An introduction to Arithmetic Chow Groups and Arakelov Theory

Morelia, Mexico

Henri Gillet

6/25/2002

Introduction These are a slightly revised version of the slides that I used for my lectures in Morelia. While they take into account the errors I found while in Morelia, together with corrections that were pointed out to me, they still contain, I am sure, many errors and omissions. In particular I was not careful about signs and constants.

Please let me know of corrections or comments - I will try to incorporate them into the text.

Lecture 1

Motivation and Basic Analogies

Two Integrals

- $$\int_{S^1 1} \log |z_0 z_1 + z_2 z_3 + z_4 z_5| = -23/15$$

  More generally:

  $$\int_{S^{(4n-1)}} \log |z_0 z_1 + z_2 z_3 + \ldots + z_{2n-2} z_{2n-1}|$$
  $$= -\mathcal{H}_{2n-1} + \frac{1}{2}\mathcal{H}_{n-1}$$

  Here $\mathcal{H}_n = \sum_{j=1}^{n} \frac{1}{j}$

- The lemniscate:

  $$x^4 + y^4 + 2x^2y^2 = 2(x^2 - y^2)$$

  has arc length:

  $$\frac{\Gamma(1/4)^2}{2^{3/2}\sqrt{\pi}}$$

Heights

Recall that if $P = (x_0, \ldots, x_n) \in \mathbb{P}^n_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{P}^n_{\mathbb{Z}}(\mathbb{Q})$ then its (logarithmic) height is:

$$h(P) := \frac{1}{2} \log(\sum_i |x_i|^2)$$

(Strictly, this is one of many equivalent ideas of height)

Observe that given any positive real number $B > 0$, there are only finitely many rational points with height less than B. In particular, given a diophantine equation, if you can prove that its solutions have bounded height then you can conclude that the equation has only finitely many solutions.

One goal: Define notion of height of an *arithmetic variety* with nice properties. To do this we shall use arithmetic intersection theory.

Points and Completions

$\mathfrak{p} \in \text{Spec}(\mathcal{O}_K)$	$x \in C$, An affine curve over $\mathbb{F}_q$
$\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}(a)$	$\text{ord}_x(f)$
$K \hookrightarrow E$, $[E : \mathbb{Q}_p] < \infty$,	$K \hookrightarrow E$, $E \simeq \mathbb{F}_{q^r}((t))$
$K \hookrightarrow \mathbb{R}$ or $\mathbb{C}$	Point in $\bar{C} - C$ $\bar{C}$ a complete smooth model of C.
The (Archimedean) absolute value $\|a\|_\infty$, $a \in \mathbb{Q}$	$A^{\text{ord}_x(f)}$, $(A > 0)$, for $x \in \bar{C} - C$; if $C = \mathbb{A}^1_{\mathbb{F}_q}$, $\text{ord}_x(f) = \deg(f)$.

Analogies: Number Fields

1. $[K : \mathbb{Q}] < \infty$

2. Set of prime ideals Spec($\mathcal{O}_K$) in the ring of integers in K

3. Product formula: $\sum_v \log |f|_v = 0$ (For $f \in K = \mathbb{Q}$, $|f|_p = p^{-v_p(a)}$, and $|f|_\infty$ is the usual absolute value.)

4. E projective $\mathcal{O}_K$ module, $<,>$ on $E \otimes_{\mathbb{Z}} \mathbb{R}$

Analogies: Function Fields

1. $[K : \mathbb{F}_p(t)] < \infty$

2. points of affine curve C with $k(C) = K$

3. $\sum_{x \in \bar{C}} \mathrm{ord}_x(f)[k(v) : \mathbb{F}_p] = 0$ - degree of a principal divisor is zero

4. E vector bundle over $\bar{C}$

The idea

V a finite dimensional vector space over $\mathbb{F}_p$:

$$\begin{aligned} \dim_p(V) &= \log_p(\#(V)) \\ [\mathbb{F}_q : \mathbb{F}_p] &= \log_p q \\ [\mathbb{F}_q : \bullet] &= \log(q) \end{aligned}$$

Here $\bullet$ is the "absolute ground field" and we think of $\mathrm{Spec}(\mathbb{Z})$ as a curve over $\bullet$!!

Exercise What is $K_*(\bullet)$?

We "compactify" $\mathrm{Spec}(\mathcal{O}_K)$ by setting $\overline{\mathrm{Spec}}(\mathcal{O}_K) =$ equal to the set of *all* valuations of K.

The "points" at ∞ on $\overline{\mathrm{Spec}}(\mathcal{O}_K)$ are simply the archimedean valuations. (This point of view goes back to Hasse.)

Adeles and Ideles

K a *global field i.e.,* $[K : \mathbb{Q}] < \infty$ or $[K : \mathbb{F}_q(t)] < \infty$.

If v = valuation of K, write K_v for the associated completion, which is a locally compact topological ring.

Adeles
$\mathbb{A}_K \subset \prod_v K_v =$

$$\{(a_v) \in \mathbb{A}_K | \#\{v | v(a_v) < 0\} < \infty\}$$

Ideles
$\mathbb{I}_K = GL_1(\mathbb{A}_K) = \{(x, y) \in \mathbb{A}_K^2 | xy = 1\}$. As a group the Ideles are simply the units in the Adeles - but with the topology coming from viewing them as $\mathbb{G}(\mathbb{A}_K)$.

The maximal compact subgroups $O_{n,v} \subset GL_n(K_v)$ are:

- $GL_n(\mathcal{O}_v)$ if v is a non-Archimedean completion of K *i.e.*, a p-adic field or a power series field.
- O_n if $K_v = \mathbb{R}$
- U_n if $K_v = \mathbb{C}$

Note All three stabilize the unit ball in K^n. In the p-adic case the unit ball is a subgroup - however not in the archimedean case!

The maximal compact subgroup of $GL_N(\mathbb{A}_K) = \prod_v O_{n,v}$

An element $\bar{E}$ of

$$GL_n(K) \setminus GL_n(\mathbb{A}_K)/(\text{maxcompact})$$

represents either:

$K =$ a number field:
A rank n projective $\mathcal{O}_K$ module E, equipped with an inner product at each archimedean place of K. $(\bar{E} = (E, (|| \, ||_v)))$

$K = K(C)$, $C/\mathbb{F}_q$ smooth affine curve:
A rank n vector bundle $\bar{E}$ on C, a smooth projective model of K, or equivalently:

A rank n projective $\mathcal{O}_K$ module E, equipped with, for each valuation v corresponding to a point at infinity.

- a free rank n sub-$\mathcal{O}_{K_v}$-module of $E \otimes_{\mathcal{O}_K} K_v$,

  or equivalently:

- a $v - adic$ norm on $E \otimes_{\mathcal{O}_K} K$

In particular,if $C/\mathbb{F}_q$ is a smooth projective curve, with function field K we have a natural isomorphism:

$$\mathrm{Pic}(C) \simeq K(C)^* \setminus \mathbb{I}_K/U_K \ .$$

This suggests:

Definition. *If K is a number field, set*

$$\mathit{Pic}(\overline{\mathrm{Spec}}(\mathcal{O}_K)) := K^* \setminus I_K/U_K \ .$$

Note that this is simply the idele class group.

Proposition. *$\mathit{Pic}(\overline{\mathrm{Spec}}(\mathcal{O}_K))$ is isomorphic to the group of isomorphism classes of pairs $(L, \{\| \ \|_\sigma\})$ consisting of:*

- *a rank 1 projective $\mathcal{O}_K$-modules L,*
- *a Hermitian inner product $\| \ \|_\sigma$ on $L \otimes_\sigma \mathbb{C}$, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, compatible with complex conjugation.*

Degrees and Volumes

E vector bundle over X - smooth projective curve over $\mathbb{F}_q$

$$\deg(E) = \deg(\det(E))$$

(Recall that determinant = top exterior power),

The map:

$$\begin{aligned} \deg : \mathrm{Pic}(X) &\to \mathbb{Z} \\ \mathcal{L} &\mapsto \deg(\mathrm{div}(s)) \end{aligned}$$

for s is a meromorphic section of $\mathcal{L}$, is induced by the map:

$$\begin{aligned} \mathbb{I}_K : &\to \mathbb{Z} \\ (a_v) : &\mapsto \sum_v \mathrm{ord}_v(a) \end{aligned}$$

If K is a number field, and $\bar{L} = (L, || \ ||)$ is a "line bundle" on Spec$(\mathcal{O}_K)$, we define its degree analogously, via

$$\begin{aligned} \mathbb{I}_K : &\ \rightarrow \ \mathbb{R} \\ (a_v) : &\ \mapsto \ \sum_v \log ||a||_v \end{aligned}$$

We have the following exact sequence:

$$\mathcal{O}_K^* \rightarrow \mathbb{R}^{r_1+r_2} \rightarrow \mathrm{Pic}(\overline{\mathrm{Spec}(\mathcal{O}_K)}) \rightarrow \mathbb{R} \oplus Cl(\mathcal{O}_K) \rightarrow 0$$

The lefthand map is the Dirichlet regulator, while the right hand map is the sum of the degree map with the map from Pic to the ideal class map which forgets the metric.

If $\bar{E} = (E, || \ ||)$ is a vector bundle on Spec$(\mathcal{O}_K)$, $\deg(\bar{E}) := \deg(\det(\bar{E}))$

Proposition. *For $\bar{E} = (E, || \ ||)$ a "vector bundle", the norms $|| \ ||_v$ induce a measure on $E_{\mathbb{R}} := E \otimes_{\mathbb{Z}} \mathbb{R}$, and*

$$\deg(\bar{E}) = -\log(vol(E_{\mathbb{R}}/E))$$

Heights again

On $\mathbb{P}^n_{\mathbb{Z}}$ the tautological line bundle $\mathcal{O}(1)$ is a quotient of $\mathcal{O}^{n+1}$, and therefore its restriction to $\mathbb{P}^n(\mathbb{C})$ inherits a hermitian inner product by orthogonal projection from the trivial bundle. Let us write $\overline{\mathcal{O}(1)}$ for this "metrized" line bundle.

Theorem. *Let $P \in \mathbb{P}^n(\mathbb{Q}) = \mathbb{P}^n(\mathbb{Z})$ be a point, and $\sigma_P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^n_{\mathbb{Z}}$ the corresponding morphism. Then*

$$h(P) = \overline{\mathit{deg}}(\sigma_P^*)\overline{\mathcal{O}(1)}$$

“Classical” Intersection of Divisors

$X/\mathbb{F}_p$ a smooth projective surface:

$$\begin{aligned} \mathrm{Pic}(X) \otimes \mathrm{Pic}(X) : &\to \mathbb{Z} \\ \mathcal{O}(C) \otimes \mathcal{O}(D) : &\to < \mathcal{O}(C), \mathcal{O}(D) > \\ &:= \deg(\mathcal{O}(C)|_D) \end{aligned}$$

Well defined because X is *projective*:
If $C - C' = \mathrm{div}(f)$:

$$\begin{aligned} < \mathcal{O}(C), \mathcal{O}(D) > &- < \mathcal{O}(C'), \mathcal{O}(D) > \\ &= \deg(\mathcal{O}(C)|_D) - \deg(\mathcal{O}(C')|_D) \\ &= \deg(\mathrm{div}(f)|_D) \\ &= 0 \end{aligned}$$

-by the product formula.

Alternative Definition

Given "prime" divisors C and D on X, *i.e.* $C \subset X$ and $D \subset X$ integral subschemes.

$$< C, D >= \sum_{i,j} (-1)^{i+j} \mathrm{dim}_{\mathbb{F}_p} H^i(X, \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_D))$$

This is really just the direct image of the product of the classes in K-theory:

$$\pi_*([\mathcal{O}_C].[\mathcal{O}_D] \in K_0(\mathbb{F}_p)) \simeq \mathbb{Z} \, .$$

where $\pi : X \to \mathrm{Spec}(\mathbb{F}_p)$.

$f : X \to C$ a family of curves over a smooth *affine* curve. Let $C \subset \bar{C}$ be a smooth compactification. A *model* of X is

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ C & \longrightarrow & \bar{C} \end{array}$$

with $\mathfrak{X}$ regular flat and projective over $\bar{C}$.

Write $\mathfrak{X}_\infty$ for (union of) fiber(s) over $\bar{C} - C$.

$\mathrm{Div}(\mathfrak{X}) = \mathrm{Div}(X) \oplus \mathrm{Div}_\infty(\mathfrak{X})$

Where $\mathrm{Div}_\infty(\mathfrak{X}) = \bigoplus_E \mathbb{Z}$,
E=component of $(\mathfrak{X})_\infty$.

Arakelov '74

Arithmetic Surface: $f : X \to \mathrm{Spec}\mathbb{Z}$ regular flat projective. Analogous to a map from an algebraic surface to an *affine* curve.

A "model" is a choice of Kähler metric on the Riemann Surface $X(\mathbb{C})$. Set:

$$\mathrm{Div}(\bar{X}) := \mathrm{Div}(X) \oplus \bigoplus_{\pi_0(X(\mathbb{C}))} \mathbb{R}$$

Definition. • *A Hermitian line bundle* $\bar{L} = (L, || \, ||)$, *where* $|| \, ||$ *is a* C^∞ *Hermitian metric on* $L|_{X(\mathbb{C})}$.

- $\bar{L}$ *is* admissible *if its curvature is harmonic w.r.t the Kähler metric* i.e. *equal to* $= a\mu$. *Here* a *is a locally constant function on* $X(\mathbb{C})$, *and* μ *is the Kähler form.*

- *If* s *meromorphic section of* L, $\overline{\mathrm{div}}(s) = div(s) + \sum_{\pi_0(X(\mathbb{C}))} a_Y$ *where* $C_1(\bar{L}) = a_Y\mu$ *for each component* Y.

There are a couple of reasons why this is a good analogy.

First of all, if we are given a smooth projective curve X over $\mathbb{Q}_p$, then a model $\mathfrak{X}$ over $\mathbb{Z}_p$ determines a metric on $X(\overline{\mathbb{Q}_p})$, in which the distance between two (distinct) points is determined by the intersection multiplicity of the corresponding sections, which may or may not meet in the special fibre. As a simple example, consider the affine line $\mathbb{A}^1_{\mathbb{Z}_p,t}$. Given two points $t = a$, and $t = b$ in $\mathbb{A}^1(\mathbb{Z}_p)$, the p-adic distance from a to b in $\mathbb{A}^1(\mathbb{Z}_p)$ is the $\text{Const}^{-v_p(a-b)}$, and it is easy to see the intersection multiplicity of the corresponding sections is $v_p(a - b)$.

Secondly, in the geometric situation, given a rational point on the generic fibre, it automatically extends to a section of (the smooth locus of) $\mathfrak{X}$ over $\mathbb{Z}$, and so the relative dualizing sheaf will pull back by this section. This in the case of an arithmetic surface over Spec($\mathbb{Z}$) we should expect that for any section of $\mathfrak{X}$ over $\mathbb{Z}$ the pull back of the relative dualizing sheaf should extend to $\overline{\text{Spec}}(\mathbb{Z})$ - *i.e.* should have an inner product. But this means that the tangent bundle to $X(\mathbb{C})$) at that point has an inner product.

Definition. *Let* $f \in K(X)^*$. $\overline{\mathrm{div}}(f) := div(f) + \sum_Y a_Y$, *where* $a_Y = -(\int_Y \log|f|)\mu_Y$

Definition. $\mathrm{Pic}(\bar{X})$ *is the group of isomorphism classes of Hermitian line bundles.*

Proposition. $\mathrm{Pic}(\bar{X}) \simeq Div(\bar{X})/\{\overline{\mathrm{div}}(f)\}$, *via the map* $\overline{\mathrm{div}}$.

Green Functions

If $\bar{X} = (X, \mu)$, and $\bar{L} = (L, ||\ ||)$ is an admissible metric, and s is a meromorphic section of L, then $\log(||s||)$ is C^∞ on $X(\mathbb{C}) - |div(s)|$.

$$\frac{1}{\pi}\partial\bar{\partial}(\log(||s||)) = \deg(\text{div}(s))\mu - \delta_{\text{div}(s)}$$

Given a divisor $D = \sum_P n_p$ on $X(\mathbb{C})$, an *admissible* Green's function is a function g_D, which is real valued and C^∞ on $X(\mathbb{C}) - |D|$, such that

$$\frac{1}{\pi}\partial\bar{\partial}(g_D) = \deg(D)\mu - \sum n_p \delta_P\ .$$

There is a canonical choice of such a function, in which $\int_{X(\mathbb{C})} g_D \mu = 0$. Also note if $D = P$ is a single point, then if z is a local parameter at P, then near P, $g_P = \log(|z|^2) + \phi$ where ϕ is C^∞.

Arakelov's Pairing

$$\begin{aligned} \mathrm{Pic}(\bar{X}) \otimes \mathrm{Pic}(\bar{X}) &\to \mathbb{R} \\ \bar{L} \otimes \bar{M} &\to <\bar{L}, \bar{M}> \end{aligned}$$

Pick s, t meromorphic sections of L and M.

$$\begin{aligned} <\bar{L}, \bar{M}> &:= <\overline{\mathrm{div}}(s), \overline{\mathrm{div}}(t)> \\ &= <\mathrm{div}(s), \mathrm{div}(t)>_f \\ &\quad + <\overline{\mathrm{div}}(s), \overline{\mathrm{div}}(t)>_\infty \end{aligned}$$

Define by linearity:

Suppose that C and D are "prime" divisors on X:

$$<C, D>_f :=$$

$$\sum_{i,j} (-1)^{i+j} \log(\#(H^i(X, \mathcal{T}\mathrm{or}_j^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_D))))$$

If Y is a component of $X(\mathbb{C})$,

$$< C, 1_Y > = < C, 1_Y >_\infty := \deg(C \cap Y) \; .$$

If Y and Z are two components of $X(\mathbb{C})$,

$$< 1_Y, 1_Z > = < 1_Y, 1_Z >_\infty := 0$$

If C and D are prime divisors on X,

$$< C, D >_\infty := \sum m_P n_Q < P, Q >_\infty$$

where $C|_{X(\mathbb{C})} = \sum m_P P$ and $D|_{X(\mathbb{C})} = \sum n_Q Q$,

Finally:

$$< P, Q >_\infty := g_P(Q)$$

Theorem. *This is well defined and symmetric.*

Lecture 2

Quick Review of Chow Groups

Definition.
Let X be a scheme:

1. $X^p = \{$*points* $x \in X$ *of codimension* p $\}$.

2. Codimension p algebraic cycles
$Z^p(X) = \bigoplus_{x \in X^p} \mathbb{Z}$

3. Codimension p K_1-chains
$R_1^p = \bigoplus_{x \in X^p} \mathbf{k}(x)^*$

4. Codimension p K_2-chains
$R_2^p = \bigoplus_{x \in X^p} K_2(\mathbf{k}(x))$

Chow groups

X a scheme,

$$CH^p(X) := \mathrm{Coker}(\mathrm{div} : R^{p-1}(X) \to Z^p(X))$$

$$R^{p-1}(X) := \bigoplus_{W \subset X} k(W)^*$$

with $W \subset X$ codim p.

- Covariant wrt **proper** morphisms

- $CH^*(X)$ is a graded ring if X is a regular variety over a field k, and contravariant for maps between such.

- If X is regular and projective, then have *intersection numbers*

- If X is a regular scheme, then we only know that $CH^*(X)_{\mathbb{Q}}$ is a ring.

Constructing the pairing

If X is regular and of finite type over a field:

- Moving Lemma (1950s)

- K-theory (Bloch's Formula) (1970s) $CH^p(X) \simeq H^p(X, K(\mathcal{O}_X)_p)$. Variations:

  - $CH^p(X) \simeq H^p(X, \mathcal{K}_p^M)$

  - Hypercohomology of Motivic Sheaves.

  All of these depend on some version of Gersten's conjecture, which can be viewed as a local moving lemma.

- Deformation to the Normal Cone (1970s). This is a purely geometric method, which uses reduction to the diagonal, and therefore depends on X being smooth.

If X is an arbitrary regular scheme:

- $CH^*(X)_{\mathbb{Q}} \simeq Gr^*_\gamma K_O(X)_{\mathbb{Q}}$

Arithmetic variety

$\pi : X \to \mathbb{Z}$ flat, projective over $\mathbb{Z}$, equidimensional and regular.

$X(\mathbb{C})$ is a complex manifold.

$A^{(p,p)}(X) := A^{(p,p)}(X(\mathbb{C}))^{F_\infty=(-1)^p}$

$D^{(p,p)}(X)$= similar space of currents.

If $\zeta = \sum_i n_i Z_i \in Z^p(X)$ then:

$$\delta_\zeta = \sum_i n_i \delta_{Z_i(\mathbb{C})} \in D^{(p,p)}(X)$$

Currents

Recall that if M is a manifold, a *current* $T \in D_p(M)$ is a continuous linear functional $T : A^p_c(M) \to \mathbb{C}$ on the space of compactly supported currents, which we equip with the topology determined by the sup-norms of partials of coefficients in local coordinate charts.

Given an orientation on M,

$$\begin{array}{rcl} A^{n-p} & \hookrightarrow & D_p \\ \alpha & \mapsto & (\omega \mapsto \int_M \alpha \wedge \omega) \end{array}$$

This is a dense embedding - currents = "very singular forms". The exterior derivative on forms extends to currents.

$$dT(\alpha) = -(-1)^{\mathrm{deg}(T)} T(d\alpha) \ .$$

More generally, any locally L^1 form ϕ defines a current $[\phi]$. *But*, $d[\phi] \neq [d\phi]$ -the difference is a "residue".

Green Currents

If X is a complex manifold of dim n, $Z \subset X$ a codim p complex analytic subspace, then:

$$\begin{aligned} \delta_Z : A^{2n-2p} &\rightarrow \mathbb{C} \\ \omega &\mapsto \int_Z \omega \end{aligned}$$

Viewed as a "form", $\delta_Z \in D^{p,p}(X)$.

Definition. *A Green current for Z is an (equivalence class of) $g_Z \in D^{p-1,p-1}(X)$ such that*

$$dd^c(g_Z) - \delta_Z = \omega$$

where ω is a C^∞ (p,p)-form.

$g_Z \simeq g'_Z$ *if* $g_Z - g'_Z = \partial(u) + \bar{\partial}(v)$

Note $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$, where $d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$

Analogies: Arithmetic Varieties

1. $f : X \to \mathrm{Spec}(\mathbb{Z})$, flat projective and regular.

2. no well defined intersection numbers

3. $\bar{X} = (X, \omega)$ - K aehler metric

4. $CH^p(\bar{X}) \otimes CH^q(\bar{X}) \to \mathbb{R}$ but $CH^*(\bar{X})$ not a ring (Arakelov '72 for surfaces)

5. $\widehat{CH}^p(X)$ (Soulé & H.G.)

Analogies: Varieties over Function field

1. $f : X \to \mathrm{Spec}(k[t])$, projective and smooth

2. no well defined intersection numbers

3. $\bar{X} \to \bar{C}$ compactification.

4. $CH^p(\bar{X}) \otimes CH^q(\bar{X}) \to \mathbb{Z}$

5. $\widehat{CH}^p(X) = \varinjlim_{\mathfrak{X}} CH^p(\mathfrak{X})$ - over *all* compactifications (Bloch, Soulé, HG)

Definition of $\widehat{CH}^p(X)$

$$\widehat{CH}^p(X) = \frac{\widehat{Z}^p(X)}{\{\widehat{\mathrm{div}}(f) \mid \{f\} \in R^{p-1}(X)\}}$$

Where:

$$\widehat{Z}^p(X) = \{(Z, g_Z) | Z \in Z^p(X), g_Z \in \frac{\mathcal{D}^{p-1,p-1}(X(\mathbb{C}))}{\mathrm{Im}(\partial) + \mathrm{Im}(\overline{\partial})}\}$$

Where g_Z is a Green's current for Z.

And:

$f \in k(W)^*$, $W \subset X$ codimension $p - 1$,

$$\log |f|^2 : \alpha \to \int_{W(\mathbb{C})} \log |f|^2 \alpha|_W$$

If $f = (f_W) \in \bigoplus_{W \subset X} k(W)^*$ is a K_1-chain, then

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), \sum_W \log |f_W|^2)$$

Some Natural Maps

$$\begin{aligned} \omega : \widehat{CH}^p(X) &\rightarrow Z^{p,p} \subset A^{p,p} \\ (Z, g_Z) &\mapsto dd^c g_Z + \delta_Z \end{aligned}$$

$$\begin{aligned} \zeta : \widehat{CH}^p(X) &\rightarrow CH^p(X) \\ (Z, g_Z) &\mapsto Z \end{aligned}$$

Notation:

- $H^{p,p}(X) := H^{p,p}(X(\mathbb{C}))^{F_\infty = (-1)^p}$

- $Z^{p,p}$ = closed (p,p)-forms

- $CH^{p,p-1}(X) := \dfrac{\text{Ker(div)}: R_1^{p-1}(X) \rightarrow Z^p(X)}{\text{Im(Tame)}: R_2^{p-2}(X) \rightarrow R_1^{p-1}(X)}$

  Same as $CH^{2p-1}(X, p)$

We also have maps $a : H^{p,p}(X) \rightarrow \widehat{CH}^p(X)$, $\alpha \mapsto (0, \alpha)$, and $\rho : CH^{p,p-1}(X) \rightarrow H^{p,p}(X)$, $\sum_W f_W \mapsto \sum_W \log |f_W|^2$.

An exact sequence

These maps form an exact sequence:

$$
\begin{aligned}
&CH^{p,p-1} \to H^{p-1,p-1}(X) \to \\
&\qquad \widehat{CH}^p(X) \to CH^p(X) \oplus Z^{p,p} \to H^{p,p}(X)
\end{aligned}
$$

Here we use that on a compact Kähler manifold:

$A^{p-2,p-1} \oplus A^{p-1,p-2} \xrightarrow{\partial + \bar\partial} A^{p-1,p-1} \xrightarrow{\partial\bar\partial} A^{p,p}$
has cohomology $H^{p-1,p-1}(X)$.

The map $\rho : CH^{p,p-1} \to H^{p-1,p-1}(X)$ is (Const)$\times$ Beilinson regulator.

Examples

$\widehat{CH}^0(X) = CH^0(X)$

$\widehat{CH}^1(\mathrm{Spec}(\mathbb{Z})) = \mathbb{R}$

$\widehat{CH}^p(X) = 0$ if $p > \dim(X)$.

If $p = \dim(X)$, then the exact sequence above becomes, since $\dim(X(\mathbb{C})) = p - 1$,

$$\begin{aligned} CH^{p,p-1} \to H^{p-1,p-1}(X) \to & \\ \widehat{CH}^p(X) \to CH^p(X) &= CH_0(X) \to 0 \end{aligned}$$

Desiderata

- Products
- Pullbacks
- Pushforward
- Chern Classes

Products

$\eta = (Y, g_Y) \in \widehat{CH}^q(X)$, $\zeta = (Z, g_Z) \in \widehat{CH}^p(X)$

Using the moving lemma for $X_{\mathbb{Q}}$, we can choose Y and Z so that they meet properly on $X_{\mathbb{Q}}$, but:

$|Y| \cap |Z|$ may have components of codimension $> p + q$,

$Y.Z :=$
$[\mathcal{O}_Y].[\mathcal{O}_Z] \in Gr_{\gamma}^{p+q} K_O^{Y \cap Z}(X)_{\mathbb{Q}} \simeq CH_{Y \cap Z}^{p+q}(X)_{\mathbb{Q}}$.

$\eta.\zeta := (Y.Z, g_Y * g_Z)$, where

$$g_Y * g_Z := g_Y.\delta_Z + \omega_Y.g_Z \ .$$

Difficulty - show that $g_Y.\delta_Z$ makes sense. Multiplying currents is generally hard - for example what might $\delta_{\{0\}} \cdot \delta_{\{0\}}$ for $\{0\} \in \mathbb{R}$, mean?

In this case, blow up Y to divisor with normal crossings, and represent g_Y by a form having singularities like $\log |z|$ near the exceptional divisor. Then define $g_Y \delta_Z := g_Y|_{Z(\mathbb{C})}$. The problem is that this is no longer a current associated to a locally L_1-form.

A better approach, by Burgos, will be sketched later, which allows one to stick to forms all throughout the discussion.

For the $*$-product described above, one must show that it is:

- Associative and commutative.

- Respects rational equivalence.

This requires a careful analysis what of what happens on divisors with normal crossings.

Pull-backs

$f : X \to Y$ a map of arithmetic varieties.

If $\zeta \in \widehat{CH}^p(Y)$, by the moving lemma one may represent $\zeta = (Z, g_Z)$ with $Z_{\mathbb{Q}}$ meeting f properly

$\Rightarrow f^*(Z) \in CH_*(f^{-1}|Z|)_{\mathbb{Q}}$ restricts to an cycle on $X_{\mathbb{Q}}$, not just a cycle class, and if g_Z is represented by a C^∞ form with log growth along Z, $f^* g_Z$ is a Green form for Z.

Note: Can remove the $\mathbb{Q}$ here by using deformation to the normal cone.

Push Forwards

If $f : X \to Y$ restricts to a *smooth* map $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$, then if $(Z, g_Z) \in \widehat{CH}^*(X)$, $f_*(g_Z)$ is automatically a Green's form for the cycle $f_*(Z)$, so

$$f_*(Z, g_Z) := (f_*(Z), f_*(g_Z))$$

If $\dim(X) = n$, there is a push forward

$$\widehat{\deg} : \widehat{CH}^n(X) \to \widehat{CH}^1(\mathrm{Spec}(\mathbb{Z})) \simeq \mathbb{R}$$

$$(\sum_P n_P P, g_Z) \mapsto \sum_P n_P \log(\#(\mathbf{k}(P))) - \int_{X(\mathbb{C})} g_Z$$

Note We did not choose a hermitian metric on $X(\mathbb{C})$ - the $\widehat{CH}^*$ are analogous to:

$f : X \to C$ a flat projective family of varieties, with X regular. Then as before, consider models

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ C & \longrightarrow & \bar{C} \end{array}$$

Definition. $\widehat{CH}^p(X) := \lim_{\to} CH^p(\mathfrak{X})$

This was developed by Bloch, Soulé & H.G. in J. Alg. Geom.

Height of a Projective Variety

$X \subset \mathbb{P}^n_{\mathbb{Q}}$ a projective variety - not necessarily smooth - dimension $= d$.

Given $\alpha \in \widehat{CH}^{d+1}(\mathbb{P}^n_{\mathbb{Z}})$.

Define $h_\alpha(X) := \widehat{\deg}(\alpha|_{\bar{X}})$

where $\bar{X} \subset \mathbb{P}^n_{\mathbb{Z}}$ is the Zariski closure.

Even if Z is singular this makes sense:

Represent $\alpha = (Y, g_Y)$, with $Y \cap X \subset \mathbb{P}^n_{\mathbb{Q}}$ empty.

$h_\alpha(X) =< \bar{X}, Y >_f + \int_{X(\mathbb{C})}$ where

$< \bar{X}, Y >_f = \sum_P n_P \log(\#(\mathbf{k}(P)))$ if we represent $Y.\bar{X} = \sum_P n_P P$

There are natural choices for α - $\widehat{C}_1(\mathcal{O}(1))^{d+1}$ or (L, g_Z) with L a linear subspace and g_L an anti-harmonic Green form.

Geometric Heights $X \subset \mathbb{P}^n_{K(C)}$ a variety projective over $K(C)$ of codimension p, with Zariski closure $\mathcal{X} \subset \mathbb{P}^n \times C$ of dimension d

Then define ht(X)= intersection number of $\mathcal{X}$ with $C_1(\mathcal{O}(1))^{n-p+1}$

This measure how "horizontal" $\mathcal{X}$ is.

If $C = \mathbb{P}^1$, then

$$CH^*(\mathbb{P}^n \times \mathbb{P}^1) \simeq \mathbb{Z}[t, \epsilon]/(t^{n+1} = \epsilon^2 = 0) \ ,$$

where t and ϵ are the class of a divisor on $\mathbb{P}^n$ and $\mathbb{P}^1$, respectively. Then

$$[\mathcal{X} = \deg(X) t^p + \mathrm{ht}(X) t^{p-1} \epsilon]$$

Lecture 3

Characteristic Classes and Heights

Heights of Divisors

$X \subset \mathbb{P}^n_{\mathbb{Q}}$ a divisor of degree d.

Choose equation: $f(x_0, \ldots, x_n) \in \mathbb{Z}[x_0, \ldots, x_n]$, coprime coeffs.$\Rightarrow$ f is equation of $\bar{X} \subset \mathbb{P}^n_{\mathbb{Z}}$.

Let $\gamma := \widehat{C}_1(\overline{\mathcal{O}(1)}) = (H, -\log||x_0||^2)$. So $\gamma^n = (\bar{P} = (1:0:\ldots:0), g_P)$,

$$\begin{aligned} h(X) &= \widehat{\deg}(\gamma^n|_{\bar{X}}) \\ &= <\bar{X}, \bar{P}>_f + \int_X g_P \\ &= <\bar{X}, \bar{P}>_f + \int_{\mathbb{P}(\mathbb{C})^n} \delta_X g_P \\ &= <(\bar{X}, g_P), \gamma^n> - \int_{\mathbb{P}(\mathbb{C})^n} g_X \omega_p \end{aligned}$$

But:
We can pick g_X so

$$(X, g_X) = \widehat{C}_1(\overline{O(d)}) = d\gamma \ ,$$

and ω_P=volume form on $\mathbb{P}(\mathbb{C})^n$. So:

$$\begin{aligned} h(X) &= d \cdot \widehat{\deg}(\gamma^{n+1}) + \int_{\mathbb{P}(\mathbb{C})^n} \log ||f||^2 \omega \\ &= dh(\mathbb{P}_{\mathbb{Q}}{}^n) + \int_{S^{2n+1}} \log |f|^2 \mu \end{aligned}$$

One may easily compute that $dh(\mathbb{P}_{\mathbb{Q}}{}^n) \in \mathbb{Q}$.

So $\int_{S^11} \log |z_0z_1 + z_2z_3 + z_4z_5|$ is (up to a rational number)

$$h(z_0z_1 + z_2z_3 + z_4z_5 = 0) = G(2,4) \subset \mathbb{P}^5$$

This height is sometimes referred to as the Faltings height. If instead we take the height with respect to the class (L, g_L) of a linear subspace together with an anti-harmonic Green current L, rather than $\widehat{C}_1(\overline{\mathcal{O}(1)})$, then the term $dh(\mathbb{P}_{\mathbb{Q}}{}^n) \in \mathbb{Q}$ is zero.

To compute the height of $\mathbb{G}(k,n)$, it is enough to compute arithmetic intersection numbers on $\mathbb{G}(k,n)$ itself, and since the curvature of $\overline{\mathcal{O}(1)}|_{\mathbb{G}(k,n)}$ is harmonic, we work in

$$CH^*(\overline{\mathbb{G}(k,n)}) \subset \widehat{CH}\mathbb{G}(k,n)$$

the *subring* where all $\omega_Z = dd^g_Z + \delta_g$ are harmonic.

Lemma. *Have exact sequences:*

$$0 \to H^{n-1,n-1}(\mathbb{G}(k,n)) \to CH^n(\overline{\mathbb{G}(k,n)}) \to CH^*(\mathbb{G}(k,n)) \to 0$$

Proof Look at

$$CH^{p,p-1} \to H^{p-1,p-1}(X) \to \widehat{CH}^p(X) \to CH^p(X) \oplus Z^{p,p} \to H^{p,p}(X)$$

Tamvakis computed the product structure, to get:

Theorem. (Tamvakis, Maillot and Cassaigne)

Heights of Grassmannians are rational.

Tamvakis also computes $\widehat{CH}$ for variety of symmetric spaces.

Maillot and Cassaigne's earlier proof:

Form the Zeta function: $Z(s) = \int_{S^{2n+1}} |f|^{2s}\mu$.

$$\int_{S^{2n+1}} \log |f|^2 \mu = Z'(0)$$

Special values of zeta function at positive integers can be computed, and determine zeta function - equal to expression involving Γ-functions

Beilinson-Bloch Height Pairing

X= smooth projective variety over $\mathbb{Q}$of dimension n. $\eta \in CH^p(X)$, $\zeta \in CH^q(X)$, $p+q = n-1$.

Suppose η and ζ are homologically ~ 0. Have height pairing: $< \eta, \zeta > \in \mathbb{R}$:

Choose (assume resolution) regular model $\mathfrak{X}$. Represent η and ζ by cycles Y, Z on $\mathfrak{X}$. Choose Green currents g_Y, g_Z such that $dd^c g_Y + \delta_Y = 0$.

Then $< \eta, \zeta > = \widehat{\deg}((Y, g_Y), (Z, g_Z))$.

Note: If $\mathfrak{X}$ has singular fibres over Spec$\mathbb{Z}$, cannot not just take Zariski closure to get Y and Z, but must add cycles supported in singular fibres.

Adding such cycles analogous to finding g_Y with $\omega_Y = 0$.

Burgos' approach to Green currents

Let W be a complex algebraic manifold and $D \subset W$ a normal crossing divisor in. Write $j : X = W \setminus D \to W$. the natural inclusion. $\mathcal{A}^*_W$= sheaf of smooth, complex forms

$\mathcal{A}^*_W(logD)$ is the sheaf of differential forms with logarithmic singularities along D is the $\mathcal{A}^*_W$-subalgebra of $j_*\mathcal{A}^*_X$, which is locally generated by the sections $\log(z_i\bar{z}_i)$,$\frac{dz_i}{z_i}$, $\frac{d\bar{z}_i}{\bar{z}_i}$ for $i = 1, \ldots, m$, where $z_1 \cdots z_m = 0$ is a local equation for D.

The complex of differential forms with logarithmic singularities along infinity is defined by $A^*_{\log}(X) = \lim_{\rightarrow} A^*_{\bar{X}_\alpha}(\log D_\alpha)$; it is a subalgebra of $A^*(X)$ - the direct limit over all compactifications.

$A^*_{\mathbb{R},\log(X)}$= real forms and

$$A^*_{\mathbb{R},\log(X)}(p) = (2\pi i)^p A^*_{\mathbb{R},\log(X)}$$

Real Deligne Cohomology

Recall that real Deligne cohomology fits into an exact sequence:

$$\frac{H^{n-1}(X,\mathbb{C})}{F^pH^{n-1}(X,\mathbb{C})+H^{n-1}(X,\mathbb{R}(p))}$$

$$\to H^n_{\mathcal{D}}(C,\mathbb{R}(p)) \to H^n(X,\mathbb{R}(p))$$

$$\to \frac{H^n(X,\mathbb{C})}{F^pH^n(X,\mathbb{C})}$$

Consider the complex

$$\mathcal{D}^n_{\log}(X,p) =$$

$$\begin{cases} A^{n-1}_{\mathbb{R},\log(X)}(p-1)\cap F^{n-p,n-p}A^{n-1}_{\mathbb{C},\log(X)}, & n\le 2p-1; \\ A^{n}_{\mathbb{R},\log(X)}(p)\cap F^{p,p}A^{n}_{\mathbb{C},\log(X)}, & n\ge 2p. \end{cases}$$

The differential is the projection of the usual exterior derivative, except from degree $2p-1$ to $2p$, when it is dd^c

The complex $\mathcal{D}^n_{\log}(X,p)$:

$$
\begin{tikzcd}
 & & & {} \arrow[d, dash, dashed] & {} \arrow[d, dash, dashed] & \\
 & & & A^{p,p+1}(p) \arrow[r, "\partial"] & A^{p+1,p+1}(p) \arrow[r, dash, dashed] & {} \\
 & & & A^{p,p}(p) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & A^{p+1,p}(p) \arrow[u, "\overline{\partial}"] \arrow[r, dash, dashed] & {} \\
 {} \arrow[r, dash, dashed] & A^{p-2,p-1}(p-1) \arrow[r, "\partial"] & A^{p-1,p-1}(p-1) \arrow[ur, "dd^c"] & & & \\
 {} \arrow[r, dash, dashed] & A^{p-2,p-2}(p-1) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] \arrow[d, dash, dashed] & A^{p-1,p-2}(p) \arrow[u, "\overline{\partial}"] \arrow[d, dash, dashed] & & & \\
 & {} & {} & & &
\end{tikzcd}
$$

Theorem. (Burgos) *There is a natural multiplicative isomorphism*

$$H^*_{\mathcal{D}}(C, \mathbb{R}(p)) \to H^*(\mathcal{D}_{\mathrm{log}}(X, p)) .$$

Truncated relative cohomology groups: This is a general procedure for constructing groups like $\widehat{CH}$ or the differential characters of Cheeger and Simons.

Definition. *Let* $f : (A, d_A) \to (B, d_B)$ *be a morphism of complexes. The associated truncated relative cohomology groups are:*

$$\begin{aligned} \widehat{H}^n(A, B) &= H^n(A, \sigma_n B) \\ &= \{(a, \tilde{b}) \in Z^n(A) \oplus \tilde{B}^{n-1} | f(a) = d_B b\} \end{aligned}$$

Here $\tilde{B}^{n-1} = B^{n-1}/Im(d_B)$, *and* $(\sigma_n B) = 0$ *if* $i > n$

Definition. *The group of codimension p arithmetic cycles*

$$\widehat{Z}^p(X) = \{(Y, g_Y) \in \widehat{Z}^p(X) \oplus \widehat{H}^{2p}_{\mathcal{D},Y}(X) | cl(g_Y) = [Y]\}$$

Where

$$\widehat{H}^{2p}_{\mathcal{D},Y}(X) = \widehat{H}^{2p}(\mathcal{D}^*_{\log}(X, p), \mathcal{D}^*_{\log}(X - Y, p))$$

I.e. A Green form is $g_Y \in A^{p-1,p-1}_{\mathbb{R},\log(U)}(p-1)$ such that $\omega_Y = dd^c(g_Y)$ is C^∞ on X, and the pair (ω_Y, g_Y) represents the cycle class of Y in

$$H^{2p}_{\mathcal{D},Y}(X, \mathbb{R}(p)) \simeq \bigoplus_{Y^{(0)}} \mathbb{R}$$

This eliminated currents, and allows the product of Green currents to be defined cohomologically.

Hermitian Vector Bundles

Definition. *Let X be an arithmetic variety. A Hermitian bundle $\bar{E} = (E, || \, ||)$ on X, is a vector bundle E on X, plus a C^∞ Hermitian metric $|| \, ||$ on $E|_{X(\mathbb{C})}$ which is compatible with complex conjugation.*

Given $\bar{E}$, one has Chern classes:

- $C_p^{\text{Chow}}(E) \in CH^p(X)$
- $C_p^{\text{Dolb}}(E) \in H^{p,p}(X(\mathbb{C}))$

These agree via the cycle class map $CH^p(X \to H^{p,p}(X(\mathbb{C}))$, by axioms for Chern classes.

Metric on E
$\Rightarrow$ *canonical choice* $C_p(E,h) \in A^{p,p}$ representing $C_p^{\text{Dolb}}(E)$.

M= complex manifold. E= holomorphic vector bundle, equipped with C^∞ Hermitian metric.

There is a unique connection $\nabla : A(E) \to A^1(E)$, such that $\nabla^{0,1} = \bar{\partial}$ and such that parallel transport is unitary.

Curvature $R := \nabla^2 = [\nabla^{0,1}, \nabla^{1,0}]$. This is a linear map $E \to A^{1,1}(E)$.

$$\det(tR - I_E) = \sum_p \pm t^p C_p(E, h)$$

Since R depends "explicitly" on h, if h_0 and h_∞ are two metrics, Bott and Chern constructed forms such that $\tilde{C}_p(E, h_0, h_\infty)$:

$$dd^c(\tilde{C}_p(E, h_0, h_\infty)) = C_p(E, h_0) - C_p(E, h_\infty)$$

If E is a line bundle, then $R = \bar{\partial}\partial(\log(h))$, and $\tilde{C}_1(E, h, h')) = \pm \log(h/h'))$.

Note that h/h' is C^∞ on M.

Write h_i, $i = 0, \infty$ for the two metrics. If $\pi : M \times \mathbb{P}^1 \to M$. $\tilde{h}$, choose metric on $\pi^* E$, $\tilde{h}|_{M \times \{i\}} = h_i$.

$$\tilde{C}_p(E, h_0, h_\infty) = \int_{\mathbb{P}^1} C_p(\pi^*, \tilde{h}) \log |z|$$

More generally, if

$$\mathcal{E} := \ 0 \to (E', h') \to (E; h) \to (E'', h'') \to 0$$

is an exact sequence of bundles, (not necessarily respecting metrics), there are classes:

$$dd^c(\tilde{C}_p(\mathcal{E})) = C_p(E, h) - C_p((E', h') \oplus (E'', h''))$$

Constructed using a bundle $(\tilde{E}, \tilde{h})$ on $M \times \mathbb{P}^1$, restricting to (E, h) at $\{0\}$, and $(E', h') \oplus (E'', h'')$ at ∞.

Characteristic Classes Recall axiomatic approach to Chern classes, via splitting principal - they are determined by:

- Line Bundles: $C_*(L) = 1 + C_1(L))$, and $C_1(L \otimes M) = C_1(L) \oplus C_1(M)$.

- If $0 \to E' \to E \to E'' \to 0$ is exact, then $C_*(E) = C_*(E')C_*(E'')$

(We assume functoriality also.)

Given a bundle E over X, there is a variety $\pi : \mathbb{F}(E) \to X$, the flag bundle, such that $CH^*(X) \hookrightarrow\hookrightarrow CH^*(\mathbb{F}(E))$ is injective, and $\pi^*(E)$ has a maximal flag.

How to construct $\widehat{C}_*(E,h)$?
Can we construct natural transformations which satisfy:

Line Bundles: $\widehat{C}_1 : \widehat{\text{Pic}}(X) \simeq \widehat{CH}^1(X)$ - isomorphism classes of Hermitian holomorphic line bundles, via:

$$(L, h) \mapsto (\text{div}(s), -log||s||_h^2)$$

Exact Sequences: Given

$$\mathcal{E} := \ 0 \to (E', h') \to (E; h) \to (E'', h'') \to 0$$

$$\widehat{C}_*(E, h) = \widehat{C}_*(E', h')\widehat{C}_*(E'', h'') + (0, \tilde{C}_*(\mathcal{E}))$$

Compatibility Compatible with Chern classes in Chow groups, and Chern-Weil forms: $\widehat{C}_p(\bar{E}) = (Z, g)$, where Z is algebraic cycle representing p-th Chern class, $dd^c(g) + \delta_g = C_p(E, h)$

Hard to use splitting principal, since $\widehat{CH}^*$ is *NOT* homotopy invariant, and so the $\widehat{CH}$ groups of Grassmannians are so easy to compute. However, with care this can be done - (by Elkik - using Segre classes, and more recently Tamvakis).

Original method: Pull bundles back from Grassmannians. $\mathbb{G}(d,n) =$ rank d quotients of trivial rank n bundle. Then use the fact that the natural map induced by the direct sum

$$\mathbb{G}(d_1,n_1) \times \mathbb{G}(d_2,n_2) \to \mathbb{G}(d,n)$$

for $d_1 + d_2 = d$ and $n_1 + n_2$ induces an isomorphism on $CH^*(\overline{\mathbb{G}(d,n)})$ in low degrees.

Definition. *X an arithmetic variety. Then $\widehat{K}_0(X)$ is generated by classes of triples $(E, || \, ||, \gamma)$ where $(E, || \, ||)$ is a Hermitian bundle, and $\gamma \in \oplus_p \tilde{A}^{p,p}(X)$*

Modulo the relation, if

$$\mathcal{E} := 0 \to (E', h') \to (E; h) \to (E'', h'') \to 0$$

is an exact sequence then:

$$[(E; h, 0)] - [(E', h', 0)] - [(E'', h'', 0)] = [(0, 0, \tilde{ch}(\mathcal{E}))]$$

Definition. $\hat{ch}(E, h, \gamma) = \hat{ch}(E, h) + (0, \gamma) \in \widehat{CH}^*(X)_{\mathbb{Q}}$

Theorem. *This is an isomorphism.*

Push-Forwards and Quillen Metric

The $\widehat{K}_0(X)$ are clearly contravariant functors in X. What about pushforward?

Fix a “Kähler” metric on the relative tangent bundle.

Let $f : X \to Y$ be a map of arithmetic varieties. (=regular, flat, projective over $\mathbb{Z}$) such that $f : X\mathbb{Q} \to \mathbb{Q}$ is smooth. Suppose that (E, h) a Hermitian bundle on X, and $R^i f_*(E) = 0$ if $i > 0$.

$\Rightarrow$ f_*E is a bundle and $f_*[E] = [f_*E]$.

Choose metric h' on f_*E - then have natural choice of γ such that the Chern character form of $[f_*E, h', \gamma]$ is *exactly equal* to

$$\int_f ch(E, h) Td(X/Y, h_{X/Y})$$

Constructed using Analytic Torsion of the Laplace Operator of E.

Lecture 4

Analytic Torsion

Let

$$E^* = 0 \to E^0 \to \ldots \to E^i \to \ldots E^n \to 0$$

be a bounded complex of f.d. vector spaces over a field k, or more generally f.g. free or projective modules over a ring k.

Definition. • *If V is an n-dimensional vector space, det$(V) = \bigwedge^n(V)$ is the determinant (line) of V.*

- *det$(E^*) := \bigotimes_i$ det$(E^i)^{(-1)^i}$*

- *If E^i is zero, its determinant is k.*

If $k = \mathbb{R}$ or $\mathbb{C}$ and the E^i have inner products, then so does det(E^*). Similarly if $k = \mathbb{Q}_p$, and $E^i = M_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with M_i free $\mathbb{Z}_p$-modules, the det(E^*) contains a rank one $\mathbb{Z}_p$ module. Equivalently, if the E^i have p-adic norms, then so does det(E^*).

E^* is acyclic $\Rightarrow$ det$(E^*) \simeq k$, more generally, if $f^* : E^* \to F^*$ is a quasi-iso, det$(E^*) \simeq$ det(F^*). If in addition, the E^i are Euclidean, or have $\mathbb{Z}_p$-structures, what is $||1||$?

Proposition. *Suppose that the E^i are Euclidean. Then*

$$||1||^{-2} = \prod det(\Delta_i)^{(-1)^i i}$$

where $\Delta_i = d^{i*}d^i + d^{i-1}d^{i-1,*}$

If $k = \mathbb{Q}_p$, and the E_i have p-adic norms, then the same is true, with d^* the transpose of d with respect to any choice of $\mathbb{Z}_p$-bases compatible with the norms.

We can extend this construction to the Dolbeault complex which computes $R^i f_*$. Let Δ_i= Hodge Laplacian on $(0, i)$-forms with coefficients in E. Using eigenvalues λ_n of Δ_i, form the zeta function:

$$\zeta(s) = \sum_n \frac{1}{\lambda_n^s}$$

$$\det(\Delta_i) := \exp(-\zeta'(0))$$

Note that if one ignores questions of convergence, then formally $\exp(-\zeta'(0)) = \prod_n \lambda_n$. This is known as the zeta-regularization of the product. Thus for example, one has (in the sense of zeta-regulariztion) that, $\infty! = \sqrt{2\pi}$.

Quillen Metric

Definition. *Given $f : X \to Y$ as before, and a bundle E on X, set $\lambda(E) :=$ det $(Rf_*(E))$.*

Makes sense because locally on Y, $Rf_*(E)$ is quasi-iso to a bounded complex of bundles.

Now set suppose that $R^i f_*(E) = 0$ if $i > 0$. The the bundle $f_*(E)$ can be identified with the harmonic forms - this puts a metric on $\lambda(E)$. Now multiply by the analytic torsion - this is the Quillen metric.

This extend to all bundles, and puts a C^∞ metric $|| \, ||_Q$ on $\lambda(E)$.

Theorem. (Bismut, G, Soulé)

$$C_1(\lambda(E), || \, ||_Q) = \left[\int_f ch(E, || \, ||) Td(T_f, || \, ||) \right]^{(1,1)}$$

I.e., Riemann-Roch is true at the level of *forms*.
Question Does this lift to an equality in $\widehat{CH}$?

Arithmetic Riemann Roch (Soulé- HG, Inv 1992 + unpublished, Bismut-Lebeau)

$\widehat{K}_0(X) := (E, h, \beta)/\sim$

Given $f : X \to Y$ proper map of arithmetic varieties, smooth on generic fibers, then can construct a direct image map: $\widehat{K}_0(X) \to \widehat{K}_0(Y)$. Definition uses (higher) analytic torsion, and depends on choice of Kähler structure on the fibration.

"Naive" Riemann Roch fails - can see this by computing the zeta function of $\mathbb{P}^1$

Must modify $\widehat{Td}$ using the R-genus. This is a purely topological characteristic class, determined by series:

$$R(X) = \sum_{m \in 2\mathbb{N}+1} [2\zeta'(-m) + \zeta(-m)\mathcal{H}_m] \frac{X^m}{m!}$$

Arithmetic Riemann Roch (more accurately, the result on Chern forms) can be used to show:

Theorem. *Suppose that* $\bar{E} = (E, h)$ *is a Hermitian bundle on the arithmetic variety* X*, and that* $\bar{L} = (L, h)$ *is a Hermitian Line bundle with* L *ample, and* $\widehat{C}_1(\bar{L})$ *"positive". Then* $E \otimes L^{\otimes n}$ *has lots of sections with sup-norm less than 1.*

Remark Proof uses Minkowski's theorem, which is the $\overline{\mathrm{Spec}(\mathbb{Z})}$-analog of Riemann's theorem. (*I.e.*, Riemann-Roch with Roch.)

Why introduce R-genus - why not try to prove that there is a direct image for bundles for which RR is true without modification?

Done by Zha, U. Chicago thesis.

But, in fact the determinant of the Laplacian is something that one does want to compute - e.g. Kronecker limit formula.

Heights again: Canonical or Faltings height of an Abelian Variety

A an abelian variety of dimension g, defined over $\mathbb{Z}$.
$\bar{\omega_A} := H^0(A, \Omega^g_{A/\mathbb{Z}})$, this is a line bundle on Spec($\mathbb{Z}$) with inner product:

$$||\alpha|| = \frac{i^{g^2}}{(2\pi)^g} \int_{A(\mathbb{C})} \alpha \wedge \bar{\alpha}$$

and Falings defined: $h_F(A) = \widehat{\deg(\bar{\omega_A}}$

Bost (1994) showed that this is essentially the same as the height of A w.r.t the canonical line bundle with an invariant metric.

For an elliptic curve with complex multiplication, this is essentially a period of the elliptic curve: *i.e.* the integral of a holomorphic 1-form around a generator of H_1.

Conjecture of Colmez (Annals, '93)

A an abelian variety with complex multiplication by a field K,

$$\frac{1}{\dim(A)} h_F(A) = \sum_{\chi} \frac{L'(\chi,0)}{L(\chi,0)} + \log(f_\chi)$$

Here χ runs through odd characters of the Galois group of K over $\mathbb{Q}$.

Colmez proves the conjecture when $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.

Conjecture of Maillot and Roessler

A, K as above. For all $n \geq 1$, we have:

$$\sum_{\sigma} \widehat{ch}^{n}(\bar{\Omega}_{A,\sigma} \oplus \bar{\Omega}^{\vee}_{A^{\vee},\sigma})\chi(\sigma)$$
$$= -((\frac{L'(\chi, 1-n)}{L'(\chi, 1-n)} + \mathcal{H}_{n-1}) \sum_{\sigma} ch^{n-1}(\Omega_{A,\sigma}))\chi(\sigma)$$

$\sigma =$ embeddings of $\mathcal{O}_K$ in $\mathbb{C}$ (roughly). χ is a non-trivial character of the Galois group of K.

This generalizes and refines the conjecture of Colmez. Maillot and Roessler proves their conjecture in the abelian case. Also hypersurfaces - new cases.

Idea: express RHS as an index - the equality then becomes an index theorem.

Where do L-functions come in?

Use group action from Complex Multiplication - get action of finite cyclic group - apply Lefschetz-Riemann-Roch Theorem of Köhler and Roessler, which is the "Arakelov" analog of a theorem of Baum, Fulton and Quart. Allows you to localize on the fixed point set.

R-genus is replaced by equivariant R-genus, is defined in part by replacing the Riemann zeta function by the Lerch zeta function, where $\zeta \in \mathbb{C}$ has absolute value 1.

$$L(\zeta, s) := \sum_k \frac{\zeta^k}{k^s}$$

Themes

- Grassmanians and other combinatorial varieties have rational height. The "conjecture" is that a variety has rational height iff it is defined over $\bullet$.

- Heights of *arithmetic varieties* with "many automorphisms" such as Abelian Varieties with Complex Multiplication have heights which are given by special values of Gamma functions and L-functions. (Results and conjectures of: Kronecker, Chowla-Selberg, Gross-Deligne, Colmez, Maillot-Roessler.)

Other topics that could have been covered

- Connection with Deformation to the normal cone. (Work of Hu)

- Arithmetic Bézout theorem (Bost, G,Soulé)

- Semi-stable bundles and geometry of numbers (Soulé)