Calculus with infinitesimals

Isaac Goldbring

UCLA

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Continuity

Suppose $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$.

Scary Definition

f is *continuous at* c if, for every $\epsilon > 0$, there is $\delta > 0$ such that whenever $d \in (a, b)$ is such that $|c - d| < \delta$, then $|f(c) - f(d)| < \epsilon$.

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Limits of sequences

Let $(a_n) = (a_0, a_1, a_2, \ldots)$ be a sequence of real numbers and let L be a real number.

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(a_n) *converges to* L if, for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - L| < \epsilon$.

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History

- Great mathematicians, including Aristotle, Leibniz, Newton, and Euler did mathematics making a liberal use of infinitely small and infinitely large numbers.
- Many mathematicians and philosophers were extremely skeptical of this method of reasoning, for what exactly was an *infinitesimal*?
- In the 19th century, Cauchy and Weierstrass rescued the calculus by providing the now familiar $\epsilon - \delta$ definitions.
- In the 1960s, Abraham Robinson noticed that one could use the techniques of *mathematical logic* to provide a rigorous foundation for the use of infinitesimals, spawning the birth of *nonstandard analysis*.
- Ever since, nonstandard analysis has found great applications to many areas of mathematics, including measure theory, functional analysis, Lie theory, probability theory, number theory, algebra, mathematical economics, and mathematical physics.

A quick start

- To start "doing" analysis in a nonstandard fashion, we need to know where we get our supply of infinitely large and infinitely small elements.
- In order to get going, let us start by *assuming* the existence of an ordered field $\mathbb{R}^*$ satisfying:
  1. The ordered field $\mathbb{R}$ of real numbers is an ordered subfield of $\mathbb{R}^*$;
  2. There is an *infinitely large* element α of $\mathbb{R}^*$ in the sense that $r < \alpha$ for every $r \in \mathbb{R}$;
  3. $\mathbb{R}^*$ behaves "logically" like $\mathbb{R}$. (Transfer Principle)
- This axiomatic approach is similar to studying $\mathbb{R}$ by only using the fact that it is a complete, ordered field. (Complete means every nonempty set bounded above has a least upper bound.)

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The transfer principle

- We assume that every subset A of $\mathbb{R}$ has an *natural extension* A^*: $A \subseteq A^* \subseteq \mathbb{R}^*$. Thus, we will have sets $\mathbb{N} \subseteq \mathbb{N}^*$, $[a, b] \subseteq [a, b]^*$, etc... We also do this for subsets of $\mathbb{R}^2$, $\mathbb{R}^3$, etc...
- Also, we assume that every function $f : A \to B$ has a natural extension $f^* : A^* \to B^*$. For example, we have $\sin^* : \mathbb{R}^* \to [-1, 1]^*$ and $\ln^* : (0, \infty)^* \to \mathbb{R}^*$.
- The transfer principle then asserts that any *elementary* property about $\mathbb{R}$ is true if and only if the corresponding property of $\mathbb{R}^*$ is true.
- To make this precise would require a detour into logic, so let us be satisfied with an example.

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An example of the transfer principle

Let $f : (a, b) \to \mathbb{R}$ be a function and let $c \in (a, b)$.
Consider the following statement:

$$(\forall \epsilon \in \mathbb{R}_+)(\exists \delta \in \mathbb{R}_+)(\forall x \in (a, b))(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)$$

which is true (in $\mathbb{R}$) if and only if f is continuous at c.

The transfer principle then requires that the above statement is true if and only if the statement

$$(\forall \epsilon \in \mathbb{R}^*_+)(\exists \delta \in \mathbb{R}^*_+)(\forall x \in (a, b)^*)(|x - c|^* < \delta \to |f^*(x) - f^*(c)|^* < \epsilon)$$

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A non-example of the transfer principle

Here is an example of something that is *not* an elementary property:

$\mathbb{R}$ is complete.

The main reason that completeness of $\mathbb{R}$ is not elementary is that to write this property down, one starts by writing $\forall A \in \mathcal{P}(\mathbb{R}) \ldots$

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New numbers

- Recall that we assumed the existence of an infinitely large element α of $\mathbb{R}^*$.
- Then $-\alpha$ is a negative infinite number.
- $\frac{1}{\alpha}$ and $-\frac{1}{\alpha}$ are positive and negative *infinitesimals* respectively.
- There is an element $N \in \mathbb{N}^*$ that is infinite. (Transfer principle) Then $N \pm k$ is an infinite natural number for every *standard* $k \in \mathbb{N}$.
- We need to be careful with arithmetic in $\mathbb{R}^*$: If $\beta \in \mathbb{R}^*$ is a positive infinitesimal, what can we say about $\alpha\beta$? If $\beta = \frac{1}{\alpha}$, then $\alpha\beta = 1$. If $\beta = \frac{1}{\alpha^2}$, then $\alpha\beta = \frac{1}{\alpha}$, an infinitesimal.
- **Exercise:** If $r \in \mathbb{R}^*$ is *finite* (that is, not infinite) and $s \in \mathbb{R}^*$ is infinitesimal, then rs is infinitesimal.

Standard parts

- For $r, s \in \mathbb{R}^*$, write $r \approx s$ if and only if $|r - s|$ is an infinitesimal. (Here, 0 is infinitesimal; it is the only standard infinitesimal.)
- For example, if ϵ is an infinitesimal, then $1 + \epsilon \approx 1$; notice that $1 + \epsilon$ is finite (and not standard if $\epsilon \neq 0$).
- **Very Important Fact:** If $r \in \mathbb{R}^*$ is finite, then there is a unique *standard* number s such that $r \approx s$; we call s the *standard part* of r and write $s = \text{st}(r)$.

Continuity

Theorem

f is continuous at c if and only if, for all $d \in (a,b)^*$, if $c \approx d$, then $f(c) \approx f(d)$.

Proof.

($\Rightarrow$) Suppose that f is continuous at c and suppose that $d \in (a,b)^*$ is such that $c \approx d$. Fix $\epsilon \in \mathbb{R}_+$. We need $|f(c) - f(d)| < \epsilon$.
Recall that there is $\delta \in \mathbb{R}_+$ such that

$$(\forall x \in (a,b))(|x - c| < \delta \rightarrow |f(x) - f(c)| < \epsilon).$$

By transfer,

$$(\forall x \in (a,b)^*)(|x - c| < \delta \rightarrow |f(x) - f(c)| < \epsilon).$$

Since $c \approx d$, $|d - c| < \delta$ is true, so $|f(d) - f(c)| < \epsilon$, as desired. □

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Uniform Continuity

Definition

Let $f : A \to \mathbb{R}$. Then f is *uniformly continuous* if, for every $\epsilon > 0$, there is $\delta > 0$ such that for every $x, y \in A$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem

$f : A \to \mathbb{R}$ is uniformly continuous if and only if for all $x, y \in A^*$, if $x \approx y$, then $f(x) \approx f(y)$.

Compare this with the nonstandard characterization of continuity: $f : A \to \mathbb{R}$ is continuous if and only if for all $x, y \in A^*$ with $x \approx y$ and y standard we have $f(x) \approx f(y)$.

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Uniform Continuity (cont'd)

Example

Consider $f : (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, a continuous function. Let $M, N \in \mathbb{N}^*$ be *infinite* and distinct. Then $\frac{1}{M}, \frac{1}{N} \in (0,1)^*$ and $\frac{1}{M} \approx \frac{1}{N}$, but $f(\frac{1}{M}) = M \not\approx N = f(\frac{1}{N})$, so f is not uniformly continuous.

Notice that for any $x \in [a,b]^*$, there is $y \in [a,b]$ such that $x \approx y$. This proves:

Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

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Limits of sequences

- Suppose that $(a_n) = (a_0, a_1, a_2, \ldots)$ is a sequence of real numbers.
- We view this as a function $a : \mathbb{N} \to \mathbb{R}$, whence we get the extension $a : \mathbb{N}^* \to \mathbb{R}^*$.
- If $N \in \mathbb{N}^*$, we will write a_N instead of $a(N)$, thinking of it as the N^{th} term of the sequence.

Theorem

(a_n) converges to L if and only if, for any infinite $N \in \mathbb{N}^*$, we have $a_N \approx L$.

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Cluster Points

Definition

L is a *cluster point* of (a_n) if, for every $\epsilon > 0$, there are infinitely many a_n in the interval $(L - \epsilon, L + \epsilon)$.

Example

If $a_n = (-1)^n(1 + \frac{1}{n})$, then -1 and 1 are the cluster points of the sequence.

Theorem

L is a cluster point of (a_n) if and only if $L \approx a_N$ for some infinite $N \in \mathbb{N}^*$.

Compare: L is the limit of (a_n) if and only if $L \approx a_N$ for every infinite $N \in \mathbb{N}^*$.

Cluster Points

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L is a *cluster point* of (a_n) if, for every $\epsilon > 0$, there are infinitely many a_n in the interval $(L-\epsilon, L+\epsilon)$.

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If $a_n = (-1)^n(1+\frac{1}{n})$, then -1 and 1 are the cluster points of the sequence.

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Bolzano-Weierstrass

- Now suppose that (a_n) is a bounded sequence, say $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$.
- Suppose that $N \in \mathbb{N}^*$ is infinite. Then by the transfer principle, $-M \leq a_N \leq M$. Thus, $\mathrm{st}(a_N)$ exists.
- By the theorem, $\mathrm{st}(a_N)$ is a cluster point of (a_n). We just proved:

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Derivatives

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Suppose that $f:(a,b)\to\mathbb{R}$ is a function and $c\in(a,b)$. Then f is differentiable at c with derivative L if and only if, for every nonzero infinitesimal ϵ:

$$\frac{f(c+\epsilon)-f(c)}{\epsilon}\approx L.$$

Suppose f is differentiable at c. Then for any nonzero infinitesimal ϵ, we have $f(c+\epsilon)-f(c)\approx f'(c)\cdot\epsilon\approx 0$. This proves:

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If f is differentiable at c, then f is continuous at c.

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Integrals

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let N be an infinite natural number. Let $\{x_0, x_1, \ldots, x_N\}$ be the partition of $[a, b]^*$ into N equal pieces. Then one can make sense out of the sum $\sum_{i=0}^{N-1} f(x_i) \cdot \frac{1}{N}$ and it turns out that

$$\int_a^b f(x)dx \approx \sum_{i=0}^{N-1} f(x_i)\frac{1}{N}.$$

Dirac delta function

"Definition"

The *Dirac delta function* is the "function" $\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}$$

further satisfying $\int_{-\infty}^{+\infty} \delta(x)dx = 1$. (Unit impulse)

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Let $N \in \mathbb{N}^*$ be infinite. Then the *Dirac delta function* is the nonstandard function $\delta : \mathbb{R}^* \to \mathbb{R}^*$ given by

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Major Accomplishments

Theorem (Bernstein-Robinson, 1966)

If H is a separable Hilbert space and $T : H \to H$ is a polynomially compact operator, then T has a nontrivial invariant subspace.

Theorem (van den Dries-Schmidt, 1984)

If $f_0(C, X), f_1(C, X), \ldots, f_m(C, X) \in \mathbb{Z}[C, X]$ are polynomials and K is an algebraically closed field, then the set

$$\{c \in K^M \mid f_0(c, X) \in (f_1(c, X), \ldots, f_m(c, X))\}$$

is a Zariski-constructible set.

Theorem (G., 2010)

Hilbert's fifth problem for local groups has a positive solution: every locally euclidean local group is locally isomorphic to a Lie group.

Cantor's construction of $\mathbb{R}$

- Let (a_n) be a sequence of rational numbers. We say that (a_n) is *Cauchy* if, for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|a_m - a_n| < \epsilon$.
- Every real number is a limit of a Cauchy sequence of rational numbers (finite decimal approximations) and, conversely, every Cauchy sequence of rational numbers converges to a real number.
- Thus, we can think of real numbers as limits of Cauchy sequences of rational numbers.
- Problem: Many Cauchy sequences have the same limit, so we do not have an identification of real numbers with Cauchy sequences.
- Fix: We say that the Cauchy sequences (a_n) and (b_n) are *equivalent* if $\lim_{n\to\infty} |a_n - b_n| = 0$. We then *define* real numbers to be equivalence classes of Cauchy sequences of rational numbers.

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A construction of $\mathbb{R}^*$

- Once again, we use sequences: We think of the sequence $1, 2, 3, \ldots, n, n+1, \ldots$ as defining an infinite element of $\mathbb{R}^*$.
- However, the sequence $\pi, e, -\ln(8), 103, 5, 6, 7, 8, 9, \ldots$ should define the same infinite number.
- More generally, the sequence (a_n) and (b_n) of *real* numbers should define the same element of $\mathbb{R}^*$ if they agree on "most" entries. But what does "most" mean?
- We want a notion of "large" subset of $\mathbb{N}$ and then declare that (a_n) and (b_n) agree on "most" entries if the set of entries they agree on is "large".
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Large and small

- For every subset $A \subseteq \mathbb{N}$, we want a way to categorize A as either small or large (but not both).
- If $A \subseteq \mathbb{N}$ is small and $B \subseteq A$, then B should also be small.
- If $A, B \subseteq \mathbb{N}$ are both small, then $A \cup B$ should also be small. (Then the intersection of two large sets is large.)
- Finite subsets of $\mathbb{N}$ should be small.

Theorem

There is a division of subsets of $\mathbb{N}$ into small and large as above.

Given such a division of subsets of $\mathbb{N}$ into small and large, the collection of large sets is called an *nonprincipal ultrafilter on* $\mathbb{N}$. For the rest of this talk, let's fix such a divison into small and large.

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- For $[a_n], [b_n] \in \mathbb{R}^*$, we define

$$[a_n] + [b_n] := [a_n + b_n] \text{ and } [a_n] \cdot [b_n] := [a_n b_n].$$

- We also say $[a_n] < [b_n]$ if and only if $\{n \in \mathbb{N} \ : \ a_n < b_n\}$ is large.
- We can view $\mathbb{R}$ as a subset of $\mathbb{R}^*$ by pretending that $r \in \mathbb{R}$ is $[(r, r, r, \ldots,)]$, the equivalence class of the constant sequence r.

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Theorem

$\mathbb{R}^*$ *satisfies the three axioms from earlier in the talk.*

We will only verify that $\mathbb{R}^*$ is an ordered field extension of $\mathbb{R}$ that contains an infinite element; the proof of the transfer principle (which goes under the name "Łos' theorem" in this context) would require too far a detour into logic to prove here. We just say how you extend sets and functions.

Definition

1. Suppose that $A \subseteq \mathbb{R}$. Then we define $A^* \subseteq \mathbb{R}^*$ by: $[a_n] \in A^*$ if and only if $\{n \in \mathbb{N} \ : \ a_n \in A\}$ is large.
2. Given $f : A \to B$, we define $f^* : A^* \to B^*$ by $f^*([a_n]) = [f(a_n)]$.

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$\mathbb{R}^*$ is an ordered field

- Most of the axioms of a field follow easily from the fact that $\mathbb{R}$ is a field. We prove here the hardest of them, namely that every nonzero element has a multiplicative inverse.
- Suppose that $[a_n]$ is not the zero element. What does this mean? Well, the zero element of $\mathbb{R}^*$ is $[(0, 0, 0, \ldots)]$, the equivalence class of the constantly 0 sequence, so $[a_n] \neq [(0, 0, \ldots)]$.
- Thus, for "most" $n \in \mathbb{N}$, $a_n \neq 0$; for these n, define $b_n := \frac{1}{a_n}$. For the other n, let b_n be any real number that you want!
- Then $a_n \cdot b_n = 1$ for "most" $n \in \mathbb{N}$, so $[a_n] \cdot [b_n] = [(1, 1, 1, \ldots)]$, which is the unit element of $\mathbb{R}^*$.

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- What about the order? Well, since

  $$\mathbb{N} = \{n \in \mathbb{N} : a_n < b_n\} \sqcup \{n \in \mathbb{N} : a_n = b_n\} \sqcup \{n \in \mathbb{N} : b_n < a_n\}$$

  and exactly one of these sets must be large, we have that $<$ is a linear order on $\mathbb{R}^*$: either $[a_n] < [b_n]$ or $[a_n] = [b_n]$ or $[b_n] < [a_n]$.
- The other axioms for an ordered field are easy to verify.
- By the way, the axioms of an ordered field are "elementary," so if we had already proved the Transfer Principle, then we would get that $\mathbb{R}^*$ is an ordered field as a consequence.

$\mathbb{R}^*$ has an infinite number

- Let $\alpha = [(1, 2, 3, \ldots)] \in \mathbb{R}^*$. We claim that α is an infinite element of $\mathbb{R}^*$. (Actually, $\alpha \in \mathbb{N}^*$)
- To see this, let $r \in \mathbb{R}$. We need $[(r, r, r, \ldots)] < [(1, 2, 3, 4, \ldots)]$.
- This is true because $\{n \in \mathbb{N} \ : \ r < n\}$ has a finite complement and thus is large!
- By the way, $\frac{1}{\alpha} = [(1, \frac{1}{2}, \frac{1}{3}, \ldots,)]$ is a positive infinitesimal.

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Concluding remarks

- In order to apply nonstandard methods to more sophisticated subjects, one needs to modify the ultrapower construction above to a more elaborate framework.
- Nonstandard analysis provides a new collection of principles that one can use in proofs, e.g. overflow, underflow, saturation, hyperfinite approximation, etc...
- Many theorems of standard mathematics have been proven by nonstandard techniques because the intuitive approach to the proof can be formalized in nonstandard analysis. However, it is a theorem that any theorem which can be proven using nonstandard analysis can also be proven *without* nonstandard analysis, although the standard proof is often completely unreadable!

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- Many theorems of standard mathematics have been proven by nonstandard techniques because the intuitive approach to the proof can be formalized in nonstandard analysis. However, it is a theorem that any theorem which can be proven using nonstandard analysis can also be proven *without* nonstandard analysis, although the standard proof is often completely unreadable!

Concluding remarks

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References

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