\check{f}^* E')_z = (f^* E')_{r(z)}$. This is well defined because the ball $B^k \subset \mathbb{R}^k$ is contractible.

For $s \in \mathbb{Z}$, denote by $W_s^*(F_s, E)$ the field of Hilbert spaces over M given by $W_s^*(F_s, E)_x = W_s^*(\widetilde{L}_x, E)$, the s-th Sobolev space of differential forms on $\widetilde{L}_x$ with coefficients in $E \,|\, \widetilde{L}_x$. Just as it does for the leafwise

L^2 forms, the compactness of M implies that these spaces do not depend on our choice of Riemannian structure. Note that $W_s^* \subset W_{s_1}^*$ if $s \geq s_1$, and set

$$W_\infty^*(F_s, E) = \bigcap_{s \in \mathbb{Z}} W_s^*(F_s, E) \text{ and } W_{-\infty}^*(F_s, E) = \bigcup_{s \in \mathbb{Z}} W_s^*(F_s, E).$$

Equip $W_\infty^*(F_s, E)$ with the induced locally convex topology.

Proposition 6.3. *For any $s \in \mathbb{Z}$, $f^{(i,\omega)}$ extends to a bounded operator from $W_s^*(F_s', E')$ to $W_s^*(F_s, E)$.*

As ω is closed, e_ω commutes with de Rham differentials. The image of $e_\omega \circ p_f^*$ is contained in the π_1-fiber compactly supported forms, so $f^{(i,\omega)} = \pi_{1,*} \circ e_\omega \circ p_f^*$ commutes with de Rham differentials. It follows immediately that the extension of $f^{(i,\omega)}$ to the L^2 forms also commutes with the closures of the de Rham differentials, so $f^{(i,\omega)}$ induces a well defined map $\widetilde{f}^* : \mathrm{H}_{(2)}^*(F_s', E') \longrightarrow \mathrm{H}_{(2)}^*(F_s, E)$ on leafwise reduced L^2 cohomology.

Proposition 6.4. $\widetilde{f}^* : \mathrm{H}_{(2)}^*(F_s', E') \longrightarrow \mathrm{H}_{(2)}^*(F_s, E)$ *does not depend on the choices of i and ω. $\widetilde{f}^*$ is an isomorphism with inverse $\widetilde{g}^*$, so if f_1 and f_2 are leafwise homotopy equivalent, then $\widetilde{f_1^*} = \widetilde{f_2^*}$.*

Recall the definition of the pairing Q from the proof of Lemma 4.1.

Proposition 6.5. *If ξ_1' and ξ_2' are closed L^2 sections of $\wedge^\ell \widetilde{L}_{f(x)}' \otimes E'$, then*

$$Q_x(\widetilde{f}^*(\xi_1'), \widetilde{f}^*(\xi_2')) = Q_{f(x)}'(\xi_1', \xi_2').$$

7. Induced bundles

We assume again that F and F' are Riemannian foliations. The restriction of $\widetilde{f}^*$ gives isomorphisms from $\mathrm{Ker}(\Delta_\ell^{E'})$, $\mathrm{Ker}(\Delta_\ell^{E'+})$, and $\mathrm{Ker}(\Delta_\ell^{E'-})$ to their images which we denote by

$$\mathrm{Im}\, \widetilde{f}^* = \widetilde{f}^*(\mathrm{Ker}(\Delta_\ell^{E'})), \quad \mathrm{Im}\, \widetilde{f_+^*} = \widetilde{f}^*(\mathrm{Ker}(\Delta_\ell^{E'+})),$$

and

$$\mathrm{Im}\, \widetilde{f_-^*} = \widetilde{f}^*(\mathrm{Ker}(\Delta_\ell^{E'-})),$$

respectively. We use similar notation for the map $\widetilde{g}^* : W_{-\infty}^*(F, E) \to W_{-\infty}^*(F', E')$.

Note that for $x \in M$, $gf(x) \neq x$ in general, which creates technical problems. To deal with this, choose a leafwise homotopy equivalence $h : M \times I \to M$ between the identity map on M and gf. Recall the smooth leafwise path γ_x from x to $gf(x)$ given by $\gamma_x(t) = h(x, t)$. It

determines the isometry $R_x : \widetilde{L}_{gf(x)} \to \widetilde{L}_x$ given by $R_x([\gamma]) = [\gamma \cdot \gamma_x]$. For any Sobolev space $W_s^*(\widetilde{L}_x, E)$, R_x determines the isometry

$$R_x^* : W_s^*(\widetilde{L}_x, E) \to W_s^*(\widetilde{L}_{gf(x)}, E).$$

In particular for $s = 0$, it gives the isometry

$$R_x^* : L^2(\widetilde{L}_x; \wedge T^* F_s \otimes E) \to L^2(\widetilde{L}_{gf(x)}; \wedge T^* F_s \otimes E).$$

We shall also consider the smooth leafwise paths $\gamma'_{x'}$ from $x' \in M'$ to $fg(x')$ given by $\gamma'_{x'}(t) = h'(x', t)$, where h' is a fixed leafwise homotopy between the identity of M' and fg. Given $x \in M$, define the isometry $R'_x : \widetilde{L}'_{f(x)} \to \widetilde{L}'_{f(x)}$ to be

$$R'_x[\gamma'] = [\gamma' \cdot f(\gamma_x)^{-1} \cdot \gamma'_{f(x)}].$$

This induces the isometry

$$R'^* : L^2(\widetilde{L}'_{f(x)}; \wedge T^* F'_s \otimes E') \to L^2(\widetilde{L}'_{f(x)}; \wedge T^* F'_s \otimes E').$$

Note that the composition

$$R'_x \circ \check{f} \circ R_x \circ \check{g} : \widetilde{L}'_{f(x)} \to \widetilde{L}'_{f(x)}$$

is homotopic to the identity map, since for $[\gamma'] \in \widetilde{L}'_{f(x)}$,

$$R'_x \circ \check{f} \circ R_x \circ \check{g}([\gamma']) \;=\; [fg(\gamma') \cdot f(\gamma_x) \cdot f(\gamma_x)^{-1} \cdot \gamma'_{f(x)}] \;=\; [fg(\gamma') \cdot \gamma'_{f(x)}].$$

Set

$$A^t(\gamma') = (\gamma'^{-1}_{r(\gamma')} \,|\, [0,t]) \cdot fg(\gamma') \cdot \gamma'_{f(x)}.$$

Then $A^0(\gamma') = fg(\gamma') \cdot \gamma'_{f(x)}$, and $A^1(\gamma') = \gamma'^{-1}_{r(\gamma')} \cdot fg(\gamma') \cdot \gamma'_{f(x)}$. Now $s(A^1(\gamma')) = s(\gamma')$ and $r(A^1(\gamma')) = r(\gamma')$, and h' provides a leafwise homotopy between $A^1(\gamma')$ and γ', so they define the same element in $\widetilde{L}'_{f(x)}$. Thus A^t induces a homotopy from $R'_x \circ \check{f} \circ R_x \circ \check{g}$ to the identity map. For $x \in M$, consider the composition

$$(P'_\ell \widetilde{g}^* R_x^* P_\ell \widetilde{f}^* R_x'^* P'_\ell)_{f(x)} : L^2(\widetilde{L}'_{f(x)}; \wedge T^* F'_s \otimes E') \to L^2(\widetilde{L}'_{f(x)}; \wedge T^* F'_s \otimes E').$$

Since $R'_x \circ \check{f} \circ R_x \circ \check{g} : \widetilde{L}'_{f(x)} \to \widetilde{L}'_{f(x)}$ is homotopic to the identity and P_ℓ^* is the identity on cohomology, it follows that $(P'_\ell \widetilde{g}^* R_x^* P_\ell \widetilde{f}^* R_x'^* P'_\ell)_{f(x)}$ induces the identity on cohomology, which is naturally isomorphic to $\mathrm{Ker}(\Delta_\ell^{E'})_{f(x)} = \mathrm{Im}(P'_\ell)_{f(x)}$. So its restriction

$$(P'_\ell \widetilde{g}^* R_x^* P_\ell \widetilde{f}^* R_x'^* P'_\ell)_{f(x)} : \mathrm{Ker}(\Delta_\ell^{E'})_{f(x)} \to \mathrm{Ker}(\Delta_\ell^{E'})_{f(x)}$$

is the identity.

We now show that $\mathrm{Im}\, \widetilde{f}_+^*$ is a smooth sub-bundle of $\mathcal{A}_{(2)}^\ell(F_s, E)$ over M/F. Set

$$\pi_+^f = \widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^* P_\ell.$$

Then for each $x \in M$,

$$(\pi_+^f)_x : L^2(\widetilde{L}_x; \wedge T^* F_s \otimes E) \to L^2(\widetilde{L}_x; \wedge T^* F_s \otimes E)$$

is bounded and leafwise smoothing since π_+' and P_ℓ are, and $R_x'^*$, R_x^*, $\widetilde{f}^*$ and $\widetilde{g}^*$ are bounded maps. We leave it to the reader to show that π_+^f is $\mathcal{G}$ invariant using the equality

$$[gf(\gamma) \cdot \gamma_x] = [\gamma_y \cdot \gamma]$$

for any $\gamma \in \mathcal{G}$ with $s(\gamma) = x$ and $r(\gamma) = y$. As above, this equality holds since the two paths start and end at the same points and a leafwise homotopy between them can be constructed using the leafwise homotopy equivalence h.

We extend π_+^f to an $\mathcal{A}^*(M)$ equivariant operator on $\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E$ in the usual way.

Proposition 7.1. $\pi_+^f : \mathcal{A}_{(2)}^\ell(F_s, E) \to \operatorname{Im} \widetilde{f}_+^*$ *is a transversely smooth idempotent.*

Proof. First we have,

$$(\pi_+^f)^2 = \widetilde{f}^* R'^* \pi_+' \widetilde{g}^* R^* P_\ell \widetilde{f}^* R'^* \pi_+' \widetilde{g}^* R^* P_\ell =$$

$$\widetilde{f}^* R'^* \pi_+' P_\ell' \widetilde{g}^* R^* P_\ell \widetilde{f}^* R'^* P_\ell' \pi_+' \widetilde{g}^* R^* P_\ell =$$

$$\widetilde{f}^* R'^* (\pi_+')^2 \widetilde{g}^* R^* P_\ell = \widetilde{f}^* R'^* \pi_+' \widetilde{g}^* R^* P_\ell = \pi_+^f,$$

since $\pi_+' = \pi_+' P_\ell' = P_\ell' \pi_+'$, and for each $x \in M$,

$$(P_\ell' \widetilde{g}^* R^* P_\ell \widetilde{f}^* R'^* P_\ell')_{f(x)} : \operatorname{Ker}(\Delta_\ell^{E'})_{f(x)} \to \operatorname{Ker}(\Delta_\ell^{E'})_{f(x)}$$

is the identity map, and $\operatorname{Ker}(\Delta_\ell^{E'}) \supset \operatorname{Im}(\pi_+')$.

As P_ℓ is transversely smooth, we need only show that $\widetilde{f}^* R'^* \pi_+' \widetilde{g}^* R^*$ is transversely smooth.

Let ∇_E and $\nabla_{E'}$ be the leafwise flat connections on E and E' and $\nabla_{F'}$ and ∇_F be the Riemannian connections on $T^* F'$ and $T^* F$, respectively. Denote by ∇^ν and ∇'^ν the quasi-connections on $C^\infty(\wedge \nu_s^* \otimes \wedge T^* F_s \otimes E)$ and $C^\infty(\wedge \nu_s'^* \otimes \wedge T^* F_s' \otimes E')$ constructed from $\nabla_F \otimes \nabla_E$, and $\nabla_{F'} \otimes \nabla_{E'}$, respectively.

Now suppose H is any $\mathcal{G}$ invariant operator of degree zero on $\wedge T^* F_s \otimes E$, e.g. $H = \widetilde{f}^* R'^* \pi_+' \widetilde{g}^* R^*$. If $X \in C^\infty(TF)$, then since H and ∇^ν are $\mathcal{G}$ invariant, $\partial_\nu^X(H) = 0$. A vector field Y on M is a Γ vector field provided that for any $X \in C^\infty(TF)$, $[X, Y] \in C^\infty(TF)$. If $Y \in C^\infty(\nu)$ is a Γ vector field, it is invariant under the parallel translation defined by F, so $\partial_\nu^Y(H)$ is $\mathcal{G}$ invariant. Globally defined Γ vector fields rarely exist. The restriction of a global vector field to an open subset will be called a local extendable vector field. Such local vector fields have all their derivatives bounded. Any local Γ vector field may, after a slight reduction in its domain of definition, be extended to a global vector field.

Finally, a bounded function (on M) times a bounded leafwise smoothing operator yields a bounded leafwise smoothing operator. With this in mind, the problem of showing that such an H is transversely smooth may be recast as follows (with the proof left to the reader).

Lemma 7.2. *Suppose H is a degree zero $\mathcal{G}$ invariant $\mathcal{A}^*(M)$ equivariant (homogeneous of degree 0) bounded leafwise smoothing operator on $\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E$. Then H is transversely smooth if and only if for all local extendable Γ vector fields $Y_1, ..., Y_m \in C^\infty(\nu)$, the operator $\partial_\nu^{Y_1}...\partial_\nu^{Y_m}(H)$ is a bounded leafwise smoothing operator on $\wedge T^*F_s \otimes E$.*

Note that $\widetilde{f}^*R'^*\pi'_+\widetilde{g}^*R^*\nabla^\nu$ makes sense as $\widetilde{f}^*R'^*\pi'_+\widetilde{g}^*R^*$ is a well defined $\mathcal{A}^*(M)$ equivariant operator on $\wedge\nu_s^* \otimes \wedge T^*F_s \otimes E$. Note further that the expression $R^*\nabla^\nu$ does not make sense in general. However, restricted to any sufficiently small transverse submanifold, gf is a diffeomorphism onto its image, so $(gf)^{-1}$ is well defined on this image. This makes it possible to prove the following.

Lemma 7.3. *Suppose $Y \in \nu_x$, and denote by $h_*(Y) \in \nu_{gf(x)}$ the parallel translate of Y along γ_x. Then*

$$\widetilde{f}^*R'^*\pi'_+\widetilde{g}^*R^*\nabla_Y^\nu = \widetilde{f}^*R'^*\pi'_+\widetilde{g}^*\nabla_{h_*(Y)}^\nu R^*.$$

If $Y' \in \nu'_{f(x)}$, then $\widetilde{f}^*R'^*\nabla_{Y'}'^\nu\pi'_+\widetilde{g}^*R^* = \widetilde{f}^*\nabla_{h'_*(Y')}'^\nu R'^*\pi'_+\widetilde{g}^*R^*$, where $h'_*(Y') \in \nu'_{f(x)}$ is the parallel translate of Y' along $f(\gamma_x)^{-1} \cdot \gamma'_{f(x)}$.

Proof. Let (U_x, γ, V) and $(U_{gf(x)}, \gamma\gamma_x^{-1}, V)$ be local charts containing $[\gamma] \in \widetilde{L}_x$, and $[\gamma\gamma_x^{-1}] \in \widetilde{L}_{gf(x)}$, respectively. To compute $\widetilde{f}^*\pi'_+\widetilde{g}^*R^*\nabla_Y^\nu$, we may restrict our attention to $s^{-1}(T)$, where T is any submanifold of M which has Y tangent to it. We may assume that $T \subset U_x$, and gf restricted to T is a diffeomorphism onto its image $gf(T)$, which is also a transverse submanifold, with $gf(T) \subset U_{gf(x)}$. Now $s^{-1}(T) \cap (U_x, \gamma, V) \simeq V$ and $s^{-1}(gf(T)) \cap (U_{gf(x)}, \gamma\gamma_x^{-1}, V) \simeq V$, and the diffeomorphisms with V are just given by the restriction of the target map r. In addition,

$$\left(\nabla_Y^\nu \mid s^{-1}(T)\right) \circ r^* = r^* \circ \left((\nabla_F \otimes \nabla_E)_{Y_\gamma}^\nu\right)$$

and

$$\left(\nabla_{h_*(Y)}^\nu \mid s^{-1}(gf(T))\right) \circ r^* = r^* \circ \left((\nabla_F \otimes \nabla_E)_{h_*(Y)_{\gamma\gamma_x^{-1}}}^\nu\right),$$

where $(\nabla_F \otimes \nabla_E)^\nu$ is the quasi-connection on $\wedge T^*F \otimes E$ over M constructed using the normal bundle ν of F, Y_γ is the parallel translation of Y along γ, and $h_*(Y)_{\gamma\gamma_x^{-1}}$ is the parallel translation of $h_*(Y)$ along $\gamma\gamma_x^{-1}$. So $Y_\gamma = h_*(Y)_{\gamma\gamma_x^{-1}}$. The restriction of R,

$$R_T : s^{-1}(gf(T)) \to s^{-1}(T)$$

is well defined, since $(gf)^{-1}$ is well defined on $gf(T)$. In fact it is a diffeomorphism which locally is just $r^{-1} \circ r$. R_T induces the map on leafwise differential forms

$$R_T^* : C^\infty(s^{-1}(T); \wedge T^* F_s \otimes E) \to C^\infty(s^{-1}(gf(T)); \wedge T^* F_s \otimes E),$$

which extends to the operator

$$R_T^* : C^\infty(s^{-1}(T); \wedge T^*(s^{-1}(T)) \otimes E) \to$$

$$C^\infty(s^{-1}(gf(T)); \wedge T^*(s^{-1}(gf(T))) \otimes E).$$

It is clear that $R_T^* \nabla_Y^\nu$ is a well defined map, and since locally $R_T = r^{-1} \circ r$, we have $R_T^* \nabla_Y^\nu = \nabla_{h_*(Y)}^\nu R_T^*$. But R_T^* is just the restriction of R^* to $s^{-1}(T)$, so $R^* \nabla_Y^\nu = \nabla_{h_*(Y)}^\nu R^*$.

The second statement is proved in the same way. q.e.d.

Proposition 7.4. *The operators $\widetilde{f}^* \nabla'^\nu - \nabla^\nu \widetilde{f}^*$ and $\widetilde{g}^* \nabla^\nu - \nabla'^\nu \widetilde{g}^*$ are leafwise differential operators (By a leafwise differential operator from L' to L, it is sometimes meant, here and in the sequel, operators generated locally by $\rho' \mapsto f^*(\frac{\partial \rho'}{\partial x_i'})$ where the $x_i's$ are leafwise coordinates on L'.), whose composition with a bounded leafwise smoothing operator is again a bounded leafwise smoothing operator.*

Proof. We will only do the proof for $\widetilde{f}^*$ as the proof for $\widetilde{g}^*$ is the same.

Let $\omega \otimes \alpha \otimes \phi \in C_c^\infty(\wedge \nu_s'^* \otimes \wedge T^* F_s' \otimes E')$, with $\omega \in s^* \mathcal{A}^k(M')$, $\alpha \in C_c^\infty(\mathcal{G}'; \wedge T^* F_s')$, and $\phi \in C_c^\infty(\mathcal{G}'; E')$. Then

$$d_s'(\omega \otimes \alpha \otimes \phi) = (-1)^k \omega \otimes d_s'(\alpha \otimes \phi).$$

Now

$$\widetilde{f}^* \nabla'^\nu(\omega \otimes \alpha \otimes \phi) =$$

$$\widetilde{f}^*(d_{M'}\omega \otimes \alpha \otimes \phi + (-1)^k \omega \otimes \nabla_{F'}^\nu \alpha \otimes \phi + (-1)^k \omega \otimes \alpha \otimes \nabla_{E'}^\nu \phi) =$$

$$d_M f^* \omega \otimes \widetilde{f}^* \alpha \otimes \widetilde{f}^* \phi + (-1)^k f^* \omega \otimes \widetilde{f}^* \nabla_{F'}^\nu \alpha \otimes \widetilde{f}^* \phi +$$

$$(-1)^k f^* \omega \otimes \widetilde{f}^* \alpha \otimes \widetilde{f}^* \nabla_{E'}^\nu \phi.$$

On the other hand,

$$\nabla^\nu \widetilde{f}^*(\omega \otimes \alpha \otimes \phi) =$$

$$d_M f^* \omega \otimes \widetilde{f}^* \alpha \otimes \widetilde{f}^* \phi + (-1)^k f^* \omega \otimes \nabla_F^\nu \widetilde{f}^* \alpha \otimes \widetilde{f}^* \phi +$$

$$(-1)^k f^* \omega \otimes \widetilde{f}^* \alpha \otimes \nabla_E^\nu \widetilde{f}^* \phi.$$

Thus

$$(\widetilde{f}^* \nabla^{\nu'} - \nabla^\nu \widetilde{f}^*)(\omega \otimes \alpha \otimes \phi) =$$

$$(-1)^k f^* \omega \otimes \left((\widetilde{f}^* \nabla_{F'}^\nu - \nabla_F^\nu \widetilde{f}^*)\alpha \otimes \widetilde{f}^* \phi + \widetilde{f}^* \alpha \otimes (\widetilde{f}^* \nabla_{E'}^\nu - \nabla_E^\nu \widetilde{f}^*)\phi \right),$$

which contains no differentiation of ω, so $\widetilde{f}^* \nabla'^\nu - \nabla^\nu \widetilde{f}^*$ is indeed a leafwise operator, as are its individual components $\widetilde{f}^* \nabla_{F'}^\nu - \nabla_F^\nu \widetilde{f}^*$ and $\widetilde{f}^* \nabla_{E'}^\nu - \nabla_E^\nu \widetilde{f}^*$.

Next consider the leafwise operator $\widetilde{f}^* \nabla^\nu_{F'} - \nabla^\nu_F \widetilde{f}^*$ on $C^\infty(\wedge T^* F'_s)$. Set

$$d_\nu = p_\nu d_\mathcal{G} \quad \text{and} \quad d'_\nu = p_{\nu'} d_{\mathcal{G}'}.$$

In local coordinates, we may write $\nabla^\nu_{F'}$ and ∇^ν_F as $p_{\nu'}(d_{\mathcal{G}'} + \Theta_{F'})$ and $p_\nu(d_\mathcal{G} + \Theta_F)$, respectively, where $\Theta_{F'}$ and Θ_F are leafwise differential operators (of order zero) with coefficients in $T^*\mathcal{G}'$ and $T^*\mathcal{G}$. Then we have

$$\widetilde{f}^* \nabla^\nu_{F'} - \nabla^\nu_F \widetilde{f}^* = \widetilde{f}^* p_{\nu'}(d_{\mathcal{G}'} + \Theta_{F'}) - p_\nu(d_\mathcal{G} + \Theta_F)\widetilde{f}^* =$$

$$\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^* + \widetilde{f}^* p_{\nu'}\Theta_{F'} - p_\nu \Theta_F \widetilde{f}^*.$$

Lemma 7.5. $\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^*$ and $\widetilde{g}^* d_\nu - d'_\nu \widetilde{g}^*$ are leafwise operators, with

$$\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^* = -\widetilde{f}^* d'_s + d_s \widetilde{f}^* \quad \text{and} \quad \widetilde{g}^* d_\nu - d'_\nu \widetilde{g}^* = -\widetilde{g}^* d_s + d'_s \widetilde{g}^*.$$

Proof. Again we prove this only for $\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^*$. As $\widetilde{f}^* \nabla^\nu_{F'} - \nabla^\nu_F \widetilde{f}^*$ and $\widetilde{f}^* p_{\nu'}\Theta_{F'} - p_\nu \Theta_F \widetilde{f}^*$ are leafwise operators, so is $\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^*$.

On $\mathcal{G} \times B^k$ we have the foliation $F_s \times B^k$ with all its baggage. In particular, we use the product metric on $\mathcal{G} \times B^k$, and we have the transverse derivative d^B_ν. Local charts on $\mathcal{G} \times B^k$ are given by subsets of the form $(U, \gamma, V) \times B^k$, where (U, γ, V) is a local chart for $\mathcal{G}$. It is clear that in these local coordinates, d_ν and d^B_ν have exactly the same form. It is then obvious from the definitions of $\pi_{1,*}$ and e_ω that

$$d_\nu(\pi_{1,*} \circ e_\omega) = (\pi_{1,*} \circ e_\omega)d^B_\nu \quad \text{and} \quad d_s(\pi_{1,*} \circ e_\omega) = (\pi_{1,*} \circ e_\omega)d^B_s,$$

where d^B_s is the leafwise derivative associated to the foliation $F_s \times B^k$. As $\widetilde{f}^* = \pi_{1,*} \circ e_\omega \circ p_f^*$, to prove that $\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^* = -\widetilde{f}^* d'_s + d_s \widetilde{f}^*$, we need only prove that

$$p_f^* d'_\nu - d^B_\nu p_f^* = -p_f^* d'_s + d^B_s p_f^*.$$

This is purely a local question, and the usual proof shows that we need only prove it for compactly supported functions on $\mathcal{G}'$.

Denote by p'_s the projection $p'_s : T\mathcal{G}' \to TF'_s$ determined by the splitting $T\mathcal{G}' = \nu'_s \oplus TF'_s$, and by $p^B_F : T(\mathcal{G} \times B^k) \to T(F_s \times B^k)$ and $p^B_\nu : T(\mathcal{G} \times B^k) \to \nu_B$, the projections determined by the splitting $T(\mathcal{G} \times B^k) = \nu_B \oplus T(F_s \times B^k)$. Let $\phi \in C^\infty_c(\mathcal{G}')$. If $X \in T(F_s \times B^k)$, then $p^B_\nu(X) = 0$ and $p_{f_*}X \in TF'_s$, so $p'_\nu p_{f_*}(X) = 0$. Thus

$$(p_f^* d'_\nu \phi - d^B_\nu p_f^* \phi)(X) = p_f^*((d'_\nu \phi)p_{f_*}(X)) - (d_{\mathcal{G} \times B^k} p_f^* \phi)p^B_\nu(X) =$$

$$p_f^*((d_{\mathcal{G}'}\phi)p'_\nu p_{f_*}(X)) = 0.$$

Next, suppose $X \in \nu_B$, the normal bundle to $F_s \times B^k$, and note that $p_{f_*}X$ is not necessarily in ν'_s. Then

$$(p_f^* d'_\nu \phi)(X) = p_f^*((d'_\nu \phi)(p_{f_*}X)) = p_f^*((d_{\mathcal{G}'}\phi)(p'_\nu p_{f_*}X)) =$$

$$p_f^*((d_{\mathcal{G}'}\phi)(p_{f_*}X)) - p_f^*((d_{\mathcal{G}'}\phi)(p'_s p_{f_*}X)) =$$
$$(d_{\mathcal{G}\times B^k}p_f^*\phi)(X) - p_f^*((d'_s\phi)(p_{f_*}X)) =$$
$$(d_{\mathcal{G}\times B^k}p_f^*\phi)(p_\nu^B X) - p_f^*((d'_s\phi)(p_{f_*}X)) = (d_\nu^B p_f^*\phi - p_f^* d'_s\phi)(X).$$

So
$$(p_f^* d'_\nu - d_\nu^B p_f^*)\phi = (-p_f^* d'_s\phi)p_\nu^B = (-p_f^* d'_s\phi)(I - p_F^B) =$$
$$-p_f^* d'_s\phi + (p_f^* d'_s\phi)p_F^B = -p_f^* d'_s\phi + d_s^B p_f^*\phi,$$

since, restricted to $T(F_s \times B^k)$, $p_f^* d'_s\phi = d_s^B p_f^*\phi$.

Thus $\widetilde{f}^* d'_\nu - d_\nu \widetilde{f}^* = -\widetilde{f}^* d'_s + d_s \widetilde{f}^*$.   q.e.d.

So
$$\widetilde{f}^*\nabla^\nu_{F'} - \nabla^\nu_F \widetilde{f}^* = d_s\widetilde{f}^* - \widetilde{f}^* d'_s + \widetilde{f}^* p_{\nu'}\Theta_{F'} - p_\nu\Theta_F\widetilde{f}^*,$$

a leafwise differential operator (of order at most one).

Finally, consider $\widetilde{f}^*\nabla^\nu_{E'} - \nabla^\nu_E\widetilde{f}^*$ acting on $C_c^\infty(E')$. In local coordinates, and with respect to local framings of E' and E, we may write $\nabla_{E'} = d_{\mathcal{G}'} + \Theta_{E'}$ and $\nabla_E = d_{\mathcal{G}} + \Theta_E$, where $\Theta_{E'}$ and Θ_E are leafwise differential operators (of order zero) with coefficients in $T^*\mathcal{G}'$ and $T^*\mathcal{G}$. Then

$$\widetilde{f}^*\nabla^\nu_{E'} - \nabla^\nu_E\widetilde{f}^* = \widetilde{f}^* p_{\nu'}\nabla_{E'} - p_\nu\nabla_E\widetilde{f}^* =$$
$$\widetilde{f}^* p_{\nu'}(d_{\mathcal{G}'} + \Theta_{E'}) - p_\nu(d_{\mathcal{G}} + \Theta_E)\widetilde{f}^* =$$
$$\widetilde{f}^* d'_\nu - d_\nu\widetilde{f}^* + \widetilde{f}^* p_{\nu'}\Theta_{E'} - p_\nu\Theta_E\widetilde{f}^* = -\widetilde{f}^* d'_s + d_s\widetilde{f}^* + \widetilde{f}^* p_{\nu'}\Theta_{E'} - p_\nu\Theta_E\widetilde{f}^*,$$

since the proof of Lemma 7.5 above extends to show that $\widetilde{f}^* d'_\nu - d_\nu\widetilde{f}^* = -\widetilde{f}^* d'_s + d_s\widetilde{f}^*$, with respect to the local framings. So

$$\widetilde{f}^*\nabla^\nu_{E'} - \nabla^\nu_E\widetilde{f}^* = d_s\widetilde{f}^* - \widetilde{f}^* d'_s + \widetilde{f}^* p_{\nu'}\Theta_{E'} - p_\nu\Theta_E\widetilde{f}^*,$$

is also a leafwise differential operator (of order at most one).

Now observe that if we use coordinates on $\mathcal{G}'$ and $\mathcal{G}$ and framings of E' and E coming from coordinates on M' and M, and framings of E' and E over M' and M, all of whose derivatives are uniformly bounded, then $d_s\widetilde{f}^* - \widetilde{f}^* d'_s + \widetilde{f}^* p_{\nu'}\Theta_{F'} - p_\nu\Theta_F\widetilde{f}^*$ and $d_s\widetilde{f}^* - \widetilde{f}^* d'_s + \widetilde{f}^* p_{\nu'}\Theta_{E'} - p_\nu\Theta_E\widetilde{f}^*$ are (at worst) order one differential operators which have all of their derivatives uniformly bounded. Thus $\widetilde{f}^*\nabla^{\nu'} - \nabla^\nu\widetilde{f}^*$ and all its derivatives define bounded operators from $W_s^*(F', E')$ to $W_{s-1}^*(F, E)$ for each s, and so their compositions with a bounded leafwise smoothing operator are again bounded leafwise smoothing operators.   q.e.d.

Note that the proof above also proves that the composition of $\Upsilon_f = \widetilde{f}^*\nabla^{\nu'} - \nabla^\nu\widetilde{f}^*$ or $\Upsilon_g = \widetilde{g}^*\nabla^\nu - \nabla^{\nu'}\widetilde{g}^*$ with a transversely smooth operator is again a transversely smooth operator. By virtue of Lemma 7.2, we will be using only local extendable Γ vector fields $Y_1, ..., Y_m$ in proving that $\widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^*$ is transversely smooth. Thus Lemma 7.3

becomes $\widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^* \nabla^\nu = \widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* \nabla^\nu R^*$ and $\widetilde{f}^* \nabla'^\nu R'^* \pi'_+ \widetilde{g}^* R^* = \widetilde{f}^* R'^* \nabla'^\nu \pi'_+ \widetilde{g}^* R^*$. Then

$$\partial_\nu(\widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^*) = [\nabla^\nu, \widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^*] =$$

$$\nabla^\nu \widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^* - \widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^* \nabla^\nu =$$

$$\widetilde{f}^* R'^* \nabla'^\nu \pi'_+ \widetilde{g}^* R^* - \widetilde{f}^* R'^* \pi'_+ \nabla'^\nu \widetilde{g}^* R^* - \Upsilon_f R'^* \pi'_+ \widetilde{g}^* - \widetilde{f}^* R'^* \pi'_+ \Upsilon_g R^*.$$

So

7.6. $\qquad \partial_\nu^{Y_1}(\widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^*) =$
$i_{\widehat{Y}_1} \widetilde{f}^* R'^* \partial_{\nu'}(\pi'_+) \widetilde{g}^* R^* - (i_{\widehat{Y}_1} \Upsilon_f) R'^* \pi'_+ \widetilde{g}^* R^* - \widetilde{f}^* R'^* \pi'_+ (i_{\widehat{Y}_1} \Upsilon_g) R^*.$

By assumption, $\partial_{\nu'}(\pi'_+)$ is a bounded leafwise smoothing operator, so $i_{\widehat{Y}_1} \widetilde{f}^* R'^* \partial_{\nu'}(\pi'_+) \widetilde{g}^* R^*$ is also. The operators $i_{\widehat{Y}_1} \Upsilon_f$, and $i_{\widehat{Y}_1} \Upsilon_g$ are leafwise operators which have all their derivatives bounded, so their composition with a bounded leafwise smoothing operator (e.g. $R'^* \pi'_+ \widetilde{g}^*$) is again a bounded leafwise smoothing operator. Thus for any local extendable Γ vector field Y_1 on M, $\partial_\nu^{Y_1}(\widetilde{f}^* R'^* \pi'_+ \widetilde{g}^* R^*)$ is a bounded leafwise smoothing operator.

To continue the induction argument, we need the following.

Lemma 7.7. *Let $Y \in C^\infty(\nu)$ be a local extendable Γ vector field. Then there is a bounded vector field Z' on $\mathcal{G}'$ so that for any $([\gamma], t) \in \mathcal{G} \times B^k$,*

$$i_{\widehat{Y}([\gamma],t)} p_f^* = p_f^* i_{Z'(p_f([\gamma],t))}.$$

Given this, then at $([\gamma], t) \in \mathcal{G} \times B^k$ we have

$$i_{\widehat{Y}_1} p_f^* R'^* \partial_{\nu'}(\pi'_+)([\gamma], t) = i_{\widehat{Y}_1([\gamma],t)} p_f^* R'^* \partial_{\nu'}(\pi'_+) =$$

$$p_f^*(R'^* i_{Z'_1(p_f([\gamma],t))} \partial_{\nu'}(\pi'_+)) = p_f^*(R'^* i_{Z'_1} \partial_{\nu'}(\pi'_+) p_f([\gamma], t)).$$

That is, $i_{\widehat{Y}_1} p_f^* R'^* \partial_{\nu'}(\pi'_+) = p_f^* R'^* i_{Z'_1} \partial_{\nu'}(\pi'_+)$, so

$$i_{\widehat{Y}_1} \widetilde{f}^* R'^* \partial_{\nu'}(\pi'_+) \widetilde{g}^* R^* = \widetilde{f}^* R'^* i_{Z'_1} \partial_{\nu'}(\pi'_+) \widetilde{g}^* R^*.$$

Lemma 7.8. *If ρ is a transversely smooth operator on $\mathcal{A}^*_{(2)}(F'_s, E')$ and Z' is a bounded vector field on $\mathcal{G}'$, then $i_{Z'} \partial_{\nu'}(\rho)$ is a transversely smooth operator.*

Proof. Since $i_{Z'} \partial_{\nu'}(\rho) = i_{p_{\nu'}(Z')} \partial_{\nu'}(\rho)$, we may assume that $Z' = \sum_j g_j \widehat{X}'_j$, where X'_j is a finite local basis for the vector fields on M' and the g_j are smooth functions which are globally bounded along with all their derivatives. Then $i_{Z'} \partial_{\nu'}(\rho) = \sum_j g_j i_{\widehat{X}'_j} \partial_{\nu'}(\rho) = \sum_j g_j \partial_{\nu'}^{X'_j}(\rho)$, which is clearly transversely smooth since the g_j and all their derivatives are globally bounded. $\qquad$ q.e.d.

Using Equation 7.6, we have

$$\partial_\nu^{Y_2}\partial_\nu^{Y_1}(\widetilde{f}^*R'^*\pi'_+\widetilde{g}^*R^*) =$$

$$\partial_\nu^{Y_2}\Big(\widetilde{f}^*R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)\widetilde{g}^*R^* - (i_{\widehat{Y}_1}\Upsilon_f)R'^*\pi'_+\widetilde{g}^*R^* - \widetilde{f}^*R'^*\pi'_+(i_{\widehat{Y}_1}\Upsilon_g)R^*\Big).$$

Repeating the argument above we get

$$\partial_\nu^{Y_2}(\widetilde{f}^*R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)\widetilde{g}^*R^*) =$$

$$i_{\widehat{Y}_2}\widetilde{f}^*R'^*\partial_{\nu'}(i_{Z'_1}\partial_{\nu'}(\pi'_+))\widetilde{g}^*R^* - (i_{\widehat{Y}_2}\Upsilon_f)R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)\widetilde{g}^*R^* -$$

$$\widetilde{f}^*R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)(i_{\widehat{Y}_2}\Upsilon_g)R^* =$$

$$\widetilde{f}^*R'^*i_{Z'_2}\partial_{\nu'}(i_{Z'_1}\partial_{\nu'}(\pi'_+))\widetilde{g}^*R^* - (i_{\widehat{Y}_2}\Upsilon_f)R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)\widetilde{g}^*R^* -$$

$$\widetilde{f}^*R'^*i_{Z'_1}\partial_{\nu'}(\pi'_+)(i_{\widehat{Y}_2}\Upsilon_g)R^*,$$

which is bounded and leafwise smoothing since $i_{Z'_1}\partial_{\nu'}(\pi'_+)$ is transversely smooth.

As $\partial_\nu^{Y_2}$ is a derivation, we have

$$\partial_\nu^{Y_2}((i_{\widehat{Y}_1}\Upsilon_f)R'^*\pi'_+\widetilde{g}^*R^*) =$$

$$\partial_\nu^{Y_2}(i_{\widehat{Y}_1}\Upsilon_f)(R'^*\pi'_+\widetilde{g}^*R^*) + (i_{\widehat{Y}_1}\Upsilon_f)\partial_\nu^{Y_2}(R'^*\pi'_+\widetilde{g}^*R^*).$$

The operators $\partial_\nu^{Y_2}(i_{\widehat{Y}_1}\Upsilon_f)$ and $i_{\widehat{Y}_1}\Upsilon_f$ composed with bounded leafwise smoothing operators produce bounded leafwise smoothing operators. As $R'^*\pi'_+\widetilde{g}^*R^*$ and $\partial_\nu^{Y_2}(R'^*\pi'_+\widetilde{g}^*R^*)$ are bounded leafwise smoothing operators, this term is a bounded leafwise smoothing operator. Similarly for the third term.

Now a straight forward induction argument finishes the proof, modulo the proof of Lemma 7.7. $\qquad\blacksquare$

Proof. To prove Lemma 7.7, we "factor through the graph". In particular, consider the map $p_{f,G} : \mathcal{G} \times B^k \to \mathcal{G} \times B^k \times \mathcal{G}'$ given by $p_{f,G}(\gamma,t) = (\gamma,t,p_f(\gamma,t))$ which is a diffeomorphism onto its image. Denote by $F'_{G,s}$ the foliation of $\mathcal{G} \times B^k \times \mathcal{G}'$ whose leaves are of the form $\widetilde{L} \times B^k \times \widetilde{L}'$, and denote by E'_G the pull back of E' under the projection $\mathcal{G} \times B^k \times \mathcal{G}' \to \mathcal{G}'$. We want to construct a transversely smooth idempotent $\pi'_{+,G}$ which will play the role of π'_+. However, $\pi'_{+,G}$ will not be acting on $\mathcal{A}^*_{(2)}(F'_{G,s}, E'_G)$ over $M \times M'$, but rather on the space denoted $\mathcal{A}^*_{(2)}(F'_{G,s}, \wedge T^*F'_s \otimes E'_G)$ over $M \times M'$, which associates to each (x,x') the Hilbert space $L^2(\widetilde{L}'_{x'}; \wedge T^*F'_s \otimes E')$. Then

$$(\pi'_{+,G})_{(x,x')} := (\pi'_+)_{x'} : L^2(\widetilde{L}'_{x'}; \wedge T^*F'_s \otimes E') \to L^2(\widetilde{L}'_{x'}; \wedge T^*F'_s \otimes E')$$

is well defined, and it is obvious that $\pi'_{+,G}$ is a transversely smooth idempotent and has image $\mathrm{Ker}(\Delta_\ell^{E'+})$.

To define the action $\widetilde{p}^*_{f,G}$ of $\widetilde{p}_{f,G}$ on $\mathcal{A}^*_{(2)}(F'_{G,s}, \wedge T^*F'_s \otimes E'_G)$, we may consider this space as a subspace of all the forms on $\mathcal{G} \times B^k \times \mathcal{G}'$ by using

the pull back of the projection $\mathcal{G} \times B^k \times \mathcal{G}' \to \mathcal{G}'$. When we do so, $\widetilde{p}^*_{f,G}$ is just the usual induced map, and on each fiber $L^2(\widetilde{L}'_{f(x)}; \wedge T^*F'_s \otimes E')$ it equals p^*_f.

Next define

$$\widetilde{g}^*_G : \mathcal{A}^*_{(2)}(F_s, E) \to \mathcal{A}^*_{(2)}(F'_{G,s}, \wedge T^*F'_s \otimes E'_G)$$

to be

$$(\widetilde{g}^*_G)_{g(x')} := (\widetilde{g}^*)_{g(x')} : L^2(\widetilde{L}_{g(x')}; \wedge T^*F_s \otimes E) \to L^2(\widetilde{L}'_{x'}; \wedge T^*F'_s \otimes E'),$$

for each $x' \in M'$.

Finally, the action of R'^* on $\mathcal{A}^*_{(2)}(F'_s, E')$ extends easily to an action on $\mathcal{A}^*_{(2)}(F'_{G,s}, \wedge T^*F'_s \otimes E'_G)$.

Then $p^*_{f,G} R'^* \pi'_{+,G} \widetilde{g}^*_G R^* = p^*_f R'^* \pi'_+ \widetilde{g}^* R^*$, and we may work with $\mathcal{G} \times B^k \times \mathcal{G}'$, $F'_{G,s}$, $p^*_{f,G}$, $\widetilde{g}^*_G$, and $\pi'_{+,G}$ in place of $\mathcal{G}'$, F', p^*_f, $\widetilde{g}^*$, and π'_+, respectively. As $p_{f,G}$ is a diffeomorphism onto its image, we may push forward vector fields such as the $\widehat{Y}_i$ on $\mathcal{G}$ (which are bounded because F is Riemannian) to bounded vector fields Z'_i on $\mathcal{G} \times B^k \times \mathcal{G}'$. Note that these vector fields are only defined along the image of $p_{f,G}$, but this is sufficient for our purposes, since things of the form

$$\widetilde{f}^* R'^* i_{Z'_2} \partial_{\nu'}(i_{Z'_1} \partial_{\nu'}(\pi'_+)) \widetilde{g}^* R^* - (i_{\widehat{Y}_2} \Upsilon_f) R'^* i_{Z'_1} \partial_{\nu'}(\pi'_+) \widetilde{g}^* R^* -$$

$$\widetilde{f}^* R'^* i_{Z'_1} \partial_{\nu'}(\pi'_+)(i_{\widehat{Y}_2} \Upsilon_g) R^*,$$

are still well defined. q.e.d.

This completes the proof that $\pi^f_+ : \mathcal{A}^\ell_{(2)}(F_s, E) \to \operatorname{Im} \widetilde{f}^*_+$ is a transversely smooth idempotent. q.e.d.

The same argument shows that $\operatorname{Im} \widetilde{f}^*_-$, and $\operatorname{Im} \widetilde{f}^*$ determine smooth bundles over M/F, denoted π^f_- and π^f respectively. In fact, we may use the proof above to prove.

Proposition 7.9. *If ρ is a transversely smooth operator on $\wedge \nu'^*_s \otimes F'_s \otimes E'$, then $\widetilde{f}^* R'^* \rho \, \widetilde{g}^* R^*$ is a transversely smooth operator on $\wedge \nu^*_s \otimes F_s \otimes E$.*

8. Induced connections

Let $\nabla' = \pi'_+ \nabla'^\nu \pi'_+$ be the connection on $\pi'_+ = \operatorname{Ker}(\Delta^{E'+}_\ell)$, determined by the quasi-connection ∇'^ν on $\wedge^\ell T^*F'_s \otimes E'$. We now prove that ∇' induces a connection ∇ on π^f_+.

Lemma 8.1. *If ξ' is a local invariant section of π'_+, then $\widetilde{f}^*(\xi')$ is a local invariant section of π^f_+.*

Proof. Recall that for $([\gamma], t) \in \mathcal{G} \times B^k$, $p_f([\gamma], t) = [P_f(\gamma, t) \cdot (f \circ \gamma)]$, the composition of the leafwise paths $P_f(\gamma, t)$ and $f \circ \gamma$, where $P_f(\gamma, t) :$ $[0, 1] \to L'_{f(s(\gamma))}$ is the leafwise path given by

$$P_f(\gamma, t)(s) = p_f(r(\gamma), st).$$

Then

$$\widetilde{f^*}(\xi')([\gamma\gamma_1]) = \pi_{1,*} \circ e_\omega((p_f^* \xi')([\gamma\gamma_1], t)) =$$

$$\pi_{1,*} \circ e_\omega(p_f^*(\xi'(P_f(\gamma\gamma_1, t) \cdot (f \circ \gamma\gamma_1)))) =$$

$$\pi_{1,*} \circ e_\omega(p_f^*(\xi'(P_f(\gamma, t) \cdot (f \circ \gamma) \cdot (f \circ \gamma_1)))) = \pi_{1,*} \circ e_\omega(p_f^*(\xi'(P_f(\gamma, t) \cdot (f \circ \gamma)))),$$

since ξ' is local invariant. But this last equals

$$\pi_{1,*} \circ e_\omega(p_f^* \xi'([\gamma], t)) = \widetilde{f^*}(\xi')([\gamma]).$$

q.e.d.

Lemma 8.2. *Any local invariant section ξ of π_+^f induces a local invariant section $\widetilde{f}^{-*}\xi$ of π'_+.*

Proof. Let T be a transversal in M on which ξ is defined. We may assume that T is so small that $f \,|\, T$ is a diffeomorphism onto its image T'. Then $(\widetilde{f^*})^{-1} : \operatorname{Im} \widetilde{f_+^*} \to \operatorname{Ker}(\Delta_\ell^{E'+})$ is well defined over T, and in fact is given by the map $R'^* P'_\ell \widetilde{g}^* R^* \,|\, T$. To see this, note that over T' the map $R'^* P'_\ell \widetilde{g}^* R^* \widetilde{f^*} : \operatorname{Ker}(\Delta_\ell^{E'}) \to \operatorname{Ker}(\Delta_\ell^{E'})$ is the identity map, since it induces the identity map on cohomology, and that $\operatorname{Ker}(\Delta_\ell^{E'+}) \subset \operatorname{Ker}(\Delta_\ell^{E'})$. For simplicity, we shall denote $R'^* P'_\ell \widetilde{g}^* R^* \,|\, T$ by $\widetilde{f}^{-*}$. For $x' \in T'$, define

$$(\widetilde{f}^{-*}\xi)(x') \equiv \widetilde{f}^{-*}(\xi(f^{-1}(x'))).$$

This gives a well defined smooth section on T'. Extend it to a local invariant section on a neighborhood of T'. We leave it to the reader to show that this construction is well defined, that is it does not depend on the choice of T. q.e.d.

In order to define the induced connection ∇, we need only define it on local invariant sections, and then extend it using (1) of Definition 5.7.

Definition 8.3. Let ξ be a local invariant section of π_+^f. Given $X \in TM$, set $X' = f_*(X)$. Define

$$\nabla_X(\xi) = \widetilde{f^*}(\nabla'_{X'}(\widetilde{f}^{-*}\xi)).$$

Extend to $\xi \in C^\infty(\wedge T^* M; \pi_+^f)$ by using (1) of Definition 5.7.

Proposition 8.4. ∇ *is a connection on π_+^f.*

Proof. We need to check that the four conditions of Definition 5.7 are satisfied.

5.7(1): For differential forms this is satisfied by definition, so we need to check it for functions. Specifically, we need that for any local function ω on M which is constant on plaques of F (i.e. local invariant functions), and for any $X \in TM$ and any local invariant section ξ of π_+^f,

$$\nabla_X(\omega\xi) = d_M\omega(X)\xi + \omega\nabla_X\xi.$$

If $X \in TF$, this is trivially true since both sides are zero. Now suppose that X is transverse to F, with $X' = f_*(X)$, and let T be a transversal of F with X tangent to T. We may assume that T is so small that f restricted to T is a diffeomorphism onto its image T', a transversal of F', with inverse $f^{-1} : T' \to T$. The vector X' is tangent to T', and thanks to Corollary 5.21, we have

$$\nabla_X(\omega\xi) = \widetilde{f}^*(\nabla'_{X'}(\widetilde{f}^{-*}(\omega\xi))) = \widetilde{f}^*(\nabla'_{X'}((\omega \circ f^{-1})\widetilde{f}^{-*}\xi)) =$$
$$\widetilde{f}^*\left[X'(\omega \circ f^{-1})\widetilde{f}^{-*}\xi + (\omega \circ f^{-1})\nabla'_{X'}\widetilde{f}^{-*}\xi\right] =$$
$$(X'(\omega \circ f^{-1}) \circ f)\widetilde{f}^*\widetilde{f}^{-*}\xi + \omega\widetilde{f}^*(\nabla'_{X'}\widetilde{f}^{-*}\xi) =$$
$$(X\omega)\xi + \omega\nabla_X\xi = d_M\omega(X)\xi + \omega\nabla_X\xi.$$

5.7(2): If $X \in TF$ then $X' \in TF'$, and as $\widetilde{f}^{-*}\xi$ is local invariant, $\nabla'_{X'}(\widetilde{f}^{-*}\xi) = 0$, so $\nabla_X(\xi) = \widetilde{f}^*(\nabla'_{X'}(\widetilde{f}^{-*}\xi)) = 0$ and ∇ is flat along F.

5.7(3): The fact that ∇ is $\mathcal{G}-$invariant is a simple exercise which is left to the reader.

5.7(4): We need to show that

$$A = \nabla\pi_+^f - \pi_+^f\nabla^\nu\pi_+^f : C_c^\infty(\wedge T^*M; \wedge T^*F_s \otimes E) \to C^\infty(\wedge T^*M; \pi_+^f)$$

is transversely smooth. Now $\pi_+^f = \widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell$ and $\nabla = \widetilde{f}^*\nabla'\widetilde{f}^{-*} = \widetilde{f}^*\nabla'R'^*P_\ell'\widetilde{g}^*R^* = \widetilde{f}^*\pi_+'\nabla'^\nu\pi_+'R'^*P_\ell'\widetilde{g}^*R^*$. Using the proof of Proposition 7.4, we have that, modulo transversely smooth operators,

$$A = \widetilde{f}^*\pi_+'\nabla'^\nu\pi_+'R'^*P_\ell'\widetilde{g}^*R^*\widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell-$$
$$\widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell\nabla^\nu\widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell =$$
$$\widetilde{f}^*\pi_+'\nabla'^\nu\pi_+'R'^*P_\ell'\widetilde{g}^*R^*\widetilde{f}^*R'^*P_\ell'\pi_+'\widetilde{g}^*R^*P_\ell-$$
$$\widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell\widetilde{f}^*\nabla'^\nu R'^*\pi_+'\widetilde{g}^*R^*P_\ell =$$
$$\widetilde{f}^*\pi_+'\nabla'^\nu\pi_+'R'^*\pi_+'\widetilde{g}^*R^*P_\ell - \widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell\widetilde{f}^*\nabla'^\nu R'^*\pi_+'\widetilde{g}^*R^*P_\ell,$$

since $P_\ell'\widetilde{g}^*R^*\widetilde{f}^*R'^*P_\ell'$ is the identity on $\text{Im}(P_\ell') = \text{Ker}(\Delta_\ell^{E'}) \supset \text{Im}(\pi_+')$. Now $R'^*\pi_+' = \pi_+'R'^*$ and $\nabla'^\nu\pi_+' = (\nabla'^\nu\pi_+')\pi_+' = \pi_+'\nabla'^\nu\pi_+'+[\nabla'^\nu, \pi_+']\pi_+'$, and $[\nabla'^\nu, \pi_+']$ is transversely smooth since π_+' is. So using Proposition 7.9, we have that modulo transversely smooth operators,

$$\widetilde{f}^*R'^*\pi_+'\widetilde{g}^*R^*P_\ell\widetilde{f}^*\nabla'^\nu R'^*\pi_+'\widetilde{g}^*R^*P_\ell =$$

$$\widetilde{f}^*R'^*\pi'_+P'_\ell\widetilde{g}^*R^*P_\ell\widetilde{f}^*\pi'_+\nabla'^\nu\pi'_+R'^*\widetilde{g}^*R^*P_\ell =$$

$$\widetilde{f}^*\pi'_+R'^*P'_\ell\widetilde{g}^*R^*P_\ell\widetilde{f}^*P'_\ell\pi'_+\nabla'^\nu\pi'_+R'^*\widetilde{g}^*R^*P_\ell = \widetilde{f}^*\pi'_+\nabla'^\nu\pi'_+R'^*\widetilde{g}^*R^*P_\ell,$$

since $R'^*P'_\ell\widetilde{g}^*R^*P_\ell\widetilde{f}^*P'_\ell$ is also the identity on $\mathrm{Im}(P'_\ell)$. As $\pi'_+R'^* = \pi'_+\pi'_+R'^* = \pi'_+R'^*\pi'_+$, $A = 0$ modulo transversely smooth operators, that is, A is transversely smooth. $\hfill$ q.e.d.

9. Leafwise homotopy invariance of the twisted higher harmonic signature

In this section we prove our main theorem that the twisted higher harmonic signature is a leafwise homotopy invariant.

Theorem 9.1. *Suppose that M is a compact Riemannian manifold, with an oriented Riemannian foliation F of dimension 2ℓ, and that E is a leafwise flat complex bundle over M with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the projection onto $\mathrm{Ker}(\Delta^E_\ell)$ for the associated foliation F_s of the homotopy groupoid of F is transversely smooth. Then $\sigma(F, E)$ is a leafwise homotopy invariant.*

Recall that the projection onto $\mathrm{Ker}(\Delta^E_\ell)$ is transversely smooth: for the (untwisted) leafwise signature operator; whenever E is a bundle associated to the normal bundle of the foliation; and whenever the leafwise parallel translation on E defined by the flat structure is a bounded map, in particular whenever the preserved metric on E is positive definite. Note also that these conditions are preserved under pull-back by a leafwise homotopy equivalence.

Suppose that M', F', and E' satisfy the hypothesis of Theorem 9.1, and that $f : M \to M'$ is a leafwise homotopy equivalence, which preserves the leafwise orientations. Set $E = f^*(E')$ with the induced leafwise flat structure and preserved metric. Assume that the projections to $\mathrm{Ker}(\Delta^E_\ell)$ and $\mathrm{Ker}(\Delta^{E'}_\ell)$ are transversely smooth. Then we need to show that

$$\mathrm{ch}_a(\pi_\pm) = f^*(\mathrm{ch}_a(\pi'_\pm)).$$

We do this in two stages. The first is to prove

Theorem 9.2. $\mathrm{ch}_a(\pi_\pm) = \mathrm{ch}_a(\pi^f_\pm)$.

Proof. Recall that $\pi^f_\pm = \widetilde{f}^*R'^*\pi'_\pm\widetilde{g}^*R^*P_\ell$, and set

$$\widehat{\pi}^{f,t}_\pm = t\pi^f_\pm + (1-t)P_\ell\pi^f_\pm.$$

A simple computation, using the fact that $\pi^f_\pm P_\ell = \pi^f_\pm$, shows that the $\widehat{\pi}^{f,t}_\pm$ are idempotents, and as P_ℓ and the $\pi^f_\pm$ are transversely smooth, the $\widehat{\pi}^{f,t}_\pm$ are smooth families of transversely smooth idempotents. It follows from Theorem 3.4 that $\mathrm{ch}_a(\widehat{\pi}^{f,0}_\pm) = \mathrm{ch}_a(\widehat{\pi}^{f,1}_\pm)$. Since $\widehat{\pi}^{f,1}_\pm = \pi^f_\pm$, we need

to show that $\mathrm{ch}_a(\widehat{\pi}_{\pm}^{f,0}) = \mathrm{ch}_a(\pi_{\pm})$. We will do only the $+$ case as the other case is the same. Set $\widehat{\pi}_{\pm}^f = \widehat{\pi}_{\pm}^{f,0}$.

Consider the pairings $\langle\,,\,\rangle$, and Q defined in Section 4. Note that $Q(d_s\alpha_1, \alpha_2) = (-1)^{\ell+1}Q(\alpha_1, d_s\alpha_2)$. Using a partition of unity and linearity, proving this reduces to considering sections of compact support of the form $\alpha = \omega \otimes \phi$, where $\omega \in C_c^\infty(\widetilde{L}; \wedge T^*\widetilde{L})$ and ϕ is a flat section of E, where it is immediate. So $\overline{B}_{(2)}^\ell(F_s, E)$ is totally isotropic under the pairing Q, and it is orthogonal to $\mathrm{Ker}(\Delta_\ell^E)$ under the pairing $\langle\,,\,\rangle$. In addition, this equation implies that Q induces a well defined pairing

$$Q : \mathrm{H}_{(2)}^\ell(F_s, E) \otimes \mathrm{H}_{(2)}^\ell(F_s, E) \to \mathcal{B}(M),$$

where $\mathcal{B}(M)$ denotes the Borel $\mathbb{C}$ valued functions on M. It further implies that P_ℓ restricted to the cocycles $Z_{(2)}^\ell(F_s, E)$ preserves Q. The subspaces $\mathrm{Ker}(\Delta_\ell^{E+})$ and $\mathrm{Ker}(\Delta_\ell^{E-})$ are orthogonal under both of the pairings, since $Q(\widehat{\tau}\alpha_1, \alpha_2) = Q(\alpha_1, \widehat{\tau}\alpha_2)$. As $\mathrm{Ker}(\Delta_\ell^E) = \mathrm{Ker}(\Delta_\ell^{E+}) \oplus \mathrm{Ker}(\Delta_\ell^{E-})$, so also $Z_{(2)}^\ell(F_s, E) = \mathrm{Ker}(\Delta_\ell^{E+}) \oplus \mathrm{Ker}(\Delta_\ell^{E-}) \oplus \overline{B}_{(2)}^\ell(F_s, E)$.

The kernels of both $\widehat{\pi}_+^f$ and π_+ contain $\mathrm{Ker}(P_\ell)$, so we may restrict our attention to $\mathrm{Im}(P_\ell) = \mathrm{Ker}(\Delta_\ell^E)$. The image of $\widehat{\pi}_+^f$ is $P_\ell(\mathrm{Im}(\widetilde{f}_+^*))$.

Lemma 9.3. $\pi_+ : P_\ell(\mathrm{Im}(\widetilde{f}_+^*)) \to \mathrm{Ker}(\Delta_\ell^{E+})$ *is an isomorphism with bounded inverse.*

Proof. By Proposition 6.5, $\widetilde{f}^*$ restricted to $\mathrm{Ker}(\Delta_\ell^{E'})$ takes the pairing Q' to the pairing Q. (Note that Q is $\pm$ definite on the $\mathrm{Im}(\pi_\pm)$ if ℓ is even, while it is $\sqrt{-1}Q$, which is $\pm$ definite on $\mathrm{Im}(\pi_\pm)$ is ℓ is odd. We will finesse this point.) Since P_ℓ (restricted to the cocycles) preserves the pairing Q, Q is positive definite on $P_\ell(\mathrm{Im}(\widetilde{f}_+^*))$. Given $0 \neq \alpha \in P_\ell(\mathrm{Im}(\widetilde{f}_+^*))$, write it (uniquely) as $\alpha = \alpha_+ + \alpha_-$, where $\alpha_\pm \in \mathrm{Ker}(\Delta_\ell^{E\pm})$. Then

$$0 < Q(\alpha, \alpha) = \langle\alpha_+, \alpha_+\rangle - \langle\alpha_-, \alpha_-\rangle \leq \langle\alpha_+, \alpha_+\rangle,$$

so $\pi_+(\alpha) = \alpha_+ \neq 0$ and $\pi_+ : P_\ell(\mathrm{Im}(\widetilde{f}_+^*)) \to \mathrm{Ker}(\Delta_\ell^{E+})$ is one-to-one.

The above inequality also implies that π_+^{-1} is bounded, with bound $\sqrt{2}$. The element $\alpha = \pi_+^{-1}(\alpha_+)$ and $||\alpha||^2 = \langle\alpha, \alpha\rangle = \langle\alpha_+, \alpha_+\rangle + \langle\alpha_-, \alpha_-\rangle = ||\alpha_+||^2 + ||\alpha_-||^2$. Since $0 < Q(\alpha, \alpha)$, $||\alpha_-||^2 < ||\alpha_+||^2$, so $||\pi_+^{-1}(\alpha_+)||^2 = ||\alpha||^2 \leq 2||\alpha_+||^2$.

Next we show that π_+ is onto. Choose $\alpha \in \mathrm{Ker}(\Delta_\ell^{E+})$ which is orthogonal to $\pi_+(P_\ell(\mathrm{Im}\,\widetilde{f}_+^*))$. The subspaces $P_\ell(\mathrm{Im}\,\widetilde{f}_+^*)$ and $P_\ell(\mathrm{Im}\,\widetilde{f}_-^*)$ are orthogonal under Q. Their direct sum is the space $\mathrm{Ker}(\Delta_\ell)$ of all harmonic forms, since $\pi_+^f + \pi_-^f$ induces the identity on cohomology. Write $\alpha = \beta_+ + \beta_-$, with $\beta_\pm \in P_\ell(\mathrm{Im}\,\widetilde{f}_\pm^*)$. Then

$$||\alpha||^2 = Q(\alpha, \alpha) = Q(\alpha, \beta_+) + Q(\alpha, \beta_-) =$$

$$Q(\alpha, \pi_+\beta_+) + Q(\alpha, \pi_-\beta_+) + Q(\alpha, \beta_-) \; = \; Q(\alpha, \beta_-).$$

The last equality is a consequence of the facts that α is Q orthogonal to $\pi_+(P_\ell(\mathrm{Im}\,\widetilde{f}_+^*))$ and that $\mathrm{Ker}(\Delta_\ell^{E+})$ and $\mathrm{Ker}(\Delta_\ell^{E-})$ are Q orthogonal. Hence we have,

$$0 \le \|\alpha\|^2 = Q(\beta_+, \beta_-) + Q(\beta_-, \beta_-) = Q(\beta_-, \beta_-) \le 0.$$

So, $\alpha = 0$, and π_+ is onto. $\hspace{4cm}$ q.e.d.

Consider the map $\pi_+^{-1} : \mathrm{Ker}(\Delta_\ell^{E+}) \to \pi_+(P_\ell(\mathrm{Im}\,\widetilde{f}_+^*))$, and define ρ_+ to be

$$\rho_+ = \pi_+^{-1} \circ \pi_+ : \mathcal{A}_{(2)}^\ell(F_s, E) \to P_\ell(\mathrm{Im}(\widetilde{f}_+^*)).$$

Then ρ_+ is an idempotent and has image $P_\ell(\mathrm{Im}\,\widetilde{f}_+^*)$, which equals $\mathrm{Im}\,\widehat{\pi}_+^f$. We claim that ρ_+ is transversely smooth. If so, then $\mathrm{ch}_a(\rho_+)$ is defined and $\mathrm{ch}_a(\rho_+) = \mathrm{ch}_a(\widehat{\pi}_+^f)$, since they have the same image. Note that ρ_+ is the projection to $P_\ell(\mathrm{Im}(\widetilde{f}_+^*))$ along $\mathrm{Ker}(\pi_+)$. With this description, it is immediate that $\rho_+ \circ \pi_+ = \rho_+$ and $\pi_+ \circ \rho_+ = \pi_+$ since π_+ is the projection to $\mathrm{Ker}(\Delta_\ell^{E+})$ along $\mathrm{Ker}(\pi_+)$. As above, we may form the smooth family of transversely smooth idempotents $t\rho_+ + (1-t)\pi_+$ which connects ρ_+ to π_+. Again, it follows from Theorem 3.4 that $\mathrm{ch}_a(\rho_+) = \mathrm{ch}_a(\pi_+)$, and we have $\mathrm{ch}_a(\pi_+) = \mathrm{ch}_a(\widehat{\pi}_+^f)$. So to finish the proof we need only show that ρ_+ is transversely smooth.

Now

$$\widehat{\pi}_\pm^f \; = \; P_\ell \pi_\pm^f \; = \; P_\ell \widetilde{f}^* R'^* \pi_\pm' \widetilde{g}^* R^* P_\ell,$$

and recalling that $P_\ell' \widetilde{g}^* R^* P_\ell \widetilde{f}^* R'^* P_\ell' = P_\ell'$ and $\pi_\pm' = \pi_\pm' P_\ell' = P_\ell' \pi_\pm'$, we have

$$(\widehat{\pi}_\pm^f)^2 \; = \; \widehat{\pi}_\pm^f \quad \text{and} \quad \widehat{\pi}_\pm^f \widehat{\pi}_\mp^f \; = \; 0.$$

These idempotents are transversely smooth since P_ℓ and the $\pi_\pm^f$ are transversely smooth. They also satisfy $\widehat{\pi}_+^f + \widehat{\pi}_-^f = P_\ell$, and their kernels both contain $\mathrm{Ker}(P_\ell)$. Finally, note that the $\mathrm{Im}(\widehat{\pi}_\pm^f) \; = \; P_\ell(\mathrm{Im}(\widetilde{f}_\pm^*))$. Next set

$$A = \pi_+ + \widehat{\pi}_-^f.$$

Lemma 9.4. *The operator A and its adjoint A^t are transversely smooth, and A is an isomorphism when restricted to $\mathrm{Ker}(\Delta_\ell^E)$.*

Proof. A is transversely smooth because both π_+ and $\widehat{\pi}_-^f$ are. As $A^t = (\pi_+ + \widehat{\pi}_-^f)^t = \pi_+ + (\widehat{\pi}_-^f)^t$, we need only show that

$$(\widehat{\pi}_-^f)^t = P_\ell R^{*t} \widetilde{g}^{*t} \pi_-' R'^{*t} \widetilde{f}^{*t} P_\ell$$

is transversely smooth. Now P_ℓ and π_-' are transversely smooth, and $R^{*t} = (R^*)^{-1}$ and $R'^{*t} = (R'^*)^{-1}$ since they are both isometries. Now

consider $\widetilde{f}^{*t}$ and $\widetilde{g}^{*t}$ restricted to the harmonic forms. Let $\alpha' \in \mathrm{Im}(P'_\ell)$ and $\alpha \in \mathrm{Im}(\pi_+)$. Then

$$\langle \alpha', \widetilde{f}^{*t}\alpha \rangle \;=\; \langle \widetilde{f}^*\alpha', \alpha \rangle \;=\; Q(\widetilde{f}^*\alpha', \widehat{*}\alpha) = \; Q(\widetilde{f}^*\alpha', \widehat{\tau}\alpha) \;=$$
$$Q(\widetilde{f}^*\alpha', \alpha) \;=\; Q(\widetilde{f}^*\alpha', \widetilde{f}^* R'^* \widetilde{g}^* R^*\alpha) \;=\; Q'(\alpha', R'^*\widetilde{g}^* R^*\alpha) \;=$$
$$Q'(\alpha', \pi'_+ R'^*\widetilde{g}^* R^*\alpha + \pi'_- R'^*\widetilde{g}^* R^*\alpha) \;=$$
$$Q'(\alpha', \widehat{\tau}\pi'_+ R'^*\widetilde{g}^* R^*\alpha - \widehat{\tau}\pi'_- R'^*\widetilde{g}^* R^*\alpha) \;=$$
$$Q'(\alpha', \widehat{*}\pi'_+ R'^*\widetilde{g}^* R^*\alpha - \widehat{*}\pi'_- R'^*\widetilde{g}^* R^*\alpha) \;=$$
$$\langle \alpha', (\pi'_+ R'^*\widetilde{g}^* R^* - \pi'_- R'^*\widetilde{g}^* R^*)\alpha \rangle.$$

So on $\mathrm{Im}(\pi_+)$, $\widetilde{f}^{*t} = \pi'_+ R'^*\widetilde{g}^* R^* - \pi'_- R'^*\widetilde{g}^* R^*$. Similarly, on $\mathrm{Im}(\pi_-)$, $\widetilde{f}^{*t} = -\pi'_+ R'^*\widetilde{g}^* R^* + \pi'_- R'^*\widetilde{g}^* R^*$. Thus on $\mathrm{Im}(P_\ell)$,

$$\widetilde{f}^{*t} \;=\; (\pi'_+ R'^*\widetilde{g}^* R^* - \pi'_- R'^*\widetilde{g}^* R^*)\pi_+ - (\pi'_+ R'^*\widetilde{g}^* R^* - \pi'_- R'^*\widetilde{g}^* R^*)\pi_- \;=$$
$$(\pi'_+ - \pi'_-)R'^*\widetilde{g}^* R^*(\pi_+ - \pi_-).$$

Similarly, $\widetilde{g}^{*t} = (\pi_+ - \pi_-)R^*\widetilde{f}^* R'^*(\pi'_+ - \pi'_-)$. As $(\pi'_+ - \pi'_-)\pi'_-(\pi'_+ - \pi'_-) = \pi'_-$, R^* commutes with $\pi_\pm$, R'^* commutes with $\pi'_\pm$, and $P_\ell\pi_\pm = \pi_\pm$, we have

$$(\widehat{\pi}^f_-)^t = (\pi_+ - \pi_-)\widetilde{f}^* R'^*\pi'_- \widetilde{g}^* R^*(\pi_+ - \pi_-),$$

which is transversely smooth.

Next, note that Q is positive definite on $\mathrm{Im}(\pi_+)$ and $\mathrm{Im}(\widehat{\pi}^f_+)$, and is negative definite on $\mathrm{Im}(\pi_-)$ and $\mathrm{Im}(\widehat{\pi}^f_-)$. So $\mathrm{Im}(\pi_\pm) \cap \mathrm{Im}(\widehat{\pi}^f_\mp) = \{0\}$. Let $\alpha \in \mathrm{Ker}(\Delta^E_\ell)$ with $A(\alpha) = 0$. Then $\pi_+(\alpha) = -\widehat{\pi}^f_-(\alpha)$ and $\pi_+(\alpha), \widehat{\pi}^f_-(\alpha) \in \mathrm{Im}(\pi_+) \cap \mathrm{Im}(\widehat{\pi}^f_-) = \{0\}$. Thus $\alpha \in \mathrm{Ker}(\pi_+) \cap \mathrm{Ker}(\widehat{\pi}^f_-) \cap \mathrm{Ker}(\Delta^E_\ell) = \mathrm{Im}(\pi_-) \cap \mathrm{Im}(\widehat{\pi}^f_+) = \{0\}$, so $\alpha = 0$ and A is one-to-one.

Now $A(\mathrm{Im}(\widehat{\pi}^f_+)) = \pi_+(P_\ell(\mathrm{Im}(\widetilde{f}^*_+))) = \mathrm{Im}(\pi_+)$, so $\mathrm{Im}(\pi_+) \subset \mathrm{Im}(A)$. Just as π_+ maps $\mathrm{Im}(\widehat{\pi}^f_+)$ isomorphically to $\mathrm{Im}(\pi_+)$, π_- maps $\mathrm{Im}(\widehat{\pi}^f_-)$ isomorphically to $\mathrm{Im}(\pi_-)$. Given $\alpha \in \mathrm{Im}(\pi_-)$, let $\beta \in \mathrm{Im}(\widehat{\pi}^f_-)$ with $\pi_-(\beta) = \alpha$, so $\beta = \pi_-(\beta) + \pi_+(\beta) = \alpha + \pi_+(\beta)$, that is $\alpha = \beta - \pi_+(\beta)$. Now $A(\beta) = \pi_+(\beta) + \widehat{\pi}^f_-(\beta) = \pi_+(\beta) + \beta$, since $\beta \in \mathrm{Im}(\widehat{\pi}^f_-)$. So $\beta \in \mathrm{Im}(A)$, since $\pi_+(\beta) \in \mathrm{Im}(\pi_+) \subset \mathrm{Im}(A)$. Thus $\alpha = \beta - \pi_+(\beta) \in \mathrm{Im}(A)$, and we have $\mathrm{Im}(\pi_-) \subset \mathrm{Im}(A)$. As A is linear and contains $\mathrm{Im}(\pi_\pm)$, it also contains $\mathrm{Im}(\pi_+) \oplus \mathrm{Im}(\pi_-) = \mathrm{Ker}(\Delta^E_\ell)$, and A is onto.   q.e.d.

Lemma 9.5. A^{-1}, the inverse of A restricted to $\mathrm{Ker}(\Delta^E_\ell)$, is a bounded isomorphism of $\mathrm{Ker}(\Delta^E_\ell)$.

Proof. A^{-1} is bounded if and only if there is a constant $C > 0$ so that $||A(\alpha)|| \geq C$ for all $x \in M$ and all $\alpha \in \mathrm{Ker}(\Delta^E_\ell)_x$ with $||\alpha|| = 1$. If not, there are sequences $x_j \in M$ and $\alpha_j \in \mathrm{Ker}(\Delta^E_\ell)_{x_j}$ with $||\alpha_j|| = 1$, and

$$\lim_{j \to \infty} ||A(\alpha_j)|| = \lim_{j \to \infty} ||\pi_+(\alpha_j) + \widehat{\pi}^f_-(\alpha_j)|| = 0,$$

that is,

$$0 = \lim_{j\to\infty} \pi_+(\alpha_j) + \widehat{\pi}^f_-(\alpha_j) = \lim_{j\to\infty} \pi_+(\alpha_j) + \pi_+(\widehat{\pi}^f_-(\alpha_j)) + \pi_-(\widehat{\pi}^f_-(\alpha_j)) =$$

$$\lim_{j\to\infty} \pi_+(\alpha_j + \widehat{\pi}^f_-(\alpha_j)) + \pi_-(\widehat{\pi}^f_-(\alpha_j)).$$

This implies that $\lim_{j\to\infty} \pi_-(\widehat{\pi}^f_-(\alpha_j)) = 0$. Now

$$0 \ \geq \ Q(\widehat{\pi}^f_-(\alpha_j), \widehat{\pi}^f_-(\alpha_j)) \ = \ ||\pi_+(\widehat{\pi}^f_-(\alpha_j))||^2 - ||\pi_-(\widehat{\pi}^f_-(\alpha_j))||^2,$$

so $\lim_{j\to\infty} \pi_+(\widehat{\pi}^f_-(\alpha_j)) = 0$, which gives that $\lim_{j\to\infty} \widehat{\pi}^f_-(\alpha_j) = 0$, so also $\lim_{j\to\infty} \pi_+(\alpha_j) = 0$. Since $\alpha_j = \pi_+(\alpha_j) + \pi_-(\alpha_j)$, we have

$$\lim_{j\to\infty} (\pi_-(\alpha_j) - \alpha_j) = 0,$$

in particular,

$$\lim_{j\to\infty} ||\pi_-(\alpha_j)|| = \lim_{j\to\infty} ||\alpha_j|| = 1.$$

Now $Q(\pi_-(\alpha_j), \pi_-(\alpha_j)) = -||\pi_-(\alpha_j)||^2$, so $\lim_{j\to\infty} Q(\pi_-(\alpha_j), \pi_-(\alpha_j)) = -1$. Since Q is continuous,

$$\lim_{j\to\infty} Q(\alpha_j, \alpha_j) = \lim_{j\to\infty} Q(\pi_-(\alpha_j), \pi_-(\alpha_j)) = -1.$$

The fact that $\lim_{j\to\infty} \widehat{\pi}^f_-(\alpha_j) = 0$ and $\alpha_j = \widehat{\pi}^f_+(\alpha_j) + \widehat{\pi}^f_-(\alpha_j)$ implies that

$$\lim_{j\to\infty} (\widehat{\pi}^f_+(\alpha_j) - \alpha_j) = 0,$$

and as above, the fact that $Q(\widehat{\pi}^f_+(\alpha_j), \widehat{\pi}^f_+(\alpha_j)) \geq 0$ implies that

$$\liminf_j Q(\alpha_j, \alpha_j) \geq 0,$$

which contradicts that fact that $\lim_{j\to\infty} Q(\alpha_j, \alpha_j) = -1$. $\qquad$ q.e.d.

Now consider the map $B = A^t A$, which is transversely smooth and is an isomorphism when restricted to $\mathrm{Ker}(\Delta^E_\ell)$. Denote by B^{-1} the composition of maps:

$$B^{-1} : \mathcal{A}^\ell_{(2)}(F_s, E) \xrightarrow{P_\ell} \mathrm{Ker}(\Delta^E_\ell) \xrightarrow{B^{-1}_\ell} \mathrm{Ker}(\Delta^E_\ell),$$

where B^{-1}_ℓ is the inverse of B restricted to $\mathrm{Ker}(\Delta^E_\ell)$. Since ρ_+ takes values in $P_\ell(\mathrm{Im}(\widetilde{f}^*_*)) = \mathrm{Im}(\widehat{\pi}^f_+)$, $A\rho_+ = \pi_+$, so $B\rho_+ = A^t\pi_+$, and $\rho_+ = B^{-1}A^t\pi_+$. Thus we are reduced to showing that B^{-1} is transversely smooth.

Restricting once again to $\mathrm{Ker}(\Delta^E_\ell)$, we have that the operator B is positive, and A and A^{-1} are bounded operators, so the spectrum of B on $\mathrm{Ker}(\Delta^E_\ell)$ is contained in a bounded positive set $[C_0, C_1]$, and since $B = P_\ell B P_\ell$, its total spectrum is contained in $[C_0, C_1] \cup 0$. Thus, for sufficiently large λ, $0 < ||P_\ell - \frac{B}{\lambda}|| \leq 1 - C_0/\lambda < 1$, and this estimate

also holds for all Sobolev norms associated to $\mathcal{A}^\ell_{(2)}(F_s, E)$. So for large λ,

$$B^{-1} \;=\; \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(P_\ell - \frac{B}{\lambda}\right)^n,$$

where $\left(P_\ell - \frac{B}{\lambda}\right)^0 = P_\ell$. For $N \in \mathbb{Z}_+$, set

$$D_N \;=\; \frac{1}{\lambda} \sum_{n=0}^{N} \left(P_\ell - \frac{B}{\lambda}\right)^n,$$

where again $\left(P_\ell - \frac{B}{\lambda}\right)^0 = P_\ell$. Then D_N is a uniformly bounded (over all N) transversely smooth operator, and it converges to B^{-1} in all Sobolev norms. Thus B^{-1} is a bounded leafwise smoothing operator.

Consider $\partial_\nu^Y D_N = \frac{1}{\lambda} \sum_{n=0}^{N} \partial_\nu^Y \left(\left(P_\ell - \frac{B}{\lambda}\right)^n\right)$, where Y is a vector field on M. For any integers k_1, k_2, and for $N > 1$,

$$||B^{-1} - D_N||_{k_1, k_2} \;\;\leq\;\; \frac{1}{\lambda} \sum_{n=N+1}^{\infty} ||\partial_\nu^Y \left(\left(P_\ell - \frac{B}{\lambda}\right)^n\right)||_{k_1, k_2} \;\;\leq$$

$$\frac{1}{\lambda} ||\partial_\nu^Y \left(P_\ell - \frac{B}{\lambda}\right)||_{k_1, k_2} \sum_{n=N+1}^{\infty} n ||P_\ell - \frac{B}{\lambda}||^{n-1} \;\;\leq$$

$$\frac{1}{\lambda} ||\partial_\nu^Y \left(P_\ell - \frac{B}{\lambda}\right)||_{k_1, k_2} \sum_{n=N+1}^{\infty} n \left(1 - \frac{C_0}{\lambda}\right)^{n-1}.$$

This converges to 0 as $N \to \infty$ as $||\partial_\nu^Y \left(P_\ell - \frac{B}{\lambda}\right)||_{k_1, k_2}$ is finite since $P_\ell - \frac{B}{\lambda}$ is transversely smooth. Thus the transverse derivative $\partial_\nu^Y D_N$ converges in all Sobolev norms, so $\lim_{N \to \infty} \partial_\nu^Y D_N$ exists, and it is bounded and leafwise smoothing.

Proposition 9.6. $\partial_\nu^Y B^{-1}$ *exists, in particular,* $\partial_\nu^Y D_N$ *converges in all Sobolev norms to* $\partial_\nu^Y B^{-1}$*, so* $\partial_\nu^Y B^{-1}$ *is a bounded leafwise smoothing operator.*

Proof. As $\partial_\nu^Y D_N$ converges in all Sobolev norms, we only need prove that $\partial_\nu^Y B^{-1}$ exists and that it equals $\lim_{N \to \infty} \partial_\nu^Y D_N$.

Recall the situation in the proof of Theorem 4.4. For y close to x in M, we have the smooth diffeomorphism $\Phi_y : \widetilde{L}_x \to \widetilde{L}_y$. Given $Y \in TM_x$, set $\gamma(t) = \exp_x(tY)$. For $z \in \widetilde{L}_x$ and t sufficiently small, say $|t| \leq \epsilon$ (which we assume from now on), we have the path $t \to \widehat{\gamma}_z(t)$, which covers $\gamma(t)$ and has tangent vector in ν_s. So for $|t| \leq \epsilon$, the diffeomorphism $\Phi_{\gamma(t)} : \widetilde{L}_x \to \widetilde{L}_{\gamma(t)}$ exists. The vector Y defines the transverse vector field $\widehat{Y}$ along $\widetilde{L}_x$, i. e. a smooth section of $\nu_s \,|\, \widetilde{L}_x$, by requiring $s_*(\widehat{Y}) = Y$. Then the operator $\partial_\nu^Y(\cdot) = [\nabla_{\widehat{Y}}^\nu, \cdot]$ can be realized

as $\partial/\partial t(\cdot)$ as follows. We may parallel translate all objects on $\widetilde{L}_x$ to $\widetilde{L}_{\gamma(t)}$ (and vice-versa) along the paths $\widehat{\gamma}_z(t)$ using the connection ∇. We will denote this parallel translation by Φ_t (and the reverse by Φ_t^{-1}). Thus any section of $\xi \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$ defines a section $\Phi_t(\xi) = \xi_t$ of $C_c^\infty(\widetilde{L}_{\gamma(t)}; \wedge^\ell T^* F_s \otimes E)$ given by

$$\xi_t(z) \;=\; \Phi_t(\xi(\Phi_{\gamma(t)}^{-1}(z))),$$

and ξ_t is smooth in t. Note that for such a local section, $\nabla_{\widehat{Y}}\xi_t = \nabla_{\widehat{Y}}^\nu \xi_t$ (as $\widehat{Y} \in \nu_s$) is defined and equals 0, since ξ_t is parallel translation along integral curves of $\widehat{Y}$ for the connection ∇. In fact, if we set $Y(t) = \gamma'(t)$, then $\nabla_{\widehat{Y}(t)}\xi_t = \nabla_{\widehat{Y}(t)}^\nu \xi_t = 0$. Further note that $\Phi_{\gamma(t)}$ is a diffeomorphism of bounded dilation and the induced action on E is also bounded, so the local operators Φ_t and Φ_t^{-1} are bounded when acting on sections of $C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, (respectively $C_c^\infty(\widetilde{L}_{\gamma(t)}; \wedge^\ell T^* F_s \otimes E)$). We denote these bounds, which are uniform in t, by $||\Phi_t||$ and $||\Phi_t^{-1}||$ respectively.

Similarly, we may parallel translate operators such as D_N from nearby leaves to $\widetilde{L}_x$ as follows. Given $\xi_1, \xi_2 \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, define the operator $D_{N,t}$ on $\widetilde{L}_x$ by

$$\langle D_{N,t}(\xi_1), \xi_2 \rangle \;=\; \langle \Phi_t^{-1}\Big(D_{N,\gamma(t)}(\xi_{1,t})\Big), \xi_2 \rangle.$$

This is well defined and smooth in t. Thus the operator $\dfrac{\partial(D_{N,t})}{\partial t}\Big|_{t=0}$ is well defined as a map from $C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$ to $C^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$. Likewise, $\nabla_{\widehat{Y}}(D_{N,\gamma(t)}(\xi_t))$ is well defined for all $\xi \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, and takes values in $C^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$. The fundamental relationship between parallel translation and the connection ∇ translates to the equation

9.7. $$\left(\dfrac{\partial(D_{N,t})}{\partial t}\Big|_{t=0}\right)(\xi) \;=\; \nabla_{\widehat{Y}}(D_{N,\gamma(t)}(\xi_t)).$$

In fact, for all $t_0 \in [-\epsilon, \epsilon]$,

$$\left(\dfrac{\partial(D_{N,t})}{\partial t}\Big|_{t=t_0}\right)(\xi) \;=\; \Phi_{t_0}^{-1}\Big(\nabla_{\widehat{Y}(t_0)}(D_{N,\gamma(t)}(\xi_t))\Big),$$

since $\Phi_t^{-1} = \Phi_{t_0}^{-1} \circ \Phi_{t,t_0}^{-1}$, where Φ_{t,t_0}^{-1} is parallel translation from $\widetilde{L}_{\gamma(t)}$ to $\widetilde{L}_{\gamma(t_0)}$.

For $\xi \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$ we have

$$\partial_\nu^Y D_N \xi \;=\; [\nabla_{\widehat{Y}}^\nu, D_N]\xi \;=\; [\nabla_{\widehat{Y}}, D_N]\xi \;=\;$$

$$\nabla_{\widehat{Y}} D_{N,\gamma(t)}(\xi_t) - D_N \nabla_{\widehat{Y}}(\xi_t) \;=\; \nabla_{\widehat{Y}} D_{N,\gamma(t)}(\xi_t),$$

since $\nabla_{\widehat{Y}}(\xi_t) = 0$. So by Equation 9.7 we have

$$\partial_\nu^Y D_N = \frac{\partial(D_{N,t})}{\partial t}\Big|_{t=0}.$$

As above, this extends to

9.8. $$\frac{\partial(D_{N,t})}{\partial t} = \Phi_t^{-1}\Big(\partial_\nu^{Y(t)} D_{N,\gamma(t)}\Big).$$

Set $B_t^{-1} = \Phi_t^{-1}(B^{-1})$, $D'_{N,t} = \Phi_t^{-1}\Big(\partial_\nu^{Y(t)} D_N\Big) = \partial(D_{N,t})/\partial t$, and $D'_t = \Phi_t^{-1}\Big(\lim_{N\to\infty}\partial_\nu^{Y(t)} D_N\Big)$. Note carefully that the following computation takes place on the leaf $\widetilde{L}_x$. For $\xi_1, \xi_2 \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, and $h \in (0, \epsilon)$, we have that

$$\left|\langle B_h^{-1}(\xi_1), \xi_2\rangle - \langle B_0^{-1}(\xi_1), \xi_2\rangle - \int_0^h \langle D'_t(\xi_1), \xi_2\rangle dt\right| \le$$

$$\left|\langle B_h^{-1}(\xi_1), \xi_2\rangle - \langle D_{N,h}(\xi_1), \xi_2\rangle\right| +$$

$$\left|\langle D_{N,h}(\xi_1), \xi_2\rangle - \langle D_{N,0}(\xi_1), \xi_2\rangle - \int_0^h \langle D'_{N,t}(\xi_1), \xi_2\rangle dt\right| +$$

$$\left|\langle D_{N,0}(\xi_1), \xi_2\rangle - \langle B_0^{-1}(\xi_1), \xi_2\rangle\right| + \left|\int_0^h \langle (D'_{N,t} - D'_t)(\xi_1), \xi_2\rangle dt\right|.$$

The first term equals

$$\left|\langle \Phi_h^{-1}(B_{\gamma(h)}^{-1} - D_{N,\gamma(h)})\Phi_h(\xi_1), \xi_2\rangle\right| \le$$

$$||\Phi_h^{-1}||\, ||B_{\gamma(h)}^{-1} - D_{N,\gamma(h)}||\, ||\Phi_h||\, ||\xi_1||\, ||\xi_2||,$$

which goes to 0 as $N \to \infty$, since $D_N \to B^{-1}$ in norm and $||\Phi_h^{-1}||$ and $||\Phi_h||$ are bounded. The second term is zero since $D'_{N,t} = \partial(D_{N,t})/\partial t$. The third term is bounded by $||D_{N,0} - B_0^{-1}||\, ||\xi_1||\, ||\xi_2||$, which goes to 0 as $N \to \infty$, since $D_N \to B^{-1}$ in norm. The last term is bounded by

$$\int_0^h ||\Phi_t^{-1}||\, ||\partial_\nu^{Y(t)} D_{N,\gamma(t)} - \lim_{\widehat{N}\to\infty}\partial_\nu^{Y(t)} D_{\widehat{N},\gamma(t)}||\, ||\Phi_t||\, ||\xi_1||\, ||\xi_2|| dt,$$

which goes to 0 as $N \to \infty$, since $||\Phi_t^{-1}||$ and $||\Phi_t||$ are uniformly bounded for $t \in [0, h]$, and $\partial_\nu^Y D_N$ converges in norm.

Thus

$$\lim_{N\to\infty}\left|\langle B_h^{-1}(\xi_1), \xi_2\rangle - \langle B_0^{-1}(\xi_1), \xi_2\rangle - \int_0^h \langle D'_t(\xi_1), \xi_2\rangle dt\right| = 0,$$

and as the expression inside the limit is independent of N, it actually equals 0. This implies that

$$\langle \lim_{h\to 0}\frac{1}{h}\Big(B_h^{-1} - B_0^{-1} - \int_0^h D'_t dt\Big)(\xi_1), \xi_2\rangle = 0,$$

for all $\xi_1, \xi_2 \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, so

$$\lim_{h \to 0} \frac{1}{h} \left(B_h^{-1} - B_0^{-1} - \int_0^h D_t' dt \right) = 0$$

as a map from $C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$ to $C^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$.

Next we have

$$\left| \langle \lim_{t \to 0} D_t'(\xi_1), \xi_2 \rangle - \langle D_0'(\xi_1), \xi_2 \rangle \right| \le \left| \langle \lim_{t \to 0} (D_t' - D_{N,t}')(\xi_1), \xi_2 \rangle \right| +$$

$$\left| \langle (\lim_{t \to 0} D_{N,t}' - D_{N,0}')(\xi_1), \xi_2 \rangle \right| + \left| \langle (D_{N,0}' - D_0')(\xi_1), \xi_2 \rangle \right| \le$$

$$\lim_{t \to 0} \|\Phi_t^{-1}\| \, \| \lim_{\widehat{N} \to \infty} \partial_\nu^{Y(t)} D_{\widehat{N}, \gamma(t)} - \partial_\nu^{Y(t)} D_{N, \gamma(t)} \| \, \|\Phi_t\| \, \|\xi_1\| \, \|\xi_2\| +$$

$$\| \lim_{t \to 0} \langle D_{N,t}'(\xi_1), \xi_2 \rangle - \langle D_{N,0}'(\xi_1), \xi_2 \rangle \| +$$

$$\| \partial_\nu^{Y(0)} D_{N, \gamma(0)} - \lim_{\widehat{N} \to \infty} \partial_\nu^{Y(0)} D_{\widehat{N}, \gamma(0)} \| \, \|\xi_1\| \, \|\xi_2\|.$$

The first and last terms can be made arbitrarily small (for $t \in [0, h]$) by choosing N sufficiently large. The middle term equals zero since $\langle D_{N,t}'(\xi_1), \xi_2 \rangle$ is continuous in t, which follows immediately from Equation 9.8 and the fact that D_N is transversely smooth. Thus,

$$0 = \langle \lim_{t \to 0} D_t'(\xi_1), \xi_2 \rangle - \langle D_0'(\xi_1), \xi_2 \rangle = \langle (\lim_{t \to 0} D_t' - D_0')(\xi_1), \xi_2 \rangle,$$

which holds for all $\xi_1, \xi_2 \in C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, so $\lim_{t \to 0} D_t' - D_0' = 0$, that is D_t' is continuous at zero. The operator $\lim_{h \to 0} \frac{1}{h} \left(\int_0^h D_t' dt \right)$ is also well defined as a map from $C_c^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$ to $C^\infty(\widetilde{L}_x; \wedge^\ell T^* F_s \otimes E)$, and as D_t' is continuous at zero, we have

$$\lim_{h \to 0} \frac{1}{h} \left(\int_0^h D_t' dt \right) = D_0'.$$

Again by the fundamental relationship between parallel translation and ∇, we have

$$\lim_{h \to 0} \frac{B_h^{-1} - B_0^{-1}}{h} = \partial_\nu^Y B^{-1},$$

so

$$\partial_\nu^Y B^{-1} = \lim_{h \to 0} \frac{B_h^{-1} - B_0^{-1}}{h} = \lim_{h \to 0} \frac{1}{h} \left(\int_0^h D_t' dt \right) = D_0' = \lim_{N \to \infty} \partial_\nu^Y D_N,$$

and $\partial_\nu^Y B^{-1}$ is a bounded leafwise smoothing operator.   q.e.d.

A boot strapping argument now finishes the proof of Theorem 9.2. Let Y_1, Y_2 be vector fields on M. As $B^{-1} B = P_\ell$ and the $\partial_\nu^{Y_i}$ are derivations, we have as usual

$$\partial_\nu^{Y_2} B^{-1} = (\partial_\nu^{Y_2} P_\ell) B^{-1} - B^{-1} (\partial_\nu^{Y_2} B) B^{-1},$$

which is in the domain of $\partial_\nu^{Y_1}$. Applying it, we find that $\partial_\nu^{Y_1}\partial_\nu^{Y_2}B^{-1}$ equals

$$(\partial_\nu^{Y_1}\partial_\nu^{Y_2}P_\ell)B^{-1} + (\partial_\nu^{Y_2}P_\ell)(\partial_\nu^{Y_1}B^{-1})-$$

$$(\partial_\nu^{Y_1}B^{-1})(\partial_\nu^{Y_2}B)B^{-1} - B^{-1}(\partial_\nu^{Y_1}\partial_\nu^{Y_2}B)B^{-1} - B^{-1}(\partial_\nu^{Y_2}B)(\partial_\nu^{Y_1}B^{-1}),$$

which is a bounded leafwise smoothing map, since B and P are transversely smooth and $\partial_\nu^{Y_1}B^{-1}$ is bounded and leafwise smoothing. Proceeding by induction, we have that for all vector fields $Y_1,...,Y_m$ on M, the operator $\partial_\nu^{Y_1}\cdots\partial_\nu^{Y_m}B^{-1}$ is bounded and leafwise smoothing, so B^{-1} is transversely smooth.

This completes the proof Theorem 9.2.   q.e.d.

Finally, we prove Theorem 9.1, that is we prove

Theorem 9.9. $\mathrm{ch}_a(\pi_\pm^f) = f^*(\mathrm{ch}_a(\pi_\pm'))$.

Proof. We will only prove that $\mathrm{ch}_a(\pi_+^f) = f^*(\mathrm{ch}_a(\pi_+'))$, as the other proof is the same. We begin by constructing special covers of M and M'. Let $\{\widehat{U}'\}$ be a finite open cover of M' by foliation charts with transversals $\widehat{T}'$. Choose the $\widehat{U}'$ so small that $g\,|\,\widehat{T}'$ is a diffeomorphism. Denote by $\rho'_{\widehat{U}'} : \widehat{U}' \to \widehat{T}'$ the projection. Let $\{U\}$ be a finite open cover of M by foliation charts with transversals T. Since the collection of open sets $f^{-1}(\widehat{U}')$ cover M, we may choose the U small enough so that for each U, there is a $\widehat{U}'_U$ with $f(U) \subset \widehat{U}'_U$. We may further assume that the U are so small that $f\,|\,T$ is a diffeomorphism. Set

$$U' = (\rho'_{\widehat{U}'_U})^{-1}(\rho'_{\widehat{U}'_U}(f(U))).$$

Then the set $\{U'\}$ is a finite open cover of M' by foliation charts, $f(U) \subset U'$, and $T' = f(T)$ is a transversal of U'. Denote the projection $\rho'_{\widehat{U}'_U}\,|\,U' \to T'$ by ρ'.

Set $V = f^{-1}(U')$, and note that V is not necessarily connected. However, $V \supset U$ whose transversal T is taken diffeomorphically onto T' by f. There is a well defined projection $\rho : V \to T$ given by $\rho = (f\,|\,T)^{-1}\circ\rho'\circ f$. Recall the connection ∇ on π_+^f (induced from the connection ∇' on π_+') which we will use to construct $\mathrm{ch}_a(\pi_+^f)$, and set $\nabla^T = \nabla\,|\,T$ with curvature θ_T. Then just as in Proposition 5.20, we have

Lemma 9.10. $\nabla\,|\,V = \rho^*(\nabla^T)$ *and* $\theta\,|\,V = \rho^*(\theta_T)$.

Proof. The proof is essentially the same. To effect it, we need to be able to define local invariant sections over V, and to do this we need families of leafwise paths such that moving along them gives the projection ρ. Given $y \in V$, choose a leafwise path $\gamma'_y : [1,2] \to U'$ from $\rho'(f(y))$ to $f(y)$. Let $h : M \times I \to M$ be a leafwise homotopy

between the identity map and $g \circ f$. In particular, $h(x, 0) = x$ and $h(x, 1) = gf(x)$. Define the leafwise path γ_y from $\rho(y)$ to y as follows:

$$\gamma_y(t) = h(\rho(y), t) \text{ for } 0 \le t \le 1; \quad \gamma_y(t) = g(\gamma_y'(t)) \text{ for } 1 \le t \le 2;$$

and

$$\gamma_y(t) = h(y, 3 - t) \text{ for } 2 \le t \le 3.$$

Since $f(\rho(y)) = \rho'(f(y))$, this does give a path from $\rho(y)$ to y. Using the γ_y, we may extend any local section defined on T to a local invariant section on all of V, and then proceed just as in the proof of Proposition 5.20. q.e.d.

The connection $\nabla^{T'}$ (which is ∇' restricted to $\pi_+' \mid T'$) and its curvature $\theta_{T'}$ satisfy $\nabla' \mid U' = \rho'^*(\nabla^{T'})$ and $\theta' \mid U' = \rho'^*(\theta_{T'})$. Set $\widehat{f} = f \mid T$ and define $\widehat{f}^*(\nabla^{T'})$ and $\widehat{f}^*(\theta_{T'})$ as follows. Let $\xi \in C^\infty(\pi_+^f \mid T)$ and suppose that X and Y are tangent to T. Set $X' = \widehat{f}_*(X) = f_*(X)$ and $Y' = \widehat{f}_*(Y) = f_*(Y)$, both of which are tangent to T'. Define

$$\widehat{f}^*(\nabla^{T'})_X \xi = \widetilde{f}^*(\nabla_{X'}^{T'}(\widetilde{f}^{-*}\xi \mid T'))$$

and

$$\left(\widehat{f}^*(\theta_{T'})(X, Y)\right)\xi = \widetilde{f}^*(\theta_{T'}(X', Y')(\widetilde{f}^{-*}\xi \mid T')).$$

Lemma 9.11. $\widehat{f}^*(\nabla^{T'}) = \nabla^T$ and $\widehat{f}^*(\theta_{T'}) = \theta_T$.

Proof. The element $\xi \in C^\infty(\pi_+^f \mid T)$ determines the local invariant sections $\widetilde{\xi}$ of π_+^f and $\widetilde{f}^{-*}\xi$ of π_+'. Then

$$\widehat{f}^*(\nabla^{T'})_X \xi = \widetilde{f}^*(\nabla_{X'}^{T'}(\widetilde{f}^{-*}\xi \mid T')) = \widetilde{f}^*(\nabla'_{X'}\widetilde{f}^{-*}\xi) = \nabla_X \widetilde{\xi} = \nabla_X^T \xi.$$

Next, using local spanning sets of $\pi_+^f \mid V$ and $\pi_+' \mid U'$, it is not difficult to show that

$$\theta_T(X, Y) = \nabla_X^T \nabla_Y^T - \nabla_Y^T \nabla_X^T - \nabla_{[X,Y]}^T,$$

and similarly for $\theta_{T'}(X', Y')$. Then

$$\nabla_X^T \nabla_Y^T \xi = \widetilde{f}^* \nabla_{X'}^{T'} \widetilde{f}^{-*} \widetilde{f}^* \nabla_{Y'}^{T'} \widetilde{f}^{-*}\xi = \widetilde{f}^* \nabla_{X'}^{T'} \nabla_{Y'}^{T'} \widetilde{f}^{-*}\xi$$

and $\nabla_Y^T \nabla_X^T \xi = \widetilde{f}^* \nabla_{Y'}^{T'} \nabla_{X'}^{T'} \widetilde{f}^{-*}\xi$. As $\widehat{f}$ is a diffeomorphism, $\widehat{f}_*([X, Y]) = [X', Y']$, so

$$\nabla_{[X,Y]}^T \xi = \widetilde{f}^* \nabla_{[X',Y']}^{T'} \widetilde{f}^{-*}\xi.$$

It follows immediately that

$$\left(\widehat{f}^*(\theta_{T'})(X, Y)\right)\xi = \widetilde{f}^* \theta_{T'}(X', Y') \widetilde{f}^{-*}\xi = \theta_T(X, Y)\xi.$$

q.e.d.

Now consider the curvature operator θ' of ∇' over U'. We may assume that $U' \simeq \mathbb{R}^p \times \mathbb{R}^q$ with coordinates $x'_1, ..., x'_n$, and that $T' = \{0\} \times \mathbb{R}^q$. Choose a local invariant spanning set $\{\xi'_i\}$ of $\pi'_+ \,|\, U'$. Recall that for sections $\alpha'_1 \otimes \phi'_1, \alpha'_2 \otimes \phi'_2$ of $\wedge T^* \widetilde{L}' \otimes E'$,

$$Q'(\alpha'_1 \otimes \phi'_1, \alpha'_2 \otimes \phi'_2) \;=\; \int_{\widetilde{L}'} \{\phi'_1, \phi'_2\} \alpha'_1 \wedge \alpha'_2 \;=\; \int_{\widetilde{L}'} (\alpha'_1 \otimes \phi'_1) \wedge (\alpha'_2 \otimes \phi'_2).$$

There are functions $a'_{i,j,k,l}$ on T' (thanks to Proposition 5.20) so that the action of θ' on a section ξ' of π'_+ is given by

$$\theta'(\xi') \;=\; \sum_{k,l=p+1}^{n} \sum_{i,j} a'_{i,j,k,l} \, Q'(\xi'_j, \xi') \, \xi'_i \, dx'_k \wedge dx'_l \;=\;$$

$$\sum_{i,j,k,l} a'_{i,j,k,l} \left[\int_{\widetilde{L}'} \xi'_j \wedge \xi' \right] \xi'_i \, dx'_k \wedge dx'_l.$$

The reason that we can represent θ' this way is because for any $\xi' \in \mathrm{Ker}(\pi'_+)$ and any $\widehat{\xi}' \in \mathrm{Im}(\pi'_+)$, $Q'(\xi', \widehat{\xi}') = 0$. This follows from the facts that $\langle \xi', \widehat{\xi}' \rangle = 0$, $Q'(\xi', \widehat{\ast}\widehat{\xi}') = \langle \xi', \widehat{\xi}' \rangle$, and $\widehat{\xi}' = \widehat{\tau}\widehat{\xi}' = \sqrt{-1}^{\ell^2} \widehat{\ast} \widehat{\xi}'$.

Let $x' \in U'$ and $y', z' \in \widetilde{L}_{x'}$. With respect to the spanning set $\{\xi'_i\}$ and the local coordinates on U', the Schwartz kernel $\Theta'_{x'}(y', z')$ of $\theta' \,|\, U'$ is given by

$$\Theta'_{x'}(y', z') = \sum_{k,l=p+1}^{n} \sum_{i,j} a'_{i,j,k,l}(\rho'(x')) \xi'_i(y') \otimes \xi'_j(z') dx'_k \wedge dx'_l.$$

We write this more succinctly as

$$\Theta' \,|\, U' = \sum_{i,j,k,l} a'_{i,j,k,l} \, \xi'_i \otimes \xi'_j \, dx'_k \wedge dx'_l.$$

Recall that $\overline{x}' \in \widetilde{L}_{x'}$ is the class of the constant path at x', that we identify M' with its image under $x' \to \overline{x}'$, and that $\int_{U'}$ is integration over the fibration $U' \to T'$. Let $\{\psi'_{U'}\}$ be a partition of unity subordinate to the special cover $\{U'\}$ of M'. Then

$$\mathrm{Tr}(\theta') \,|\, T' = \int_{U'} \psi'_{U'}(x') \sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x')) \xi'_i(\overline{x}') \wedge \xi'_j(\overline{x}') \, dx'_k \wedge dx'_l.$$

Note that we do not multiply the integrand by the leafwise volume form dx', since this is already incorporated in it by our use of the leafwise differential forms ξ'_i in the Schwartz kernel Θ' of θ'. In particular, being very precise,

$$\Theta'_{x'}(y', z') \;=\; \sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x')) \xi'_i(y') \otimes i_{\mathrm{vol}(z')}[\xi'_j(z') \wedge (\cdot)] \, dx'_k \wedge dx'_l,$$

where $\mathrm{vol}(z')$ is the oriented unit length vector in $(\wedge^{2\ell} TF_s)_{z'}$. Then

$$\mathrm{tr}(\Theta'_{x'}(\overline{x}', \overline{x}')) \, dx' =$$

$$\sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x'))(i_{\mathrm{vol}(\overline{x}')}[\xi'_i(\overline{x}') \wedge \xi'_j(\overline{x}')]) dx' \, dx'_k \wedge dx'_l =$$

$$\sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x'))\xi'_i(\overline{x}') \wedge \xi'_j(\overline{x}') \, dx'_k \wedge dx'_l.$$

To avoid notational overload, we will not be this precise.

The $\mathcal{G}'$ invariance of θ' allows us to compute $\mathrm{Tr}(\theta')$ as follows. Denote the plaque of x' in U' by $P_{x'}$. Let $j' : P_{x'} \to \widetilde{L}_{x'}$ be the map given by: $j'(w')$ is the class of any leafwise path in $P_{x'}$ from x' to w'. Then the value of $\mathrm{Tr}(\theta')$ at $\rho'(x') \in T'$ is given by

$$\mathrm{Tr}(\theta')(\rho'(x')) =$$

$$\int_{j'(P_{x'})} \psi'_{U'}(j'^{-1}(y')) \sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x'))\xi'_i(y') \wedge \xi'_j(y') \,|\, j'(P_{x'}) \, dx'_k \wedge dx'_l.$$

Abusing notation once again by identifying $P_{x'}$ with its image under j', we have that at $\rho'(x') \in T'$,

$$\mathrm{Tr}(\theta')(\rho'(x')) = \int_{P_{x'}} \psi'_{U'}(y') \sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x'))\xi'_i(y') \wedge \xi'_j(y') \, dx'_k \wedge dx'_l =$$

$$\sum_{i,j,k,l} a'_{i,j,k,l}(\rho'(x'))\Big[\int_{P_{x'}} \psi'_{U'}(y')\xi'_i(y') \wedge \xi'_j(y')\Big] dx'_k \wedge dx'_l.$$

Similar remarks apply to all powers of θ'.

We now return to our analysis on $V = f^{-1}(U')$, where we have the normal coordinates $x_{p+1}, ..., x_n$ given by $x_i = x'_i \circ f \circ \rho$, so $dx_i = f^*(dx'_i)$. If we set $\xi_i = \widetilde{f}^*(\xi'_i)$, then the ξ_i are a spanning set of $\pi^f_+ \,|\, V$. Set $a_{i,j,k,l} = a'_{i,j,k,l} \circ f \circ \rho$, where $\rho : V \to T$. Using Lemma 9.11 along with Proposition 6.5, the Schwartz kernel $\Theta_x(y,z)$ of $\theta \,|\, V$ is given by

$$\Theta_x(y,z) = \sum_{i,j,k,l} a_{i,j,k,l}(\rho(x))\xi_i(y) \otimes \xi_j(z) \, dx_k \wedge dx_l,$$

and the action $\theta \,|\, V$ is

$$\theta(\xi) = \sum_{k,l=p+1}^{n} \sum_{i,j} a_{i,j,k,l} \, Q(\xi_j, \xi) \, \xi_i \, dx_k \wedge dx_l =$$

$$\sum_{i,j,k,l} a_{i,j,k,l} \Big[\int_{\widetilde{L}} \xi_j \wedge \xi\Big] \xi_i \, dx_k \wedge dx_l.$$

That is

$$\Theta = \widetilde{f}^*\Theta'.$$

We are interested in the Schwartz kernels Θ'^k and Θ^k of the operators θ'^k and θ^k. These are given by

$$\Theta'^k_{x'}(y', z') =$$

$$\int_{\widetilde{L}_{x'}} \int_{\widetilde{L}_{x'}} \cdots \int_{\widetilde{L}_{x'}} \Theta'_{x'}(y', w'_1) \wedge \Theta'_{x'}(w'_1, w'_2) \wedge \ldots \wedge \Theta'_{x'}(w'_{k-1}, z')$$

and

$$\Theta^k_x(y, z) = \int_{\widetilde{L}_x} \int_{\widetilde{L}_x} \cdots \int_{\widetilde{L}_x} \Theta_x(y, w_1) \wedge \Theta_x(w_1, w_2) \wedge \ldots \wedge \Theta_x(w_{k-1}, z),$$

where the integration is done over repeated variables. Using Proposition 6.5 again, we have immediately that

$$\Theta^k = \widetilde{f}^*(\Theta'^k).$$

For each $\psi'_{U'}$ in the partition of unity subordinate to $\{U'\}$, set $\psi_V = \psi'_{U'} \circ f$, which gives a partition of unity subordinate to the open cover $\{V\}$ of M. Denote by $\int_V$ integration over the fibration $\rho : V \to T$. Recall the map $i : M \to \mathcal{G}$ given by $i(x) = \overline{x}$, the class of the constant path at x.

Lemma 9.12. $\operatorname{Tr}(\theta^k) = \sum_V \int_V \psi_V\, i^* \operatorname{tr}(\Theta^k).$

Proof. It suffices to show that for any differential form ω on M, $\int_F \psi_V \omega$ and $\int_V \psi_V \omega$ define the same Haefliger form. Let $W_0, ..., W_m$ be an open cover of M by foliation charts, with transversals $S_0, ..., S_m$. We may assume that $W_0, ..., W_k$ are the only elements which intersect the support of ψ_V non-trivially, and that these sets are subsets of V. Let $\widehat{\psi}_0, ..., \widehat{\psi}_m$ be a partition of unity subordinate to the W_j. We require that $W_0 = U$ and $S_0 = T$. Recall that $\rho' : U' \to T'$ is the projection. For $j = 1, ..., k$, choose a point $y_j \in S_j$. Then $\rho'(f(y_j)) = f(\rho(y_j))$, and, as in the proof of Lemma 9.10, we define the leafwise path γ_j from $\rho(y_j)$ to y_j. By construction, the holonomy map h_j induced by the leafwise path γ_j (which has domain possibly a proper subset of S_0) has range **all** of S_j. In addition, for each S_j, the map $h_j^{-1} : S_j \to S_0 = T$ is just the restriction to S_j of the projection $\rho : V \to T$. Then the Haefliger classes

$$\int_F \psi_V \omega \;\equiv\; \sum_{j=0}^{k} \int_{W_j} \widehat{\psi}_j \psi_V \omega \;=\; \int_{W_0} \widehat{\psi}_0 \psi_V \omega + \sum_{j=1}^{k} h_j^* \Big(\int_{W_j} \widehat{\psi}_j \psi_V \omega \Big) \;=\;$$

$$\int_{W_0} \widehat{\psi}_0 \psi_V \omega \;+\; \sum_{j=1}^{k} h_j^* \Big(\int_{W_j} \widehat{\psi}_j \psi_V \omega \Big).$$

The Haefliger form $\displaystyle\int_{W_0} \widehat{\psi}_0 \psi_V \omega + \sum_{j=1}^{k} h_j^* \Big(\int_{W_j} \widehat{\psi}_j \psi_V \omega \Big)$ is supported on $S_0 = T$, and it follows immediately from the fact that $h_j^{-1} : S_j \to S_0$ is just $\rho : S_j \to T$, that it equals $\displaystyle\int_V \psi_V \omega$.   q.e.d.

Now $\mathrm{ch}_a(\pi_+^f) = \Big[\mathrm{Tr}\Big(\pi_+^f + \sum_{k=1}^{[n/2]} \frac{(-1)^k \theta^k}{(2i\pi)^k k!} \Big) \Big]$, and by Theorem 9.2 this equals $\mathrm{ch}_a(\pi_+)$, which is independent of the Bott form ω used to construct $\widetilde{f}^*$. Let ϕ be a smooth even function on $\mathbb{R}$, decreasing on $[0,1]$ with $\phi(0) = 1$ and $\phi(x) = 0$ for $|x| \geq 1$, and let ω be the Bott form which is a multiple of $\phi(x_1)...\phi(x_k)dx_1...dx_k$. For $t > 0$, let $q_t : \mathbb{R}^k \to \mathbb{R}^k$ be the diffeomorphism $q_t(x) = x/t$. Denote by ω_t the smooth family of Bott forms given by $\omega_t = q_t^* \omega$, and denote by $\widetilde{f}_t^*$ the map constructed using ω_t. Then for all $t > 0$ and $k \geq 1$, we have

$$\Big[\mathrm{Tr}(\theta^k) \Big] \;=\; \Big[\sum_V \int_V \psi_V \; i^* \, \mathrm{tr}(\Theta^k) \Big] \;=\;$$

$$\Big[\sum_V \int_V f \circ \psi'_{U'} \; i^* \, \mathrm{tr}(\widetilde{f}_t^*(\Theta'^k)) \Big] \;=\; \Big[\sum_V \lim_{t \to 0} \int_V f^*(\psi'_{U'}) \; i^* \widetilde{f}_t^*(\mathrm{tr}\,\Theta'^k) \Big].$$

We may use the ω_t to construct the family of maps f_t^* (analogous to the family $\widetilde{f}_t^*$), defined on the original foliation F. As both $\widetilde{f}_t^*$ and f_t^* are locally constructed and $\mathrm{tr}\,\Theta'^k$ is $\mathcal{G}'$ invariant, it is clear that

$$i^* \widetilde{f}_t^*(\mathrm{tr}\,\Theta'^k) \;=\; f_t^*(i'^* \, \mathrm{tr}\,\Theta'^k).$$

Thus

$$\Big[\sum_V \lim_{t \to 0} \int_V f^*(\psi'_{U'}) \; i^* \widetilde{f}_t^*(\mathrm{tr}\,\Theta'^k) \Big] = \Big[\sum_V \lim_{t \to 0} \int_V f^*(\psi'_{U'}) f_t^*(i'^* \, \mathrm{tr}\,\Theta'^k) \Big].$$

It is a classical result that on each plaque in V, the compactly supported forms $f^*(\psi'_{U'}) f_t^*(i'^* \, \mathrm{tr}\,\Theta'^k)$ are bounded independently of $t \in [0,1]$, and converge pointwise to $f^*(\psi'_{U'}) f^*(i'^* \, \mathrm{tr}\,\Theta'^k) = f^*(\psi'_{U'} i'^* \, \mathrm{tr}\,\Theta'^k)$. By the Dominated Convergence Theorem, we have

$$\Big[\mathrm{Tr}(\theta^k) \Big] \;=\; \Big[\sum_V \int_V \lim_{t \to 0} f^*(\psi'_{U'}) f_t^*(i'^* \, \mathrm{tr}\,\Theta'^k) \Big] =$$

$$\Big[\sum_{U'} \int_{f^{-1}(U')} f^*(\psi'_{U'} i'^* \, \mathrm{tr}(\Theta'^k)) \Big] =$$

$$\Big[f^* \sum_{U'} \int_{U'} \psi'_{U'} i'^* \, \mathrm{tr}(\Theta'^k) \Big] = f^* \Big[\mathrm{Tr}(\theta'^k) \Big].$$

As $\mathrm{ch}_a(\pi'_+) = \left[\mathrm{Tr}\left(\pi'_+ + \sum_{k=1}^{[n/2]} \frac{(-1)^k \theta'^k}{(2i\pi)^k k!} \right) \right]$, to finish the proof that

$\mathrm{ch}_a(\pi^f_+) = f^*(\mathrm{ch}_a(\pi'_+))$, we must show that $\left[\mathrm{Tr}(\pi^f_+) \right] = f^* \left[\mathrm{Tr}(\pi'_+) \right]$. Just as we did with θ', we may write the Schwartz kernel of $\pi'_+ \,|\, U'$ as

$$(\pi'_+)_{x'}(y', z') = \sum_{i,j} b'_{i,j}(\rho'(x'))\xi'_i(y') \otimes \xi'_j(z'),$$

where the $b'_{i,j}$ are functions on T', and the action of π'_+ on a section ξ' is given by

$$\pi'_+(\xi') = \sum_{i,j} b'_{i,j} \, Q'(\xi'_j, \xi') \, \xi'_i.$$

Set $b_{i,j} = \widetilde{f}^* b'_{i,j} = b'_{i,j} \circ f \circ \rho$ and $\xi_i = \widetilde{f}^*(\xi'_i)$, and consider the operator $\widetilde{\pi}^f_+$ on $\mathcal{A}^\ell_{(2)}(F_s, E)$, where $\widetilde{\pi}^f_+ \,|\, V = \sum_{i,j} b_{i,j}\xi_i \otimes \xi_j$, which acts by

$$\widetilde{\pi}^f_+(\xi) = \sum_{i,j} b_{i,j} \, Q(\xi_j, \xi)\xi_i.$$

Then $\widetilde{\pi}^f_+$ is a $\mathcal{G}$ invariant idempotent, has image equal to $\mathrm{Im}(\pi^f_+)$, and has a smooth Schwartz kernel. In general $\widetilde{\pi}^f_+ \neq \pi^f_+$ because forms of the type $\delta_s\beta$, which are in the kernel of π^f_+, are not necessarily in the kernel of $\widetilde{\pi}^f_+$. However, since $\widetilde{\pi}^f_+$ has smooth Schwartz kernel, $\mathrm{Tr}(\widetilde{\pi}^f_+)$ is well defined, and its Schwartz kernel is just $\widetilde{f}^*$ of the Schwartz kernel of π'_+. Arguing as we did for θ^k, we get $\left[\mathrm{Tr}(\widetilde{\pi}^f_+) \right] = f^* \left[\mathrm{Tr}(\pi'_+) \right]$.

Lemma 9.13. $\qquad \left[\mathrm{Tr}(\pi^f_+) \right] = \left[\mathrm{Tr}(\widetilde{\pi}^f_+) \right].$

Proof. Since $\mathrm{Im}(\pi^f_+) = \mathrm{Im}(\widetilde{\pi}^f_+)$ and both are idempotents, we need only show that $\widetilde{\pi}^f_+$ is transversely smooth, and then apply Lemma 3.5.

We will use the notation of Section 6. Let K' be Schwartz kernel of a $\mathcal{G}'$ invariant bounded leafwise smoothing operator on $\mathcal{A}^\ell_{(2)}(F'_s, E')$, which is given locally, with respect to a local invariant spanning set $\{\xi'_i\}$ of $\mathcal{A}^\ell_{(2)}(F'_s, E')$, by $K' = \sum_{i,j} b'_{i,j}\xi'_i \otimes \xi'_j$, with the action given by

$$K'(\xi') = \sum_{i,j} b'_{i,j} \, Q'(\xi'_j, \xi') \, \xi'_i.$$

Now consider the operators $\widetilde{f}^*K'$ on $\mathcal{A}^\ell_{(2)}(F_s, E)$ and $\widetilde{p}^*_f K'$ on $\mathcal{A}^\ell_{(2)}(F_s \times B^k, p^*_f E')$, with local Schwartz kernels

$$\widetilde{f}^*K' = \sum_{i,j} \widetilde{f}^* b'_{i,j}\widetilde{f}^*\xi'_i \otimes \widetilde{f}^*\xi'_j, \text{ and } \widetilde{p}^*_f K' = \sum_{i,j} p^*_f b'_{i,j}(p^*_f\xi'_i \wedge \omega)\otimes(p^*_f\xi'_j \wedge \omega),$$

where ω is a Bott form on B^k. Recall that $\pi_{1,*}$ is integration over the fiber of the projection $\pi_1 : \mathcal{G} \times B^k \to \mathcal{G}$, and $p_{f,*}$ is integration over the fiber of the submersion $p_f : \mathcal{G} \times B^k \to \mathcal{G}'$. Straight forward computations show that for $\xi \in \mathcal{A}_{(2)}^\ell(F_s, E)$ and $\widetilde{\xi} \in \mathcal{A}_{(2)}^\ell(F_s \times B^k, p_f^* E')$,

$$\widetilde{f}^* K'(\xi) \; = \; \pi_{1,*}\left(\widetilde{p}_f^* K'(\pi_1^* \xi)\right) \text{ and } \widetilde{p}_f^* K'(\widetilde{\xi}) \; = \; p_f^*\left(K'(p_{f,*}(\omega \wedge \widetilde{\xi}))\right) \wedge \omega.$$

The maps $\pi_{1,*}$, π_1^*, p_f^*, $p_{f,*}$, and $\wedge \omega$ are all bounded maps, and K' is bounded and leafwise smoothing. Thus $\widetilde{f}^* K'$ is a bounded leafwise smoothing operator. Applying this to $K' = \pi_+'$, we have that $\widetilde{\pi}_+^f$ is a bounded leafwise smoothing operator.

Using Proposition 7.4, it is easy to show that $\partial_\nu^Y \widetilde{\pi}_+^f = [A(Y), \widetilde{\pi}_+^f] + \widetilde{f}^*(i_{Z'} \partial_{\nu'} \pi_+')$, where Y and Z' are as in Lemma 7.8, and $A(Y)$ is a leafwise operator whose composition with a bounded leafwise smoothing operator is again a bounded leafwise smoothing operator. Applying the argument above to $i_{Z'} \partial_{\nu'} \pi_+'$, we have that $\partial_\nu^Y \widetilde{\pi}_+^f$ is also a bounded leafwise smoothing operator. An obvious induction argument finishes the proof. q.e.d.

Thus $\left[\mathrm{Tr}(\pi_+^f) \right] = \left[\mathrm{Tr}(\widetilde{\pi}_+^f) \right] = f^*\left[\mathrm{Tr}(\pi_+') \right]$, and we are done. q.e.d.

10. The twisted leafwise signature operator and the twisted higher Betti classes

In this section we give some immediate consequences of our results. In particular, we show that the twisted higher harmonic signature equals the (graded) Chern-Connes character in Haefliger cohomology of the "index bundle" of the twisted leafwise signature operator, that is the (graded) Chern-Connes character $\mathrm{ch}_a(P)$ of the projection P onto all the twisted leafwise harmonic forms. We conjecture a cohomological formula for this Chern-Connes character, which has already been proven in some cases. We also indicate how our methods prove that the twisted higher Betti numbers are leafwise homotopy invariants.

Consider the first order leafwise operator $D^E = d_s + \delta_s$, which is formally self adjoint and satisfies $(D^E)^2 = \Delta^E$. Because of this, the kernel of D^E is the same as the kernel of Δ^E. Recall the ± 1 eigenspaces $\mathcal{A}_\pm^*(F_s, E)$ of the involution $\widehat{\tau}$ of $\mathcal{A}_{(2)}^*(F_s, E)$, and that

$$D^E \widehat{\tau} = -\widehat{\tau} D^E,$$

so we have the operators $D^{E\pm} : \mathcal{A}_\pm^*(F_s, E) \to \mathcal{A}_\mp^*(F_s, E)$, and D^{E+} is designated the twisted leafwise signature operator.

Denote by $P_\pm$ the projections onto the $\mathrm{Ker}(D^{E\pm})$. We assume that the projection P to $\mathrm{Ker}(\Delta^E)$ is transversely smooth, so the $P_\pm$ are also. Then the (graded) Chern-Connes character of the index bundle of the

twisted leafwise signature operator, $\mathrm{ch}_a(P)$, is defined and is given by

$$\mathrm{ch}_a(P) = \mathrm{ch}_a(P_+) - \mathrm{ch}_a(P_-) =$$

$$\left[\mathrm{ch}_a \Big(\sum_{j=0}^{\ell-1} P_j + \tau P_j \Big) + \mathrm{ch}_a \Big(\frac{1}{2}(P_\ell + \tau P_\ell) \Big) \right] -$$

$$\left[\mathrm{ch}_a \Big(\sum_{j=0}^{\ell-1} P_j - \tau P_j \Big) + \mathrm{ch}_a \Big(\frac{1}{2}(P_\ell - \tau P_\ell) \Big) \right].$$

As in the case of compact manifolds, we have

Theorem 10.1. *Suppose that M is a compact Riemannian manifold, with an oriented Riemannian foliation F of dimension 2ℓ, and that E is a leafwise flat complex bundle over M with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the projection P onto $\mathrm{Ker}(\Delta^E)$ for the associated foliation F_s of the homotopy groupoid of F is transversely smooth. Then, the (graded) Chern-Connes character $\mathrm{ch}_a(P)$ of the index bundle of the twisted leafwise signature operator equals the twisted higher harmonic signature of F, that is*

$$\mathrm{ch}_a(P) = \sigma(F, E).$$

Proof. As ch_a is linear and $\frac{1}{2}(P_\ell \pm \tau P_\ell) = \pi_\pm$, we need only show that

$$\mathrm{ch}_a(P_j + \tau P_j) = \mathrm{ch}_a(P_j - \tau P_j),$$

for $j = 0, ..., \ell - 1$. Set $P_t = P_j + t\tau P_j$ where $-1 \leq t \leq 1$. Then P_t is a smooth family of $\mathcal{G}$ invariant transversely smooth idempotents (since $P_j \tau P_j = 0$ for $j = 0, ..., \ell - 1$) which connects $P_j + \tau P_j$ to $P_j - \tau P_j$. It follows from Theorem 3.4 that $\mathrm{ch}_a(P_j + \tau P_j) = \mathrm{ch}_a(P_j - \tau P_j)$.   q.e.d.

Corollary 10.2. *Under the hypothesis of Theorem 10.1, the (graded) Chern-Connes character $\mathrm{ch}_a(P)$ of the index bundle of the leafwise signature operator with coefficients in E is a leafwise homotopy invariant.*

The operator D^{E+} is elliptic along the leaves of F_s, and so produces, via a now classical construction due to Connes [**C81**], a $K-$theory invariant $\mathrm{Ind}_a(D^{E+})$, the index of the operator D^{E+}, which has a Chern-Connes character $\mathrm{ch}_a(\mathrm{Ind}_a(D^{E+})) \in \mathrm{H}_c^*(M/F)$, [**BH04**].

Conjecture 10.3. *Under the hypothesis of Theorem 10.1,*

$$\mathrm{ch}_a(\mathrm{Ind}_a(D^{E+})) = \mathrm{ch}_a(P) \quad \in \quad \mathrm{H}_c^*(M/F).$$

This conjecture has been proven when the spectrum of D^{E+} is reasonably well behaved, see [**H95**, **HL99**, **BH08**], where it is proven for the holonomy groupoid. The proofs extend immediately to the homotopy groupoid. It also holds for both groupoids, without any extra assumptions, whenever the projection P belongs to Connes' C^*-algebra of the

foliation for the groupoid in question. In particular, it holds for the holonomy groupoid case for any foliation whose leaves are the fibers of a fibration between closed manifolds, provided that P is transversely smooth.

Recently, Azzali, Goette and Schick have announced, [**AGS**], that they have proven it for smooth proper submersions $V \to B$ with the fiberwise action (freely and properly discontinuous) of a discrete group Γ such that the quotient $V/\Gamma \to B$ is a fibration with compact fiber, but only for bundles E which are globally flat. Conjecture 10.3 should follow immediately for the homotopy groupoid provided that their result extends to bundles which are only leafwise flat.

Recall, [**BH04**, **GL03**], that in Haefliger cohomology,

$$\mathrm{ch}_a(\mathrm{Ind}_a(D^{E+})) \;=\; \int_F \mathbb{L}(TF)\,\mathrm{ch}_2(E),$$

where $\mathbb{L}(TF)$ is the characteristic class of TF associated with the multiplicative sequence $\prod_j x_j/\tanh(x_j)$, and $\mathrm{ch}_2(E) = \sum_k 2^k \,\mathrm{ch}_k(E)$.

Corollary 10.4. *Under the hypothesis of Theorem* 10.1, *and assuming Conjecture* 10.3, $\int_F \mathbb{L}(TF)\,\mathrm{ch}_2(E)$ *is a leafwise homotopy invariant.*

Finally we have the following.

Definition 10.5. Assume the hypothesis of Theorem 10.1, but now F may have arbitrary dimension. For $0 \le j \le p = \dim(F)$, define the j-th twisted higher Betti class $\beta_j(F, E)$ by

$$\beta_j(F, E) = \mathrm{ch}_a(P_j) \quad \in \quad \mathrm{H}_c^*(M/F).$$

It is an interesting exercise to show that, just as in the case of compact fibrations, the bundle defined by the projection onto the leafwise harmonics (in the case $E = M \times \mathbb{C}$) is a flat bundle. That is, it admits a connection whose curvature is zero, so there are no higher terms in the $\beta_j(F, M \times \mathbb{C})$. This is not the case in general.

Theorem 10.6. *(Compare [**HL91**]) Under the hypothesis of Theorem* 10.1 *with F allowed to have arbitrary dimension, the twisted higher Betti classes $\beta_j(F, E)$, are leafwise homotopy invariants.*

Proof. We only give a sketch here of the proof of the second statement. Let $f : (M, F) \to (M', F')$ be a smooth leafwise homotopy equivalence with smooth homotopy inverse g. The pull-back bundle $f^*(P_j')$ is a smooth bundle since it can be realized by the transversely smooth idempotent $P_j^f = f^* R'^* P_j' g^* R^* P_j$. It can be endowed with the pull-back connection under f of the connection $P_j' \nabla'^\nu P_j'$, and hence the Chern-Connes character of $f^*(P_j')$ is given by

$$\mathrm{ch}_a(f^*(P_j')) = f^* \,\mathrm{ch}_a(P_j') = f^* \beta_j(F', E').$$

As in the proof of our main theorem, show that $P_j : f^*(\mathrm{Ker}(\Delta_j^{E'})) \to$ $\mathrm{Ker}(\Delta_j^E)$ is an isomorphism and that $Q_j^f = P_j P_j^f$ is a smooth idempotent with image $\mathrm{Ker}(\Delta_j^E)$, hence its Chern-Connes character coincides with the Betti class $\beta_j(F, E)$. As $Q_j^f P_j^f = Q_j^f$ and $P_j^f Q_j^f = P_j^f$, the family $Q_t = tQ_j^f + (1-t)P_j^f$ is a smooth homotopy by transversely smooth idempotents from Q_j^f to P_j^f. Therefore P_j^f and Q_j^f have same Chern-Connes character.   q.e.d.

11. Consequences of the Main Theorem

In this section, we derive some important consequences of Theorem 9.1. In particular, we re-derive some classic results for the Novikov conjecture, and then give some general results for the Novikov conjecture for groups and for foliations.

Example 11.1 (Lusztig, [**Lu72**]).

Let N be a compact connected even dimensional Riemannian manifold. Set $W = \mathrm{H}^1(N; \mathbb{R}/\mathbb{Z})$, and recall the natural (onto) map $h_1 : W \to \mathrm{Hom}(\mathrm{H}_1(N; \mathbb{Z}); \mathbb{R}/\mathbb{Z})$. Choose a base point $x_o \in N$. Then there is the natural (onto) homomorphism $h : W \to \mathrm{Hom}(\pi_1(N, x_o); \mathbb{R}/\mathbb{Z})$ given by composing h_1 with the natural map $\pi_1(N, x_o) \to H_1(N, \mathbb{Z})$. Thus for each element $w \in W$, we have the homomorphism $h(w) : \pi_1(N, x_o) \to \mathbb{R}/\mathbb{Z}$, which we may compose with the map $x \to \exp(2\pi i x)$ to obtain the homomorphism $h_w : \pi_1(N, x_o) \to S^1 \subset \mathbb{C}$. Denote by $\widetilde{N}$ the universal covering of N. $\pi_1(N, x_o)$ acts on $\widetilde{N}$ in the usual way, and on $\widetilde{N} \times W \times \mathbb{C}$ as follows. Let $\beta \in \pi_1(N, x_o)$, and $(x, w, z) \in \widetilde{N} \times W \times \mathbb{C}$, and define

$$\beta \cdot (x, w, z) = (\beta x, w, h_w(\beta)z).$$

Set

$$E = (\widetilde{N} \times W \times \mathbb{C})/\pi_1(N, x_o),$$

a complex bundle over $(\widetilde{N} \times W)/\pi_1(N, x_o) = N \times W$, which is leafwise flat for the foliation F given by the fibration $M \equiv N \times W \to W$. It is obvious that the usual metric on $\mathbb{C}$ defines a positive definite metric on E which is preserved by the leafwise flat structure. As $\mathrm{H}_1(N; \mathbb{R}/\mathbb{Z})$ is the abelianization of $\pi_1(N, x_o)$, h is onto, and it is natural to call E the universal flat $\mathbb{C}$ bundle for N. Then M, F, and E satisfy the hypothesis of Theorem 9.1, since the preserved metric is positive definite.

Note that if $f : N \to N'$ is a homotopy equivalence, then there is a natural extension of f to $f : (M, F) \to (M', F')$ which is a leafwise homotopy equivalence, and $f^*E' = E$. Thus $\sigma(F, E)$ is a homotopy invariant of the manifold N.

By [**BH04**] (and assuming Conjecture 10.3 if necessary), we have that

$$\sigma(F, E) \;=\; \int_N \mathbb{L}(TF)\,\mathrm{ch}_2(E) \quad \in \quad \mathrm{H}_c^*(M/F) = \mathrm{H}^*(\mathrm{H}^1(N; \mathbb{R}/\mathbb{Z}); \mathbb{R}).$$

To relate this to Lusztig's theorem on Novikov conjecture, suppose that $\pi_1(N, x_0) = \mathbb{Z}^n$. Denote by $g : N \to B\mathbb{Z}^n = \mathbb{T}^n$ the map classifying the universal cover $\widetilde{N} \to N$ (as a $\mathbb{Z}^n$ bundle), and let $\alpha_1, ..., \alpha_n$ be the natural basis of $\mathrm{H}^1(\mathbb{T}^n; \mathbb{R})$.

Proposition 11.2. $\mathrm{ch}_2(E) \;=\; \displaystyle\prod_{i=1}^{n}(1 + 2g^*(\alpha_i) \otimes \alpha_i).$

Theorem 11.3 (Lusztig, [**Lu72**]). *The Novikov conjecture is true for any compact manifold with fundamental group $\mathbb{Z}^n$.*

Proof.

$$\sigma(F, E) \;=\; \int_N \mathbb{L}(TF)\,\mathrm{ch}_2(E) =$$

$$\sum_{i_1 < \cdots < i_k} 2^k \Big[\int_N \mathbb{L}(TN)g^*(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k})\Big]\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}$$

is a homotopy invariant, so each of the $\displaystyle\int_N \mathbb{L}(TN)g^*(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k})$ is a homotopy invariant.   q.e.d.

Proof. (of Proposition 11.2) As $\pi_1(N, x_0) = \mathbb{Z}^n$, $W = \mathrm{H}^1(N; \mathbb{R}/\mathbb{Z}) \simeq \mathbb{T}^n$. The bundle $E \to N \times \mathbb{T}^n$ is the pull back by $g \times \mathrm{id} : N \times \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{T}^n$ of the bundle $\widehat{E}_n \to \mathbb{T}^n \times \mathbb{T}^n$ which is given as follows. Let $\xi \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$, and $(x, w, z) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}$, and define

$$\xi \cdot (x, w, z) = (x + \xi, w, (\xi w) \cdot z),$$

where

$$(\xi w) \cdot z = (\exp(2\pi i \xi_1 w_1)z_1, \ldots, \exp(2\pi i \xi_n w_n)z_n).$$

Then

$$\widehat{E}_n = (\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C})/\mathbb{Z}^n.$$

Note that $\widehat{E}_n = E_1 \otimes \cdots \otimes E_n$, where E_j is the pull back by the projection $\mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T} \times \mathbb{T}$ onto the j-th coordinates of the bundle $\widehat{E}_1$. As $\mathrm{ch}_2(\widehat{E}_n) \;=\; \displaystyle\prod_{j=1}^{n} \mathrm{ch}_2(E_j)$, we need only show that

$$\mathrm{ch}_2(\widehat{E}_1) \;=\; \prod_{i=1}^{n}(1 + 2\alpha \otimes \alpha),$$

where α is the natural generator of $\mathrm{H}^1(\mathbb{T}; \mathbb{R})$. That is, $c_1(\widehat{E}_1)$ is the natural generator of $\mathrm{H}^2(\mathbb{T}^2; \mathbb{R})$. This is a classical direct computation in the theory of characteristic classes.   q.e.d.

We can extend the previous example to the fundamental group Γ of the closed oriented surface S_g of genus $g \geq 2$. This is a well known theorem which follows from the results of many people, the first probably being Lusztig.

Theorem 11.4. *The Novikov conjecture is true for any compact manifold with fundamental group Γ.*

Proof. The space of equivalence classes of representations of Γ in $U(1)$ is easily seen to be a torus $\mathbb{T}^{2g}$ of dimension $2g$. Form the fiberwise flat line bundle E over the total space of the trivial fibration $\pi_2^{S_g} : S_g \times \mathbb{T}^{2g} \to \mathbb{T}^{2g}$ given by

$$(x, \theta; u) \sim (x\gamma, \theta; h_\theta(\gamma)(u)), \quad x \in \mathbb{H}^2, \ \theta \in \mathbb{T}^{2g}, \ u \in \mathbb{C}, \ \gamma \in \Gamma,$$

where $h_\theta : \Gamma \to U(1)$ is the corresponding homomorphism as in 11.1. Denote by $\pi_1^{S_g} : S_g \times \mathbb{T}^{2g} \to S_g$ the other projection. Then for any cohomology class $y \in H^*(\mathbb{T}^{2g}; \mathbb{R})$, the cohomology class in $H^*(S_g; \mathbb{R}) = H^*(B\Gamma; \mathbb{R})$ given by

$$x = \pi_{1,*}^{S_g} \left[(\pi_2^{S_g})^* y \wedge \mathrm{ch}(E) \right]$$

satisfies the Novikov conjecture. This can be seen as follows. Let N be a smooth closed manifold with fundamental group Γ and denote by $\varphi : N \to S_g = B\Gamma$ a smooth classifying map. Notice that the harmonic signature of the foliated manifold $M = N \times \mathbb{T}^{2g}$ (with foliation given by the fibers of the projection $\pi_2^N : N \times \mathbb{T}^{2g} \to \mathbb{T}^{2g}$) twisted by the fiberwise flat bundle $(\varphi \times \mathrm{id})^* E$, is given in $H^*(\mathbb{T}^{2g}; \mathbb{R})$ by the formula

$$\sigma(M, F; (\varphi \times \mathrm{id})^* E) = \pi_{2,*}^N \left[(\pi_1^N)^* \mathbb{L}(TN) \cup (\varphi \times \mathrm{id})^* \mathrm{ch}(E) \right].$$

Clearly, for any cohomology class $y \in H^*(\mathbb{T}^{2g}; \mathbb{R})$, we get the homotopy invariance of

$$\int_{\mathbb{T}^{2g}} y \cup \pi_{2,*}^N \left[(\pi_1^N)^* \mathbb{L}(TN) \cup (\varphi \times \mathrm{id})^* \mathrm{ch}(E) \right] =$$

$$\int_N \mathbb{L}(TN) \pi_{1,*}^N \left[(\pi_2^N)^* y \wedge (\varphi \times \mathrm{id})^* \mathrm{ch}(E) \right].$$

But $(\pi_2^N)^* y = (\varphi \times \mathrm{id})^* (\pi_2^{S_g})^* y$ and therefore

$$\pi_{1,*}^N \left[(\pi_2^N)^* y \wedge (\varphi \times \mathrm{id})^* \mathrm{ch}(E) \right] = (\pi_{1,*}^N \circ (\varphi \times \mathrm{id})^*) \left[(\pi_2^{S_g})^* y \wedge \mathrm{ch}(E) \right].$$

The conclusion follows using that $\pi_{1,*}^N \circ (\varphi \times \mathrm{id})^* = \varphi^* \circ \pi_{1,*}^{S_g}$.

Thus we need only show that every class $x \in H^*(S_g; \mathbb{R})$ has the given form. We may write $S_g = \natural_g \mathbb{T}^2$ as the union

$$S_g = H_1 \cup H_2 \cup \cdots \cup H_g,$$

where H_1 and H_g are $\mathbb{T}^2$ with a disc removed, and the other H_j are $\mathbb{T}^2$ with two discs removed. There are natural inclusions $g_j : H_j \to \mathbb{T}^2_j \subset \mathbb{T}^{2g}$. On $\mathbb{T}^{2g} \times \mathbb{T}^{2g}$ we have the bundle $\widehat{E}_{2g}$. Consider the natural map

$$h_j = g_j \times I : H_j \times \mathbb{T}^{2g} \to \mathbb{T}^2_j \times \mathbb{T}^{2g} \subset \mathbb{T}^{2g} \times \mathbb{T}^{2g}.$$

Then $E \,|\, H_j \times \mathbb{T}^{2g} = h_j^*(\widehat{E}_{2g})$. Note also that on a neighborhood of the boundary of H_j, the bundle E is trivial, and the trivialization is independent of j. Thus we may construct a connection on E by using a partition of unity and the local connections given on the $H_j \times \mathbb{T}^{2g}$ by the pull back under h_j of the connection used on $\widehat{E}_{2g}$ and the local flat connections on the neighborhoods of the boundaries of the H_j. Thus on the complement of a collar neighborhood of the boundary of $H_j \times \mathbb{T}^{2g}$, $\mathrm{ch}(E) = h_j^*(\mathrm{ch}(\widehat{E}_{2g}))$, and on a neighborhood of the boundary, $\mathrm{ch}(E) = 0$. Now on $\mathbb{T}^{2g} \times \mathbb{T}^{2g}$ we have the one dimensional cohomology classes $[dx_j^1]$, $[dx_j^2]$, $[dw_j^1]$ and $[dw_j^2]$ which are dual to the natural generators of $\mathrm{H}_1(\mathbb{T}^2_j; \mathbb{R})$. The $[dx_j^k]$ live on the first factor of $\mathbb{T}^{2g} \times \mathbb{T}^{2g}$, and the $[dw_j^k]$ on the second. In addition,

$$\mathrm{ch}(\widehat{E}_{2g}) = \prod_{i=1}^{2g}(1 + [dx_i^1] \wedge [dw_i^1])(1 + [dx_i^2] \wedge [dw_i^2]).$$

Set $y_j = \prod_{i \neq j}[dw_i^1] \wedge [dw_i^2]$. Denote by γ_j^1 and γ_j^2 the elements of $\mathrm{H}_1(S_g; \mathbb{R})$ corresponding to the natural generators of $\mathrm{H}_1(\mathbb{T}^2_j; \mathbb{R})$. Then

$$\left(\pi_{1,*}\left[\pi_2^* y_j \wedge [dw_j^k] \wedge \mathrm{ch}(E)\right] \,|\, H_j \times \mathbb{T}^{2g}\right)(\gamma_j^m) = h_j^*([dx_j^k])(\gamma_j^m) = \delta_m^k,$$

while for $i \neq j$,

$$\left(\pi_{1,*}\left[\pi_2^* y_j \wedge [dw_j^k] \wedge \mathrm{ch}(E)\right] \,|\, H_i \times \mathbb{T}^{2g}\right)(\gamma_i^m) = h_i^*([dx_j^k])(\gamma_i^m) = 0,$$

as $h_i^*([dx_j^k]) = 0$.

Thus each element of $\mathrm{H}^1(S_g; \mathbb{R})$ has the required form. It is not difficult to see that $\pi_{1,*}[\pi_2^* y_j \wedge \mathrm{ch}(E)]$ gives a non-zero two dimensional class of the required form, so we have the theorem.

<div align="right">q.e.d.</div>

Here is another version of Lusztig's construction, see [**Lu72**] and [**G96**]. Let E be a flat $U(p,q)$ bundle over N (that is a flat bundle given by a map $\rho : \pi_1(N) \to U(p,q)$). Then E is a leafwise flat complex bundle over N with an indefinite non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Write $E = E^+ \oplus E^-$, where the indefinite metric is positive $\pm$ on $E^\pm$.

Theorem 11.5.

$$\int_N \mathbb{L}(TN)(\mathrm{ch}_2(E^+) - \mathrm{ch}_2(E^-))$$

is a homotopy invariant of N.

Proof. If N is odd dimensional, this is zero, so assume that N is even dimensional. Let F be the foliation of N with one leaf, namely N. The holonomy groupoid of F is just $\mathcal{G} = N \times N$, and the projection onto the leafwise harmonic forms is the same on each N. Thus the hypothesis of Theorem 9.1 are satisfied and Conjecture 10.3 holds, giving the result.

<div align="right">q.e.d.</div>

This may be recast as follows. Let $\rho : \Gamma \to U(p,q)$ be a homomorphism of a finitely presented group. Given any manifold N and homomorphism $\psi : \pi_1(N) \to \Gamma$, we may construct the bundle $E = E^+ \oplus E^- \to N$. This construction is natural under pull-back maps, i.e. given any map $f : N' \to N$ we can form the bundle $E' = E'^+ \oplus E'^- \to N'$ using the homomorphism $\rho \circ \psi \circ f_*$ where $f_* : \pi_1(N') \to \pi_1(N)$ is the induced map. Then $E'^{\pm} = f^*(E^{\pm})$, and so this construction determines two universal bundles E_ρ^+ and E_ρ^- over $B\Gamma$.

Theorem 11.6. *Let $\rho : \Gamma \to U(p,q)$ be a homomorphism of a finitely presented group. Then*

$$\mathrm{ch}(E_\rho^+) - \mathrm{ch}(E_\rho^-) \quad \in \quad \mathrm{H}^*(B\Gamma; \mathbb{R})$$

satisfies the Novikov conjecture.

Note that the universal $\mathbb{C}^{p+q}$ bundle $EU(p,q) \times_{U(p,q)} \mathbb{C}^{p+q} \to BU(p,q)$ splits as $EU(p,q) \times_{U(p,q)} \mathbb{C}^{p+q} = E_{p,q}^+ \oplus E_{p,q}^-$, and for any map $f : N \to BU(p,q)$ classifying a bundle E with splitting $E = E^+ \oplus E^-$, $f^*(E_{p,q}^{\pm}) = E^{\pm}$. The map $\rho : \Gamma \to U(p,q)$ induces $B\rho : B\Gamma \to BU(p,q)$, and $\mathrm{ch}(E_\rho^{\pm}) = B\rho^*(\mathrm{ch}(E_{p,q}^{\pm}))$. Now $U(p) \times U(q)$ is a maximal compact subgroup of $U(p,q)$, so the inclusion $i : BU(p) \times BU(q) \to BU(p,q)$ induces an isomorphism in cohomology. That is

$$\mathrm{H}^*(BU(p,q); \mathbb{R}) \;=\; \mathrm{H}^*(BU(p); \mathbb{R}) \otimes \mathrm{H}^*(BU(q); \mathbb{R}).$$

It is not difficult to see that under this isomorphism

$$\mathrm{ch}(E_{p,q}^+) \;=\; \mathrm{ch}(E_p) \quad \text{and} \quad \mathrm{ch}(E_{p,q}^-) \;=\; \mathrm{ch}(E_q),$$

where $E_p \to BU(p)$ and $E_q \to BU(q)$ are the universal bundles. Thus we have

Theorem 11.7. *Let $\rho : \Gamma \to U(p,q)$ be a homomorphism of a finitely presented group. Then*

$$(B\rho)^*(i^*)^{-1}\Big(\mathrm{ch}(E_p) - \mathrm{ch}(E_q)\Big) \quad \in \quad \mathrm{H}^*(B\Gamma; \mathbb{R})$$

satisfies the Novikov conjecture.

Of course, this follows immediately from the well known fact that the Novikov conjecture is true for subgroups of Lie groups. The main input here is the possibility to use (complementary) families of representations

giving rise to interesting foliations. To this end, we have the following generalization of the Lusztig construction. It would be a very interesting application to use this construction to shed more light on the series of some discrete groups sitting in $U(p,q)$. Note that, for a given Lie group H, the space $\mathrm{Hom}(\Gamma, H)$ is well understood for abelian groups Γ and has been intensively studied when Γ is a higher genus surface group and H is $PSL(2, \mathbb{R})$ or $PU(1,2)$, see [**Go85**]. Other examples of Γ and H have also been studied by other authors and they all fit into the case of $H = U(p,q)$, see [**G96**] for a survey.

Example 11.8 (Foliation Lusztig Example)**.**

Let K be a compact Riemannian manifold without boundary, and $g : \pi_1(N) \to \mathrm{Iso}(K)$ a homomorphism to the isometries of K. Denote by $\mathrm{Hom}_c(\pi_1(N), U(p,q))$ the set of homomorphisms from $\pi_1(N)$ to $U(p,q)$ which have image contained in a compact subgroup. Let

$$h : K \to \mathrm{Hom}_c(\pi_1(N), U(p,q))$$

be a weakly uniformly continuous smooth g-cocycle. Smoothness of h means that for any $\gamma \in \pi_1(N)$, $w \to h_w(\gamma)$ is a smooth function from K to $U(p,q)$. Weak uniform continuity of h means the following. Denote the norm on $U(p,q)$ by $|| \cdot ||$. Given $w_1, w_2 \in K$, define

$$d_W(w_1, w_2) \;=\; \max_{A_1}\Big[\min_{A_2} ||A_1 - A_2||\Big],$$

where $A_i \in \overline{h_{w_i}(\pi_1(N))}$, the closure of the image of $\pi_1(N)$ under h_{w_i}. Then, h is weakly uniformly continuous if $d_W(w_1, w_2) \to 0$ as $w_1 \to w_2$.

That h is a g-cocycle means that for $\gamma_1, \gamma_2 \in \pi_1(N)$ and $w \in K$,

$$h_{g_{\gamma_2}(w)}(\gamma_1) h_w(\gamma_2) \;=\; h_w(\gamma_1 \gamma_2).$$

Then we may form

$$E \;=\; \widetilde{N} \times K \times \mathbb{C}^{p+q}/\pi_1(N),$$

where the action of $\gamma \in \pi_1(N)$ on $(x, w, z) \in \widetilde{N} \times K \times \mathbb{C}^{p+q}$ is given by

$$\gamma(x, w, z) \;=\; (\gamma(x), g_\gamma(w), h_w(\gamma)z).$$

Then E is a $\mathbb{C}^{p+q}$ bundle over $\widetilde{N} \times_{\pi_1(N)} K$.

Now we have the Riemannian foliation F of the flat fiber bundle $\widetilde{N} \times_{\pi_1(N)} K \to N$, whose leaves consist of the images of the $\widetilde{N} \times \{w\}$. The bundle E is leafwise flat and the (indefinite) inner product is preserved by the flat structure. Again write $E = E^+ \oplus E^-$, where the indefinite metric is $\pm$ definite on $E^\pm$. The parallel translation along the leaves of F is bounded since the closure of the union of all the images, $\overline{\bigcup_K h_w(\pi_1(N))}$ is a compact subset of $U(p,q)$. This follows easily from the facts that K is compact, each $\overline{h_w(\pi_1(N))}$ is compact, and h is weakly uniformly continuous. (We conjecture that continuity of h and

compactness of K imply compactness of $\overline{\bigcup_K h_w(\pi_1(N))}$.) As above, the hypothesis of Theorem 9.1 are satisfied, and we have

Theorem 11.9. *Assume Conjecture* 10.3. *Then for every* g, h *and* K *as above,*

$$\int_F \mathbb{L}(TF)(\mathrm{ch}_2(E^+) - \mathrm{ch}_2(E^-))$$

is a homotopy invariant of N.

Note that we may view this Haefliger form as living on a single fiber K of the bundle $\widetilde{N} \times_{\pi_1(N)} K \to N$. This is because we may take fundamental domains of N in the various leaves to integrate over (when we do integration over the fiber to get to Haefliger cohomology), and these fundamental domains are indexed by any fiber K. Thus we may integrate over K to obtain

Corollary 11.10. *Assume Conjecture* 10.3. *Then for every* g, h *and* K *as above, the real number*

$$\int_K \int_F \mathbb{L}(TF)(\mathrm{ch}_2(E^+) - \mathrm{ch}_2(E^-))$$

is a homotopy invariant of N.

As above, we may recast this result in terms of the Novikov conjecture. Let $\Gamma = \pi_1(N)$ and let g, h and K be as in Example 11.8. The construction of the bundle $E \to \widetilde{N} \times_\Gamma K$ and its splitting $E = E^+ \oplus E^-$ are natural with respect to pull-back maps, so this construction defines the universal bundle

$$E_B = E\Gamma \times K \times \mathbb{C}^{p+q}/\Gamma,$$

where the action of $\gamma \in \Gamma$ on $E\Gamma \times K \times \mathbb{C}^{p+q}$ is given as above by $\gamma(x, w, z) = (\gamma(x), g_\gamma(w), h_w(\gamma)z)$. Then E_B is a $\mathbb{C}^{p+q}$ bundle over $E\Gamma \times_\Gamma K$, and it splits as $E_B = E_B^+ \oplus E_B^-$. If $\varphi : N \to B\Gamma$ classifies the universal cover $\widetilde{N} \to N$, with induced map $\widetilde{\varphi} : \widetilde{N} \to E\Gamma$, then $\widetilde{\varphi} \times \mathrm{id}_K : \widetilde{N} \times K \to E\Gamma \times K$ descends to the map $\widetilde{\varphi} \times_\Gamma \mathrm{id}_K : \widetilde{N} \times_\Gamma K \to E\Gamma \times_\Gamma K$, and $(\widetilde{\varphi} \times_\Gamma \mathrm{id}_K)^*(E_B^\pm) = E^\pm$.

Proposition 11.11. *Assume Conjecture* 10.3, *and denote by* $\pi_1^\Gamma : E\Gamma \times_\Gamma K \to B\Gamma$ *the projection. Then*

$$\pi_{1,*}^\Gamma(\mathrm{ch}([E^+] - [E^-]))$$

satisfies the Novikov conjecture.

Proof. This follows immediately since a direct inspection shows that in the cohomology of N

$$\pi_{1,*}^N \circ (\widetilde{\varphi} \times_\Gamma \mathrm{id}_K)^* = \varphi^* \circ \pi_{1,*}^\Gamma.$$

q.e.d.

Remark 11.12. Example 11.8 can be easily generalized to the following situation. Let E_0 be a complex vector bundle over K which is endowed with a (possibly indefinite) non-degenerate metric $\{\cdot, \cdot\}$. Assume that the vector bundle E_0 is a Γ-equivariant vector bundle and that the action of Γ preserves $\{\cdot, \cdot\}$. Then the vector bundle

$$E := \tilde{N} \times_\Gamma E_0 \to \tilde{N} \times_\Gamma K,$$

is easily seen to be a complex bundle with a well defined (possibly indefinite) non-degenerate metric, which admits a leafwise flat connection preserving that metric. Hence (assuming Conjecture 10.3), we get in this way more general cohomology classes which satisfy the Novikov conjecture.

Applications to the BC Novikov conjecture. We now explain how Theorem 9.1 can be used to investigate the Baum-Connes Novikov conjecture, that is the Novikov conjecture for foliations. We do this by generalizing the construction in Example 11.8. Choose a complete smooth transversal T to the foliation (M, F) and denote by $B\mathcal{G}_T^T$ the classifying space of the groupoid $\mathcal{G}_T^T$ which is the reduced (to T) homotopy groupoid. $\mathcal{G}_T^T$ consists of elements of $\mathcal{G}$ which start and end on T. It is well known that $B\mathcal{G}_T^T$ classifies free and proper actions of $\mathcal{G}_T^T$, so that the principal $\mathcal{G}_T^T$ bundle $\mathcal{G}_T$ (which consists of elements of $\mathcal{G}$ which start on T) over M is the pull-back, by a (up to homotopy well defined) map $\varphi : M \to B\mathcal{G}_T^T$, of a universal $\mathcal{G}_T^T$ bundle $E\mathcal{G}_T^T$ over $B\mathcal{G}_T^T$. More precisely, we have an action of $\mathcal{G}_T^T$ on $E\mathcal{G}_T^T$ on the right $E\mathcal{G}_T^T \times_{s_B} \mathcal{G}_T^T \to E\mathcal{G}_T^T$, denoted $x\gamma$ for $(x, \gamma) \in E\mathcal{G}_T^T \times_{s_B} \mathcal{G}_T^T$, where

$$E\mathcal{G}_T^T \times_{s_B} \mathcal{G}_T^T := \{(x, \gamma) \in E\mathcal{G}_T^T \times \mathcal{G}_T^T, s_B(x) = r(\gamma)\},$$

and $s_B : E\mathcal{G}_T^T \to T$, $r_B : E\mathcal{G}_T^T \to B\mathcal{G}_T^T$ satisfy

$$s_B \circ \widetilde{\varphi} = s, \quad s_B(x\gamma) = s(\gamma) \quad \text{and} \quad r_B \circ \widetilde{\varphi} = \varphi \circ r.$$

where $s : \mathcal{G}_T \to T$ and $r : \mathcal{G}_T \to M$ are the source and range maps, and $\widetilde{\varphi} : \mathcal{G}_T \to E\mathcal{G}_T^T$ is the $\mathcal{G}_T^T$-equivariant classifying map which covers φ. So we have

$$T \xleftarrow{s_B} E\mathcal{G}_T^T \xrightarrow{r_B} B\mathcal{G}_T^T.$$

The fibers of the submersion s_B are contractible and this identifies the universal principal bundle $E\mathcal{G}_T^T$, see [**C94**], pages 126-127.

Definition 11.13. A $\mathcal{G}_T^T$-equivariant Hermitian bundle $(E_0, \{\cdot, \cdot\})$ is a complex vector bundle $\pi_0 : E_0 \to T$ endowed with a (possibly indefinite) non-degenerate metric $\{\cdot, \cdot\}$ together with an action of $\mathcal{G}_T^T$ which preserves the metric.

So if we set

$$\mathcal{G}_T^T \times_T E_0 := \{(\alpha, u) \in \mathcal{G}_T^T \times E_0, s(\alpha) = \pi_0(u)\} = (s \,|\, \mathcal{G}_T^T)^* E_0,$$

then there is a smooth map $h : \mathcal{G}_T^T \times_T E_0 \to E_0$ such that $\pi_0 \circ h(\alpha, u) = r(\alpha)$, and for any $\alpha \in \mathcal{G}_T^T$ the map $h_\alpha(u) := h(\alpha, u)$ is a linear map from $E_{0,s(\alpha)}$ to $E_{0,r(\alpha)}$ which preserves the metric $\{\cdot, \cdot\}$. It is understood that h is an action in the sense that

$$h_{\alpha\beta} = h_\alpha \circ h_\beta, \quad \text{if } r(\beta) = s(\alpha).$$

Given a $\mathcal{G}_T^T$ Hermitian bundle $(E_0, \{\cdot, \cdot\})$, we define a Hermitian bundle over the classifying space $B\mathcal{G}_T^T$ whose total space is

$$E = E\mathcal{G}_T^T \times_{\mathcal{G}_T^T} E_0.$$

Here E is the quotient manifold where we have identified (x, u) with $(x\alpha, h(\alpha^{-1}, u))$ for any $\alpha \in \mathcal{G}_T^T$ such that

$$s(\alpha) = \pi_0(u) \text{ and } r(\alpha) = s_B(x).$$

Note that Example 11.8 falls into this class where we take $T = K$, a single fiber of $\widetilde{N} \times_{\pi_1(N)} K$ and where the Hermitian bundle E_0 is trivial and equivariant through the cocycle h. Finally, for general Riemannian foliations, the holonomy action of $\mathcal{G}_T^T$ on the transverse bundle to the foliation, and on all functorially defined bundles obtained from it, gives an example of a $\mathcal{G}_T^T$-equivariant Hermitian bundle.

Definition 11.14. If $(E_0, \{\cdot, \cdot\})$ is a $\mathcal{G}_T^T$-equivariant Hermitian bundle, the vector bundle E over the classifying space $B\mathcal{G}_T^T$ is called a Hermitian leafwise flat bundle.

This terminology is explained by the following. Recall that $\varphi : M \to B\mathcal{G}_T^T$ is a classifying map for the foliation F.

Lemma 11.15. *The complex vector bundle $\varphi^* E$ over M admits a leafwise flat structure which preserves the induced (possibly indefinite) metric.*

Proof. We may assume that the vector bundle $\varphi^* E$ is smooth and is isomorphic to $\mathcal{G}_T \times_{\mathcal{G}_T^T} E_0$. Since the action of $\mathcal{G}_T^T$ preserves the metric $\{\cdot, \cdot\}$, there is a well defined metric on $E \to M$ which is induced from $\{\cdot, \cdot\}$. The usual proof, using for instance properness of the action of $\mathcal{G}_T^T$ on $\mathcal{G}_T$, allows the construction of a connection on E which is leafwise flat and which preserves the (possibly indefinite) non-degenerate metric on E.   q.e.d.

As usual, the complex bundle E splits into a direct sum of unitary vector bundles $E = E^+ \oplus E^-$ which are not leafwise flat in general. We say that the leafwise flat bundle E is bounded if the leafwise parallel translation along the leafwise flat connection of E is a bounded map.

Theorem 11.16. *Assume that the foliation (M, F) is Riemannian, oriented, and transversely oriented, and assume Conjecture 10.3. Then for any Hermitian bounded leafwise flat bundle E over $B\mathcal{G}_T^T$, the Chern*

character $\mathrm{ch}(E^+) - \mathrm{ch}(E^-) \in H^*(B\mathcal{G}_T^T;\mathbb{R})$ *satisfies the BC Novikov conjecture.*

Proof. The bundle $\varphi^* E$ is a leafwise Hermitian flat bundle for the smooth foliation (M, F), and by our assumption of boundedness, the parallel translation along the leaves is bounded, so the projection onto the twisted leafwise harmonics is transversely smooth. Suppose $f : (M', F') \to (M, F)$ is a leafwise oriented, leafwise homotopy equivalence (which also preserves the transverse orientations). Then $f^*(\varphi^* E) = (\varphi \circ f)^* E$ is also a bounded leafwise Hermitian flat bundle, so projection onto the twisted leafwise harmonics for (M', F') is also transversely smooth. Applying Theorem 9.1, we get that in $H_c^*(M'/F')$,

$$\sigma(M', F'; (\varphi \circ f)^*([E^+] - [E^-])) = f^* \sigma(M, F; \varphi^*([E^+] - [E^-])).$$

Since the foliation is transversely oriented, there is a well defined transverse fundamental class, namely the holonomy invariant closed current $[M'/F']$ which is given by integration over the transversals of (M', F'). Applying $[M'/F']$ to the above equality and using the fact that $[M'/F'] \circ f^* = [M/F]$ (since f preserves the transverse orientations) we get

$$\left\langle [M'/F'], \sigma(M', F'; (\varphi \circ f)^*([E^+] - [E^-])) \right\rangle =$$
$$\left\langle [M/F], \sigma(M, F; \varphi^*([E^+] - [E^-])) \right\rangle.$$

But Conjecture 10.3 gives

$$\sigma(M, F; \varphi^*([E^+] - [E^-])) = \int_F \mathbb{L}(TF) \wedge \varphi^* \mathrm{ch}([E^+] - [E^-]),$$

and

$$\sigma(M', F'; (\varphi \circ f)^*([E^+] - [E^-])) = \int_{F'} \mathbb{L}(TF') \wedge (\varphi \circ f)^* \mathrm{ch}([E^+] - [E^-]).$$

Since $[M/F] \circ \int_F = \int_M$ and $[M'/F'] \circ \int_{F'} = \int_{M'}$, the conclusion follows, namely

$$\int_M \mathbb{L}(TF) \wedge \varphi^* \mathrm{ch}([E^+] - [E^-]) = \int_{M'} \mathbb{L}(TF') \wedge (\varphi \circ f)^* \mathrm{ch}([E^+] - [E^-]).$$

q.e.d.

12. Appendix

We give brief outlines for the proofs of the results of Section 6.

Let $f : M \to M'$, $g : M' \to M$, $E' \to M'$ and E be as in Section 6. Then there are leafwise homotopies $h : M \times I \to M$ and $h' : M' \times I \to M'$ with $I = [0, 1]$, so that for all $x \in M, x' \in M'$

$$h(x, 0) = x, \ h(x, 1) = g \circ f(x), \ h'(x', 0) = x', \ \text{and} \ h'(x', 1) = f \circ g(x').$$

Proof of Proposition 6.1. If $U \subset M$ and $U' \subset M'$ are foliation charts in good covers, with transversals T and T', and $f(U) \subset U'$, then it is not difficult to show that f induces $\widehat{f} : T \to T'$, which is a diffeomorphism onto its image. Define $\widehat{f}^*$ on Haefliger forms in the obvious way, and show that it is well defined. It follows easily that the induced map on Haefliger cohomology $f^* : \mathrm{H}_c^*(M'/F') \to \mathrm{H}_c^*(M/F)$ is an isomorphism with inverse g^*.   q.e.d.

As f is a homotopy equivalence between M and M', the dimensions of M and M' are the same. Since $\widehat{f}$ is a local diffeomorphism, it follows immediately that the codimensions of F and F' are the same, and so the dimensions of F and F' are also the same.

Proof of Proposition 6.2. Smoothness is obvious. Let $m(P)$ be bound on the diameter of any plaque in a finite good cover $\mathcal{U}$ of M, so also a bound for any plaque in the corresponding cover of $\mathcal{G}$. Let $\mathcal{U}'$ be a finite good cover of M', such that for each $U' \in \mathcal{U}'$ there is $U \in \mathcal{U}$ so that $g(U') \subset U$. Use the fact that the Lebesgue covering distance of $\mathcal{U}'$ is positive (say $\epsilon > 0$), and the fact that $\check{g}(\widetilde{L}') \cap (U, \gamma, V)$ consists of at most one plaque of $\check{g}(\widetilde{L}')$, to show that if z_t' is a path in $\widetilde{L}'$ of length less than C, then $\check{g} \circ z_t'$ is a path in $\check{g}(\widetilde{L}')$ of length less than $m(P)C/\epsilon$.

Let $z_0, z_1 \in \widetilde{L}_x$ with $\check{f}(z_i) = z_i'$, whose leafwise distance $d_{\widetilde{L}_x'}(z_0', z_1') \leq C$. Use the result above and the fact that composition on the right by any path is an isometry, to show that

$$d_{\widetilde{L}_x}((\check{g} \circ z_0') \cdot \gamma_x, (\check{g} \circ z_1') \cdot \gamma_x) \ \leq \ m(P)C/\epsilon,$$

where $\gamma_x(t) = h(x, t)$. Then show $d_{\widetilde{L}_x}(z_i, [(\check{g} \circ z_i') \cdot \gamma_x]) \leq \ \mathrm{length}(\gamma_{r(z_i)})$, which by Lemma 3.16 of [**HL91**] is bounded, say by B. Conclude that

$$d_{\widetilde{L}_x}(z_0, z_1) \ \leq \ 2B + m(P)C/\epsilon,$$

so $\check{f}$ is leafwise uniformly proper.   q.e.d.

Proof of Proposition 6.3. For this proof only, for $\xi_i = \alpha_i \otimes \phi_i \in \mathcal{A}_c^*(F_s, E)$, set

$$\xi_1 \wedge \xi_2 = (\phi_1, \phi_2)\alpha_1 \wedge \alpha_2 \quad \text{and} \quad \xi_1 \wedge *\xi_2 = (\phi_1, \phi_2)\alpha_1 \wedge *\alpha_2,$$

and extend linearly, where $(\cdot, \cdot)$ is the positive definite metric on E. Similarly for $\mathcal{A}_c^*(F_s', E')$.

Lemma 3.17 of [**HL91**] implies that f is leafwise uniformly proper. Because $\check{f}$ is also leafwise uniformly proper and $M \times B^k$ is compact, both the maps denoted p_f are also leafwise uniformly proper. Show that we may assume that the metric on each $L_x \times B^k$ (respectively $\widetilde{L}_x \times B^k$) is the product of a fiberwise metric for the submersion p_f and the pull-back under p_f of the metric on $L_{f(x)}'$ (respectively $\widetilde{L}_{f(x)}'$). Denote by $d\,\mathrm{vol}_{\mathrm{vert}}$ the canonical k form on both $L \times B^k$ and $\widetilde{L} \times B^k$ whose restriction to

the oriented fibers of p_f is the volume form. Simple computations show that for any $\alpha_1, \alpha_2 \in \wedge^\ell T^* F'_s$,

12.1. $p_f^* \alpha_1 \wedge *p_f^* \alpha_2 = d\,\mathrm{vol}_{\mathrm{vert}} \wedge p_f^*(\alpha_1 \wedge *'\alpha_2)$,

and

12.2. $d\,\mathrm{vol}_{\mathrm{vert}} \wedge p_f^* \alpha_1 \wedge *(d\,\mathrm{vol}_{\mathrm{vert}} \wedge p_f^* \alpha_2) = d\,\mathrm{vol}_{\mathrm{vert}} \wedge p_f^*(\alpha_1 \wedge *'\alpha_2)$.

Since p_f is leafwise uniformly proper,

$$C = \sup_{[\gamma'] \in \mathcal{G}'} \int_{p_f^{-1}([\gamma'])} d\,\mathrm{vol}_{\mathrm{vert}} < +\infty.$$

Use 12.1, to show that for $\xi \in \mathcal{A}_c^\ell(\widetilde{L}'_{f(x)}, E')$, $\|p_f^*(\xi)\|_0^2 \leq C\|\xi\|_0^2$, so p_f^* defines a bounded operator from $W_0^\ell(F'_s, E')$ to $W_0^\ell(F_s \times B^k, p_f^* E')$.

Choose a sub-bundle $\widehat{H} \subset TF \oplus TB^k$ so that for each L_x, it is a horizontal distribution for the submersion $p_f : L_x \times B^k \to L'_{f(x)}$. $\widehat{H}$ determines a sub-bundle H of $TF_s \oplus TB^k$, which is a horizontal distribution for the submersion $p_f : \widetilde{L}_x \times B^k \to \widetilde{L}'_{f(x)}$. Choose a finite collection of leafwise vector fields $\widehat{Y}_1, \ldots, \widehat{Y}_N$ on M' which generate $C^\infty(TF')$ over $C^\infty(M')$. Lift these to leafwise (for F'_s) vector fields $Y_1, \ldots, Y_N$ on $\mathcal{G}'$, and lift these latter to sections of H, denoted $X_1, \ldots, X_N$. If X^{vert} is a vertical vector field on $\widetilde{L} \times B^k$ with respect to p_f, then $i_{X^{\mathrm{vert}}} \circ p_f^* = 0$. Modulo such vector fields, the X_i generate $T\widetilde{L} \oplus TB^k$ over $C^\infty(\widetilde{L} \times B^k)$. In addition $i_{X_j} \circ p_f^* = p_f^* \circ i_{Y_j}$. Thus for any $\xi \in \mathcal{A}_c^\ell(\widetilde{L}'_{f(x)}, E')$, any $Y_K = Y_{k_1} \wedge \cdots \wedge Y_{k_\ell}$, and any $j_1, \ldots, j_m$, with $j_i \in \{1, \ldots, N\}$,

$$\|i_{X_{j_1}} d \cdots i_{X_{j_m}} d(p_f^*(\xi)(Y_K))\|_0 = \|p_f^*(i_{Y_{j_1}} d \cdots i_{Y_{j_m}} d(\xi(Y_K)))\|_0 \leq$$

$$\sqrt{C}\|i_{Y_{j_1}} d \cdots i_{Y_{j_m}} d(\xi(Y_K))\|_0.$$

A classical argument then shows that for any $s \geq 1$, p_f^* extends to a uniformly bounded operator from $W_s^\ell(\widetilde{L}'_{f(x)}, E')$ to $W_s^\ell(\widetilde{L}_x \times B^k, p_f^* E')$, that is a bounded operator from $W_s^\ell(F'_s, E')$ to $W_s^\ell(F_s \times B^k, p_f^* E')$.

The operator e_ω maps $W_s^\ell(\widetilde{L}_x \times B^k, p_f^* E')$ to $W_s^{k+\ell}(\widetilde{L}_x \times B^k, p_f^* E')$ and is uniformly bounded, since ω and all its derivatives are bounded. Thus for $s \geq 0$, $e_\omega \circ p_f^*$ is a bounded operator from $W_s^\ell(F'_s, E')$ to $W_s^{k+\ell}(F_s \times B^k, p_f^* E')$.

For $s < 0$, dualize the argument above. Denote by $p_{f,*}$ integration of fiber compactly supported forms along the fibers of the submersion p_f. Show that for any $\alpha \in \mathcal{A}_c^{k+\ell}(\widetilde{L}_x \times B^k)$,

12.3. $$p_{f,*}\alpha \wedge *'p_{f,*}\alpha \leq C\,p_{f,*}(\alpha \wedge *\alpha),$$

where oriented volume elements of $\widetilde{L}'_{f(x)}$ at a point are identified with $\mathbb{R}_+^*$. Do this by first reducing to the case $\alpha = d\,\mathrm{vol}_{\mathrm{vert}} \wedge \alpha_3$, with $\alpha_3 \in$

$C_c^\infty(p_f^*(\wedge^\ell T^*\widetilde{L}'_{f(x)}))$. Choose a finite collection of sections $\widehat{\beta}_1, \ldots, \widehat{\beta}_r$ of $\wedge^\ell T^* F'$, so that $\widehat{\beta}_i \wedge *'\widehat{\beta}_j = 0$ if $i \neq j$, and the $\widehat{\beta}_i$ generate $C^\infty(\wedge^\ell T^* F')$ over $C^\infty(M')$. Lift these to sections β_i of $\wedge^\ell T^* F'_s$. Write $\alpha = \sum_i g_i \, d\,\mathrm{vol}_{\mathrm{vert}} \wedge p_f^* \beta_i$. Then $p_{f,*}\alpha \wedge *'p_{f,*}\alpha = \sum_i [p_{f,*}(g_i \, d\,\mathrm{vol}_{\mathrm{vert}})]^2 \beta_i \wedge *'\beta_i$, and thanks to 12.2,

$$p_{f,*}(\alpha \wedge *\alpha) = \sum_i p_{f,*}(g_i^2 \, d\,\mathrm{vol}_{\mathrm{vert}})\beta_i \wedge *'\beta_i \geq$$

$$\sum_i \frac{[p_{f,*}(g_i \cdot 1 \, d\,\mathrm{vol}_{\mathrm{vert}})]^2}{p_{f,*}(1 \, d\,\mathrm{vol}_{\mathrm{vert}})}\beta_i \wedge '\beta_i \geq \frac{1}{C}\sum_i [p_{f,*}(g_i \, d\,\mathrm{vol}_{\mathrm{vert}})]^2\beta_i \wedge *'\beta_i =$$

$$\frac{1}{C} p_{f,*}\alpha \wedge *'p_{f,*}\alpha,$$

proving 12.3. Note that the second to last inequality is just Cauchy-Schwartz. Conclude that for all $\alpha \in \mathcal{A}_c^{k+\ell}(\widetilde{L}_x \times B^k)$, $\|p_{f,*}\alpha\|_0^2 \leq C\|\alpha\|_0^2$. Using the facts that $p_{f,*}$ commutes with the de Rham differentials, $p_{f,*} \circ i_{X^{\mathrm{vert}}} = 0$ and $i_{Y_j} \circ p_{f,*} = p_{f,*} \circ i_{X_j}$, it is easy to deduce, just as above, that for $s \geq 0$, $p_{f,*} \circ e_\omega$ extends to a uniformly bounded operator (say with bound C_s) from $W_s^\ell(\widetilde{L}_x \times B^k, p_f^* E')$ to $W_s^\ell(\widetilde{L}'_{f(x)}, E')$. For $\xi' \in W_s^\ell(\widetilde{L}'_{f(x)}, E')$ with $s < 0$,

$$\|(e_\omega \circ p_f^*)(\xi')\|_s = \sup_\xi \frac{|\langle \xi', (p_{f,*} \circ e_\omega)(\xi)\rangle|}{\|\xi\|_{-s}} \leq$$

$$\sup_\xi \frac{\|\xi'\|_s \|(p_{f,*} \circ e_\omega)(\xi)\|_{-s}}{\|\xi\|_{-s}} \leq C_s\|\xi'\|_s,$$

where $\xi \in W_{-s}^\ell(\widetilde{L}_x \times B^k, p_f^* E')$. Thus for any $s < 0$ (and so for all $s \in \mathbb{Z}$), $e_\omega \circ p_f^*$ is a uniformly bounded operator from $W_s^\ell(\widetilde{L}'_{f(x)}, E')$ to $W_s^{k+\ell}(\widetilde{L}_x \times B^k, p_f^* E')$, so $e_\omega \circ p_f^*$ is a bounded operator from $W_s^\ell(F'_s, E')$ to $W_s^\ell(F_s \times B^k, p_f^* E')$.

For all $s \in \mathbb{Z}$, the image of $e_\omega \circ p_f^*$ consists of π_1-fiber compactly supported distributional forms. The argument above for $p_{f,*}$ applied to $\pi_{1,*}$ shows that it is uniformly bounded as a map from $\mathrm{Im}(e_\omega \circ p_f^*) \subset W_s^{k+\ell}(\widetilde{L}_x \times B^k, p_f^* E')$ to $W_s^\ell(\widetilde{L}_x, E)$. Thus for all $s \in \mathbb{Z}$, $f^{(i,\omega)}$ extends to a bounded operator from $W_s^\ell(F'_s, E')$ to $W_s^\ell(F_s, E)$. q.e.d.

Proof of Proposition 6.4. To prove this, we switch our point of view to that in [**HL91**], and give another construction of the map $\widetilde{f}^*$. Let $K = \bigcup_{\widetilde{L}} K_{\widetilde{L}}$ be a bounded leafwise triangulation of F_s (see [**HL91**]) induced from a bounded leafwise triangulation to F. A simplicial k-cochain φ on $K_{\widetilde{L}}$ with coefficients in E assigns to each k-simplex σ of $K_{\widetilde{L}}$ an element $\varphi(\sigma) \in E_\sigma$, the fiber of E over the barycenter of σ.

To define the co-boundary map δ, use the flat structure of E. The ℓ^p cohomology of this is denoted $\mathrm{H}^*_{\triangle,p}(\widetilde{L},E)$. The classical Whitney and de Rham maps, [**W57**], extend to well defined isomorphisms

$$\mathrm{H}^*_{\triangle,p}(\widetilde{L},E) \xrightarrow{W} \mathrm{H}^*_{(2)}(\widetilde{L},E) \quad \text{and} \quad \mathrm{H}^*_{\triangle,p}(\widetilde{L},E) \xleftarrow{\int} \mathrm{H}^*_{(2)}(\widetilde{L},E),$$

(which are inverses of each other), with bounds independent of $\widetilde{L}$, for $p = 1, 2$. See [**HL91**] for $p = 2$, and [**GKS88**] for $p = 1$.

Any oriented leafwise simplicial approximation of $\check{f}$, [**HL91**], defines a map on simplicial cohomology denoted $f^*_\triangle$. Set $f^*_D = W \circ f^*_\triangle \circ \int$. It suffices to show that $\widetilde{f}^* = f^*_D : \mathrm{H}^*_{(2)}(F'_s, E') \longrightarrow \mathrm{H}^*_{(2)}(F_s, E)$, since then, for any choice of i and ω, $\widetilde{f}^* = f^*_D$, so they are all the same. The other properties of $\widetilde{f}^*$ follow from these same properties for f^*_D which are easy to prove using classical arguments.

Consider the following diagram, where we write $\widetilde{L}_B$ for $\widetilde{L} \times B^k$. Classical results imply that it commutes.

$$\mathrm{H}^*_{\triangle,2}(\widetilde{L}',E') \xrightarrow{p^*_{f,\triangle}} \mathrm{H}^*_{\triangle,2}(\widetilde{L}_B, p^*_f E') \xrightarrow{[\beta]\cup} \mathrm{H}^{*+k}_{\triangle,2,c}(\widetilde{L}_B, p^*_f E') \xrightarrow{\cap[B^k]} \mathrm{H}^*_{\triangle,2}(\widetilde{L},E)$$

$$\downarrow W \qquad\qquad \downarrow W \qquad\qquad \downarrow W \qquad\qquad \downarrow W$$

$$\mathrm{H}^*_{(2)}(\widetilde{L}',E') \xrightarrow{p^*_f} \mathrm{H}^*_{(2)}(\widetilde{L}_B, p^*_f E') \xrightarrow{[\omega]\wedge} \mathrm{H}^{*+k}_{(2),c}(\widetilde{L}_B, p^*_f E') \xrightarrow{\pi_{1,*}} \mathrm{H}^*_{(2)}(\widetilde{L},E).$$

The subscript c means cohomology with fiber compact supports, and β is the simplicial k cocycle $\int \omega$ on B^k. The bottom line of this diagram is $\widetilde{f}^*$, and $W^{-1} = \int$, so it suffices to show that

$$\cap[B^k] \circ [\beta] \cup \circ\, p^*_{f,\triangle} \;=\; f^*_\triangle.$$

But, $([\beta]\cup) \circ p^*_{f,\triangle} \;=\; ([\beta]\cup) \circ \pi^*_{1,\triangle} \circ i^*_\triangle \circ p^*_{f,\triangle} \;=\; ([\beta]\cup) \circ \pi^*_{1,\triangle} \circ f^*_\triangle$, where $\pi^*_{1,\triangle}$ and $i^*_\triangle$ are induced by the projection and zero section of the bundle $\widetilde{L} \times B^k \to \widetilde{L}$. By the Thom Isomorphism Theorem, $([\beta]\cup) \circ \pi^*_{1,\triangle}$ is an isomorphism whose inverse is precisely $\cap[B^k]$. q.e.d.

Proof of Proposition 6.5. Use the cup product

$$(\varphi_1 \cup \varphi_2)(\sigma) = \frac{1}{(2\ell+1)!} \sum_{i,j} \{\varphi_1(\sigma_i), \varphi_2(\sigma_j)\},$$

where σ_i and σ_j are certain faces of the 2ℓ simplex σ (see [**LS03**], Equation (3.30)), $\{\cdot, \cdot\}$ is the (possibly indefinite) metric, and we identify $\varphi_1(\sigma_i) \in E_{\sigma_i}$ and $\varphi_2(\sigma_j) \in E_{\sigma_j}$ with their images in E_σ.

Proposition 12.4 ([**LS03**]). *For any ℓ^2 simplicial cochains φ_1 and φ_2 on $\widetilde{L}$ with coefficients in E,*

$$Q(W(\varphi_1), W(\varphi_2)) \quad = \quad \int_{\widetilde{L}} W(\varphi_1 \cup \varphi_2).$$

Their proof extends immediately to this case. Using the metric $\{\cdot, \cdot\}$ guarantees that for simplicial ℓ^2 cohomology classes Ξ_1 and Ξ_2,

$$Q(W(\Xi_1), W(\Xi_2)) \;=\; \int_{\widetilde{L}} W(\Xi_1 \sqcup \Xi_2) \;=\; \langle [\widetilde{L}], \Xi_1 \sqcup \Xi_2 \rangle.$$

For any L^2 cohomology classes Ψ'_1 and Ψ'_2 on $\widetilde{L}'$ with coefficients in E',

$$\langle [\widetilde{L}'], \Big(\int \Psi'_1 \Big) \sqcup \Big(\int \Psi'_2 \Big) \rangle \;=\; \int_{\widetilde{L}'} \Psi'_1 \wedge \Psi'_2,$$

and for any classes $\Xi'_1, \Xi'_2 \in H^*_{\triangle, 2}(\widetilde{L}', E')$, we have $f^*_\triangle \Xi'_1 \sqcup f^*_\triangle \Xi'_2 = f^*_\triangle(\Xi'_1 \sqcup \Xi'_2)$ in $H^*_{\triangle, 1}(\widetilde{L}, E)$. Recall that if $\xi'_1 = \alpha'_1 \otimes \phi'_1$ and $\xi'_2 = \alpha'_2 \otimes \phi'_2$, then $\xi'_1 \wedge \xi'_2 = \{\phi'_1, \phi'_2\}\alpha'_1 \wedge \alpha'_2$, and we extend to all ξ' by linearity. Let Ψ'_1 and Ψ'_2 be the cohomology classes determined by ξ'_1 and ξ'_2. Substituting $\widetilde{f}^*_D$ for $\widetilde{f}^*$, we have

$$Q_x(\widetilde{f}^*_D(\xi'_1), \widetilde{f}^*_D(\xi'_2)) \;=\; \int_{\widetilde{L}_x} \Big(W \circ f^*_\triangle \circ \int \Psi'_1 \Big) \wedge \Big(W \circ f^*_\triangle \circ \int \Psi'_2 \Big) \;=\;$$

$$\int_{\widetilde{L}_x} W \Big(\Big(f^*_\triangle \circ \int \Psi'_1 \Big) \sqcup \Big(f^*_\triangle \circ \int \Psi'_2 \Big) \Big) \;=\; \langle [\widetilde{L}_x], f^*_\triangle \Big(\int \Psi'_1 \sqcup \int \Psi'_2 \Big) \rangle \;=\;$$

$$\langle [f_{\triangle, *}\widetilde{L}_x], \Big(\int \Psi'_1 \sqcup \int \Psi'_2 \Big) \rangle \;=\; \int_{\widetilde{L}'_{f(x)}} \Psi'_1 \wedge \Psi'_2 \;=\; Q'_{f(x)}(\xi'_1, \xi'_2).$$

<div align="right">q.e.d.</div>

References

[AGS]   S. Azzali, S. Goette & T. Schick. *Personal communication, preprint to appear.*

[BC85]   P. Baum & A. Connes. *Leafwise homotopy equivalence and rational Pontrjagin classes*, Foliations (Tokyo, 1983) 1–14, Adv. Stud. Pure Math., **5**, North-Holland, Amsterdam, 1985, MR 0877325, Zbl 0641.57008.

[BC00]   P. Baum & A. Connes. *Geometric K-theory for Lie groups and foliations*, Enseign. Math. (2) **46** (2000) 3–42, MR 1769535, Zbl 0985.46042.

[BH04]   M-T. Benameur & J. L. Heitsch. *Index theory and Non-Commutative Geometry I. Higher Families Index Theory*, K-Theory **33** (2004) 151–183, MR 2131748, Zbl 1074.58009, *Corrigendum*, ibid **36** (2005) 397–402, MR 2275014, Zbl 1108.58020.

[BH08]   M-T. Benameur & J. L. Heitsch. *Index theory and Non-Commutative Geometry II. Dirac Operators and Index Bundles*, J. of K-Theory **1** (2008) 305–356, MR 2434189, Zbl 1155.58009.

[BH]   M-T. Benameur & J. L. Heitsch. *The Twisted Higher Harmonic Signature for Foliations. The leafwise almost flat case*, in preparation.

[BN94]   J.-L. Brylinski & V. Nistor. *Cyclic cohomology of etale groupoids*, *K*-Theory **8** (1994) 341–365, MR 1300545, Zbl 0812.19003.

[Ca]   P. Carrillo-Rousse *Indices analytiques à support compact pour des groupoides de Lie*. PhD Thesis, Paris 7, 2007, available at people.math.jussieu.fr/∼carrillo/mathematiques.html

[C79]   A. Connes. *Sur la théorie de l'intégration non commutative*, Lect. Notes in Math. **725**, 1979, MR 0548112, Zbl 0412.46053.

[C81]   A. Connes. *A survey of foliations and operator algebras*, Operator Algebras and Applications, Part I, Proc. Sympos. Pure Math **38**, Amer. Math. Soc. (1982) 521-628, MR 0679730, Zbl 0531.57023.

[C86]   A. Connes. *Cyclic cohomology and the transverse fundamental class of a foliation*, Geometric Methods in Operator Algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., **123** 52–144, Longman Sci. Tech., Harlow, 1986, MR 0866491, Zbl 0647.46054.

[C94]   A. Connes. *Noncommutative Geometry*, Academic Press, New York, 1994, MR 1303779, Zbl 0818.46076.

[CGM90]   A. Connes, M. Gromov & H. Moscovici. *Conjecture de Novikov et fibrés presque plats*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 5, 273–277, MR 1042862, Zbl 0693.53007.

[CGM93]   A. Connes, M. Gromov & H. Moscovici. *Group cohomology with Lipschitz control and higher signatures*, Geom. Funct. Anal. **3** (1993) 1–78, MR 1204787, Zbl 0789.58069.

[CM90]   A. Connes & H. Moscovici. *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990) 345–388, MR 1066176, Zbl 0819.46058.

[Cu04]   J. Cuntz. *Cyclic theory and the bivariant Chern-Connes character*, Noncommutative Geometry, 73–135, Lecture Notes in Math. **1831**, Springer, Berlin, 2004, MR 2058473, Zbl 1053.46047.

[CuQ97]   J. Cuntz & D. Quillen. *Excision in bivariant periodic cyclic cohomology*. Invent. Math. **127** (1997) 67–98, MR 1423026, Zbl 0889.46054.

[D77]   J. Dodziuk. *de Rham-Hodge theory for L^2-cohomology of infinite coverings*, Topology **16** (1977) 157–165, MR 0445560, Zbl 0348.58001.

[EMS76]   R. Edwards, K. Millett & D. Sullivan. *Foliations with all leaves compact*, Topology **16** (1977) 13–32, MR 0438353, Zbl 0356.57022.

[EH86]   A. El Kacimi-Alaoui & G. Hector, *Decomposition de Hodge basique pour un feuilletage riemannien*. Ann. Inst. Fourier (Grenoble) **36** (1986), no. 3, 207–227, MR 0865667, Zbl 0586.5701.

[EHS85]   A. El Kacimi-Alaoui, G. Hector & V. Sergiescu, *La cohomologie basique d'un feuilletage riemannien est de dimension finie*. Math. Z. **188** (1985), no. 4, 593–599, MR 0774559, Zbl 0536.57013.

[Ep76]   D. B. A. Epstein. *Foliations with all leaves compact*, Ann. Inst. Fourier (Grenoble) **26** (1976) 265–282, MR 0420652, Zbl 0313.57017.

[Go85]   W. Goldman. *Representations of fundamental groups of surfaces*, Geometry and Topology (College Park, Md., 1983/84), 95–117, Lecture Notes in Math. **1167**, Springer, Berlin, 1985, MR 0827264, Zbl 0575.5702.

[GKS88]   V. M. Gol'dshtein, V. I. Kuz'minov & I. A. Shvedov. *The de Rham isomorphism of the L_p- cohomology of noncompact Riemannian manifolds.*

Sibirsk. Mat. Zh. **29** (1988) 34-44. Translation in Siberian Math. J. **29** (1988) 190–197, MR 0941121, Zbl 0668.58051.

[GR97]   D. Gong & M. Rothenberg. *Analytic torsion forms for noncompact fiber bundles*, MPIM preprint 1997-105. Available at www.mpim-bonn.mpg.de/preprints.

[GL03]   A. Gorokhovsky & J. Lott. *Local index theory over étale groupoids*, J. Reine Angew Math. **560** (2003) 151–198, MR 1992804, Zbl 1034.58021.

[G96]   M. Gromov. *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional analysis on the eve of the 21st century, Vol. II, 1–213, Progr. Math., **132**, Birkhauser Boston, 1996, MR 1389019, Zbl 0945.53022.

[Ha80]   A. Haefliger. *Some remarks on foliations with minimal leaves*, J. Diff. Geo. **15** (1980) 269–284, MR 0614370, Zbl 0444.57016.

[H95]   J. L. Heitsch. *Bismut superconnections and the Chern character for Dirac operators on foliated manifolds*, K-Theory **9** (1995) 507–528, MR 1361580, Zbl 0843.58117.

[HL90]   J. L. Heitsch & C. Lazarov. *A Lefschetz theorem for foliated manifolds*, Topology **29** (1990) 127–162, MR 1056266, Zbl 0706.58067.

[HL91]   J. L. Heitsch & C. Lazarov. *Homotopy invariance of foliation Betti numbers*, Invent. Math. **104** (1991) 321–347, MR 1098613, Zbl 0756.46040.

[HL99]   J. L. Heitsch & C. Lazarov. *A general families index theorem*, K-Theory **18** (1999) 181–202, MR 1711704, Zbl 0938.58021.

[HL02]   J. L. Heitsch & C. Lazarov. *Riemann-Roch-Grothendieck and torsion for foliations*, J. Geo. Anal. **12** (2002) 437–468, MR 1901750, Zbl 1032.58017.

[HgK01] N. Higson & G. Kasparov. *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001) 23–74, MR 1821144, Zbl 0988.19003.

[HiS92] M. Hilsum & G. Skandalis. *Invariance par homotopie de la signature à coefficients dans un fibré presque plat*, J. Reine Angew. Math. **423** (1992) 73–99, MR 1142484, Zbl 0731.55013.

[Hu93]   S. Hurder. *Exotic theory for foliated spaces* (1993), available at www.math.uic.edu/∼hurder/publications.html.

[KT83]   F. W. Kamber & P. Tondeur, *Duality for Riemannian foliations. Singularities, Part 1* (Arcata, Calif., 1981), 609–618, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983, MR 0713097, Zbl 0523.57019.

[KaM85] J. Kaminker & J. G. Miller. *Homotopy invariance of the analytic index of signature operators over C^*-algebras*, J. Operator Theory **14** (1985) 113-127, MR 0789380, Zbl 0614.46062.

[K88]   G. Kasparov. *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988) 147–201, MR 0918241, Zbl 0647.46053.

[KS03]   G. Kasparov & G. Skandalis. *Groups acting properly on "bolic" spaces and the Novikov conjecture*, Ann. of Math. (2) **158** (2003) 165–206, MR 1998480, Zbl 1029.19003.

[La02]   V. Lafforgue. *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes*, Ivent. Math. **149** (2002) 1–95, MR 1914617, Zbl 1084.19003.

[LMN05] R. Lauter, B. Monthubert & V. Nistor. *Spectral invariance for certain algebras of pseudodifferential operators*, J. Inst. Math. Jussieu **4** (2005) 405–442, MR 2197064, Zbl 1088.35087.

[LS03] W. Lück & T. Schick. *Various L^2-signatures and a topological L^2-signature theorem*, High Dimensional Manifold Topology, 362–399, World Sci. Publ., River Edge, NJ, 2003, MR 2048728, Zbl 1044.58026.

[Lu72] G. Lusztig. *Novikov's higher signature and families of elliptic operators*, J. Diff. Geo. **7** (1972) 229–256, MR 0322889, Zbl 0265.57009.

[Me] R. Meyer. *Cyclic cohomology theories and nilpotent extensions*, PhD thesis, Muenster.

[M78] A. S. Miscenko. *The theory of elliptic operators over C^*-algebras*, Dokl. Akad. Nauk SSSR **239** (1978) 1289–1291, MR 0494321, Zbl 0402.58008.

[N97] V. Nistor. *Super-connections and non-commutative geometry*, Cyclic Cohomology and Noncommutative Geometry (Waterloo, ON, 1995), 115–136, Fields Inst. Commun. **17** Amer. Math. Soc., Providence, RI, 1997, MR 1478706, Zbl 0904.46052.

[P95] M. Puschnigg. *A survey of asymptotic cyclic cohomology*, Cyclic Cohomology and Noncommutative Geometry (Waterloo, ON, 1995), 155-168, Fields Inst. Commun. **17** Amer. Math. Soc., Providence, RI, 1997, MR 1478708, Zbl 0898.46063.

[W57] H. Whitney. *Geometric Integration Theory*, Princeton University Press, Princeton, N. J., 1957, MR 0087148, Zbl 0083.28204.

[T99] J. L. Tu. *La conjecture de Novikov pour les feuilletages hyperboliques*, K-Theory **16** (1999) 129–184, MR 1671260, Zbl 0932.19005.

[W83] H.E. Winkelnkemper. *The graph of a foliation*, Ann. Global Anal. Geo. **1** (1983) 51–75, MR 0739904, Zbl 0526.53039.

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