Hilbert's Fifth Problem for Local Groups

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The Ohio State University Logic Seminar
October 30, 2008

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Hilbert's Fifth Problem

Definition

A topological group G is **locally euclidean** if there is a neighborhood of the identity homeomorphic to some $\mathbb{R}^n$.

Definition

G is a **Lie group** if G is a real analytic manifold which is also a group such that the maps $(x, y) \mapsto xy : G \times G \to G$ and $x \mapsto x^{-1} : G \to G$ are real analytic maps.

Hilbert's Fifth Problem (H5)

If G is a locally euclidean topological group, is there a real analytic structure on G compatible with the topology so that the group operations become real analytic?

Positive Answers to H5

- Linear Case: G can be continuously embedded into $\mathrm{Gl}_n(\mathbb{R})$ for some n (von Neumann)
- Abelian Case (Pontrjagin)
- Compact Case (Weyl)
- Full Solution: Gleason, Montgomery, Zippin (1952)

Theorem

For a locally compact group G, the following are equivalent:

1. *G is locally euclidean;*
2. *G has **no small subgroups**, i.e. there is a neighborhood of the identity containing no nontrivial subgroups of G;*
3. *G is a Lie group.*

- Nonstandard Exposition of the Full Solution: Hirschfeld (1990)

The Local H5

The Local H5

Is every locally euclidean local group locally isomorphic to a Lie group?

Jacoby gave a “proof” for the positive solution of this in 1957, which was later invalidated by Plaut (for reasons we will mention later). We adapt the nonstandard methods employed by Hirschfeld to give a nonstandard proof of a positive solution of the Local H5.

An Application of the Local H5

Theorem (Brown, Houston)

Every locally euclidean cancellative semigroup can be endowed with a real analytic structure so that the multiplication is a real analytic function.

The proof of the theorem involves defining a locally euclidean local group for certain pairs of open sets in the semigroup. By the positive solution to the Local H5, these local groups can be given a real analytic structure and a uniformization procedure permits a local isomorphism from the semigroup into a fixed Lie group.

What is a local group?

Definition

A **local group** is a tuple $(G, 1, \iota, p)$ where:

- G is a hausdorff topological space with distinguished element $1 \in G$;
- $\iota : \Lambda \to G$ is continuous, where $\Lambda \subseteq G$ is open;
- $p : \Omega \to G$ is continuous, where $\Omega \subseteq G \times G$ is open;
- $1 \in \Lambda$, $\{1\} \times G \subseteq \Omega$, $G \times \{1\} \subseteq \Omega$;
- $p(1, x) = p(x, 1) = x$;
- if $x \in \Lambda$, then $(x, \iota(x)) \in \Omega$, $(\iota(x), x) \in \Omega$, and
$$p(x, \iota(x)) = p(\iota(x), x) = 1;$$
- if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then
$$p(p(x, y), z) = p(x, p(y, z)).$$

Further Notions Associated to Local Groups

Definition

Suppose G is a local group.

1. G is **locally euclidean** if there is an open neighborhood of 1 in G homeomorphic to some $\mathbb{R}^n$;
2. G is a **local Lie group** if G admits a real analytic structure such that the maps ι and p are real analytic;
3. Let U be an open neighborhood of 1 in G. Then the **restriction of G to U** is the local group $G|U := (U, 1, \iota|\Lambda_U, p|\Omega_U)$, where
$$\Lambda_U := \Lambda \cap U \cap \iota^{-1}(U) \text{ and } \Omega_U := \Omega \cap (U \times U) \cap p^{-1}(U).$$
4. G is **globalizable** if there is a topological group H and an open neighborhood U of 1_H in H such that $G = H|U$.

The Local H5-Two Forms

Local H5-First Form

If G is a locally euclidean local group, then some restriction of G is a local Lie group.

Local H5-Second Form

If G is a locally euclidean local group, then some restriction of G is globalizable.

The equivalence of the two forms follows from the positive solution to the original H5 and the fact that every local Lie group has a restriction which is the restriction of a global Lie group.

What was wrong with Jacoby's Proof?

Jacoby essentially assumes that every local group G is **globally associative**, which means that given any finite sequence of elements from G, if there are two ways of introducing parentheses such that both products thus formed exist, then the two products are in fact equal. This is not true in general. In fact:

Theorem (Malcev)

A local group is globally associative if and only if it is globalizable.

Olver constructs local Lie groups which are associative up to sequences of length n for a given n but which are not globally associative.

A Way Around Global Associativity

It will be the case that we can "unambiguously multiply" sequences of elements of any length, provided the elements are close enough to the identity, as we now explain.

Definition

Let $a_1, \ldots, a_n, b \in G$ with $n \geq 1$. We define the notion $(a_1, \ldots, a_n) \to b$ by induction on n as follows:

- $(a_1) \to b$ iff $a_1 = b$;
- $(a_1, \ldots, a_{n+1}) \to b$ iff for every $i \in \{1, \ldots, n\}$, there exists $b_i', b_i'' \in G$ such that $(a_1, \ldots, a_i) \to b_i'$, $(a_{i+1}, \ldots, a_{n+1}) \to b_i''$, $(b_i', b_i'') \in \Omega$, and $b_i' \cdot b_i'' = b$.

We say that $a_1 \cdots a_n$ is **defined** if there exists a (necessarily unique) $b \in G$ such that $(a_1 \ldots, a_n) \to b$. If $a_i = a$ for each i, we say that a^n is defined.

The Sets $\mathcal{U}_n$

Terminology

- If $W \subseteq \Lambda$ and $\iota(W) \subseteq W$, we say that W is **symmetric.**
- For a set A, $A^{\times n} := \underbrace{A \times \cdots \times A}_{n \text{ times}}$;

Lemma

For each $n \geq 1$, there is an open symmetric neighborhood $\mathcal{U}_n$ of 1 such that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ and for all $(a_1, \ldots, a_n) \in \mathcal{U}_n^{\times n}$, $a_1 \cdots a_n$ is defined.

Further Restrictions

To make life easier, we assume the following two conditions hold of our local group G:

- $\Lambda = G$, i.e. every element has an inverse;
- if $(x, y) \in \Omega$, then $(y^{-1}, x^{-1}) \in \Omega$ and $(xy)^{-1} = y^{-1}x^{-1}$.

It can be shown any local group has a restriction which has these properties. Hence, by passing to this restriction, we lose no generality in our attempt to solve the Local H5.

Local 1-parameter Subgroups

The following objects are the "star of the show" in the whole story. They will end up forming the Lie algebra of our soon to be local Lie group.

Definition

A **local 1-parameter subgroup of G** (or **local 1-ps of G**) is a continuous map $X : (-r, r) \to G$, for some $r \in (0, \infty]$, such that for all $r_1, r_2 \in (-r, r)$, if $r_1 + r_2 \in (-r, r)$, then $(X(r_1), X(r_2)) \in \Omega$ and $X(r_1 + r_2) = X(r_1) \cdot X(r_2)$.

Two local 1-parameter subgroups X and Y of G are said to be **equivalent** if there is $r > 0$ such that

$$(-r, r) \subseteq \mathrm{domain}(X) \cap \mathrm{domain}(Y)$$

and

$$X|(-r, r) = Y|(-r, r).$$

The Space $L(G)$

- We let $L(G)$ be the space of equivalence classes of local 1-parameter subgroups of G. We often write $\mathbb{X}$ for an element of $L(G)$ and write $X \in \mathbb{X}$ if $\mathbb{X} = [X]$.
- We let $O : \mathbb{R} \to G$ be the trivial 1-ps of G, i.e. $O(t) = 1$ for all $t \in \mathbb{R}$. We denote the class of O by $\mathbb{O}$.
- We have a **scalar multiplication map** $(s, \mathbb{X}) \mapsto s\mathbb{X} : \mathbb{R} \times L(G) \to L(G)$ given as follows: If $s = 0$, then $s \cdot \mathbb{X} := \mathbb{O}$. Otherwise, suppose $X \in \mathbb{X}$ and $\text{domain}(X) = (-r, r)$. Then let $sX : (\frac{-r}{|s|}, \frac{r}{|s|}) \to G$ be defined by $(sX)(t) = X(st)$ and set $s\mathbb{X} := [sX]$.
- $\text{domain}(\mathbb{X}) := \bigcup_{X \in \mathbb{X}} \text{domain}(X)$ and $\mathbb{X}(t) := X(t)$ for any $X \in \mathbb{X}$ with $t \in \text{domain}(X)$.

The Space $L(G)$ (continued)

We will also consider a topology on $L(G)$ as follows:

- Recall that for a locally compact space P and a hausdorff space Q, the space $C(P, Q)$ can be endowed with the **compact-open topology**, which has as a subbasis sets of the form

$$D_{V,U} := \{f \in C(P, Q) \mid f(V) \subseteq U\},$$

  where V ranges over compact subsets of P and Q ranges over open subsets of Q.
- In an analogous fashion, we will give $L(G)$ the topology generated by the sets of the form

$$D_{V,U} := \{\mathbb{X} \in L(G) \mid V \subseteq \text{domain}(\mathbb{X}) \text{ and } \mathbb{X}(V) \subseteq U\},$$

  where V ranges over compact subsets of $(-2, 2)$ and U ranges over all open subsets of U.

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Nonstandard Extensions

- We form a many-sorted structure by taking as sorts sets relevant to our work and we name all elements of these basic sorts.
- For example, we will have a sort for $\mathbb{N}$ and a sort for $\mathbb{R}$ and a sort for G, as well as sorts for $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{N} \times G)$, $\mathcal{P}(\mathbb{N} \times \mathbb{N} \times G)$, etc...
- Then relevant objects in the story are elements of these sorts. For example, a sequence $a_0, \ldots, a_n$ from G is the set $\{(i, a_i) \mid 0 \leq i \leq n\} \in \mathcal{P}(\mathbb{N} \times G)$.
- Whenever X and $\mathcal{P}(X)$ are among our basic sets, we include the membership relation between elements of X and elements of $\mathcal{P}(X)$ as a basic relation in our language.

Nonstandard Extensions (cont'd)

- We now go to a big elementary extension of our many-sorted structure and we decorate the sorts in this extension with $*$, i.e. $\mathbb{N}^*$, $\mathbb{R}^*$, etc...
- Suppose X and $\mathcal{P}(X)$ are basic sorts. Since the original structure satisfies the sentence

$$\forall y, z \in \mathcal{P}(X)(y = z \leftrightarrow \forall a \in X(a \in y \leftrightarrow a \in z)),$$

  we see that an element $y \in \mathcal{P}(X)^*$ is determined uniquely by the set of $a \in X^*$ with $a \in y$.
- We may thus replace each $y \in \mathcal{P}(X)^*$ by the set $\{a \in X^* \mid a \in^* y\}$, thus identifying $\mathcal{P}(X)^*$ with a subset of $\mathcal{P}(X^*)$. The elements of $\mathcal{P}(X)^*$ are called **internal** subsets of X^*. Subsets of X^* that are not internal are called **external**.

Useful Nonstandard Tools

The solution of the Local H5 is full of uses of the following elementary tools of nonstandard analysis. Below, $A \subseteq \mathbb{N}^*$ is *internal* and $\nu > \mathbb{N}$ means $\nu \in \mathbb{N}^* \setminus \mathbb{N}$.

1. **Internal Induction**: If A contains 0 and is closed under successor, then $A = \mathbb{N}^*$.
2. **Overspill**: Suppose there is $n \in \mathbb{N}$ such that A contains all elements of $\mathbb{N} \geq n$. Then there is $\nu > \mathbb{N}$ such that $[n, \nu] \subseteq A$.
3. **Underspill**: Suppose there is $\nu > \mathbb{N}$ such that for all $\eta > \mathbb{N}$ with $\eta \leq \nu$, η belongs to A. Then there is $n \in \mathbb{N}$ such that every $m \in \mathbb{N}$ with $m \geq n$ belongs to A.

Internal or External?

- (**Internal Definition Principle**) Formulas with internal parameters define internal sets.
- For $r \in \mathbb{R}^*$, $\{r\}$ is internal.
- For $r, s \in \mathbb{R}^*$ with $r < s$, the interval $(r, s) := \{x \in \mathbb{R}^* \mid r < x < s\}$ is internal.
- If $\sigma \in \mathbb{N}^*$, then $\{x \in \mathbb{N}^* \mid x \leq \sigma\}$ is internal, a so-called **hyperfinite set**.
- $\mathbb{N}$ is external by internal induction (or overspill).
- $\mathbb{N}^* \setminus \mathbb{N}$ is external by underspill (or from the fact that $\mathbb{N}$ is external).
- For $r \in \mathbb{R}$, the set $\mu(r) := \{s \in \mathbb{R}^* \mid s \text{ is infinitely close to } r\}$ is external.

Nonstandard Topology

Suppose S is a hausdorff space such that S and its powerset are basic sorts. If $s \in S$, then the *monad of s* is

$$\mu(s) := \bigcap \{U^* \mid U \text{ is a neighborhood of } s\}.$$

We also define the set of *nearstandard points* to be

$$S^*_{ns} := \bigcup_{s \in S} \mu(s).$$

Since S is hausdorff, $\mu(s) \cap \mu(s') = \emptyset$ for distinct $s, s' \in S$. So, for $s^* \in S^*_{ns}$, we can define *the standard part of* s^*, $\mathrm{st}(s^*)$, to be the unique $s \in S$ with $s^* \in \mu(s)$. For $s_1, s_2 \in S^*_{ns}$, put $s_1 \sim s_2$ if $\mathrm{st}(s_1) = \mathrm{st}(s_2)$.

Nonstandard Topology (cont'd)

Basic Facts

- If S' is another hausdorff space, $f : S \to S'$ and $s \in S$, then f is continuous at s iff $f(\mu(s)) \subseteq \mu(f(s))$.
- $\mathcal{O} \subseteq S$ is open iff for every $s \in \mathcal{O}$, $\mu(s) \subseteq \mathcal{O}^*$.
- $F \subseteq S$ is closed iff for every $s \in F^*$, if $s \in S^*_{ns}$, then $\mathrm{st}(s) \in F$.
- $K \subseteq S$ is compact iff $K^* \subseteq S^*_{ns}$.

G^*

From now on, μ will denote $\mu(1) \subseteq G^*$. Note that $\mu \times \mu \subseteq \Omega^*$ and that μ **is a group with the induced multiplication from** G^*. We will use the following easy facts frequently:

- If $(a, b) \in \Omega$ and $a' \in \mu(a)$ and $b' \in \mu(b)$, then $(a', b') \in \Omega^*$ and $a'b' \in \mu(ab)$.
- If $a \in G^*_{ns}$ and $b \in \mu$, then $(a, b), (b, a) \in \Omega^*$ and $ab, ba \sim a$.
- For any $a \in G$ and $a' \in \mu(a)$, if a^n is defined for some $n \in \mathbb{N}$, then $(a')^n$ is defined and $(a')^n \in \mu(a^n)$. In particular, the partial function $p_n : G \rightharpoonup G$ given by $p_n(a) = a^n$, if a^n is defined, has an open domain and is continuous.

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NSS Local Groups

Definition

A **subgroup** H of G is a subset $H \subseteq G$ such that $1 \in H$, $H \times H \subseteq \Omega$, and for all $x, y \in H$, $xy \in H$ and $x^{-1} \in H$.

G is said to have **no small subgroups**, abbreviated G is **NSS**, if there is a neighborhood of 1 in G containing no nontrivial subgroups.

Lemma

*G is NSS if and only if μ contains no nontrivial **internal** subgroups of G^*.*

As in the solution of the H5, we will show for a locally compact G, G NSS implies G has a restriction which is a local Lie group and then G locally euclidean implies G NSS.

From now on, we assume that G is locally compact.

NSS implies Lie: Proof Sketch

- Show that $L(G)$ is a finite-dimensional real vector space.
- Show that there is a local group morphism $G \to \mathrm{Aut}(L(G))$, given by the *local adjoint representation*.
- Show that (a restriction of) the kernel of this local group morphism is abelian.
- Use a theorem of Kuranishi to conclude that a restriction of G is a local Lie group.

Special Neighborhoods

In the rest of this section, we assume G is NSS. Furthermore, we fix a **special neighborhood** $\mathcal{U}$ of G:

- $\mathcal{U}$ is a compact symmetric neighborhood of 1 in G;
- $\mathcal{U} \subseteq \mathcal{U}_2$;
- $\mathcal{U}$ contains no nontrivial subgroups of G;
- for all $x, y \in \mathcal{U}$, if $x^2 = y^2$, then $x = y$.

It is a lemma of Kuranishi that such neighborhoods exist.

Orders of Growth

We will use the Landau notation as follows: If $\nu \in \mathbb{N}^*$ and $i \in \mathbb{Z}^*$, then we write:

- $i = o(\nu)$ if $|i| < \frac{\nu}{n}$ for every $n \in \mathbb{N}^{>0}$;
- $i = O(\nu)$ if $|i| < n\nu$ for some $n \in \mathbb{N}^{>0}$.

For $\nu > \mathbb{N}$, we define:

$$G(\nu) := \{a \in \mu \mid a^i \text{ is defined and } a^i \in \mu \text{ for all } i = o(\nu)\}$$

$$G^o(\nu) := \{a \in \mu \mid a^i \text{ is defined and } a^i \in \mu \text{ for all } i = O(\nu)\}.$$

Note that $G^o(\nu) \subseteq G(\nu)$ and that both sets are symmetric.

Infinitesimal Generators of Local 1-parameter Subgroups

Suppose $\sigma > \mathbb{N}$ and that $a \in G(\sigma)$. Let

$$\Sigma_a := \{r \in \mathbb{R}^{>0} \mid a^{[r\sigma]} \text{ is defined and } a^i \in \mathcal{U}^* \text{ if } |i| \leq [r\sigma]\}.$$

One can use underspill to show that $\Sigma_a \neq \emptyset$. Let $r_a := \sup \Sigma_a$ and let $X_a : (-r_a, r_a) \to G$ be defined by $X_a(t) := \mathrm{st}(a^{[t\sigma]})$. Then:

- X_a is a local 1-ps of G;
- $L(G) := \{[X_a] \mid a \in G(\sigma)\}$;
- $[X_a] = \mathbb{O}$ iff $a \in G^o(\sigma)$.

An Addition on $L(G)$

We would like to define an abelian group operation $+$ on $L(G)$ so that $L(G)$ becomes a real topological vector space.

Idea

Fix $\sigma > \mathbb{N}$. Since $L(G) := \{[X_a] \mid a \in G(\sigma)\}$, we should try to define $[X_a] + [X_b] := [X_{ab}]$.

Theorem

1. *$G(\sigma)$ and $G^o(\sigma)$ are normal subgroups of μ;*
2. *if $a \in G(\sigma)$ and $b \in \mu$, then $aba^{-1}b^{-1} \in G^o(\sigma)$;*
3. *$G(\sigma)/G^o(\sigma)$ is abelian;*
4. *The map $\mathbb{X} \mapsto \mathbb{X}(\frac{1}{\sigma})G^o(\sigma) : L(G) \to G(\sigma)/G^o(\sigma)$ is a bijection.*

Local Gleason-Yamabe Lemmas

The trickiest part of the whole story is proving the preceding theorem. The key to doing this is proving a series of extremely technical lemmas which are the local versions of a series of lemmas proven by Gleason and Yamabe. The vague idea is to have a local action of G on the set of real valued continuous functions on G with support contained in some small neighborhood of 1 and then use a **local Haar measure** to smooth out the action. Roughly speaking, the most important nonstandard consequence of this work is the following lemma.

Part of the Proof of Lemma 6.1

Lemma (6.1)

Suppose $\nu > \mathbb{N}$ and $a_1, \ldots, a_\nu \in \mu$ is an internal sequence such that $a_i \in G^o(\nu)$ for all $i \in \{1, \ldots, \nu\}$. Then $a_1 \cdots a_\nu$ is defined and $a_1 \cdots a_\nu \in \mu$.

The proof that $a_1 \cdots a_\nu \in \mu$ if $a_1 \cdots a_\nu$ is defined mimics the proof of the global version of the lemma and is a consequence of the Gleason-Yamabe Lemmas. We will now indicate how to prove that $a_1 \cdots a_\nu$ is defined.

Fix an internal $E \subseteq \mu$. We will prove the following:

Claim

If $b_1, \ldots, b_\nu$ is an internal sequence such that b_i^j is defined and $b_i^j \in E$ for all $i, j \in \{1, \ldots, \nu\}$, then $b_1 \cdots b_\nu$ is defined.

Part of the Proof of Lemma 6.1 (cont'd)

Claim

If $b_1, \ldots, b_\nu$ is an internal sequence such that b_i^j is defined and $b_i^j \in E$ for all $i, j \in \{1, \ldots, \nu\}$, then $b_1 \cdots b_\nu$ is defined.

We prove the claim by internal induction on ν. At the induction step, the induction hypothesis allows us to assume that $b_1 \cdots b_\nu$ is defined and $b_i \cdots b_{\nu+1}$ is defined for every $i \in \{2, \cdots, \nu+1\}$. By the first part of the proof of Lemma 6.1, we can conclude, for every $i \in \{1, \ldots, \nu\}$

$$(b_1 \cdots b_i, b_{i+1} \cdots b_{\nu+1}) \in \mu \times \mu \subseteq \Omega^*,$$

which by the transfer of the associativity condition is enough to conclude that $b_1 \cdots b_{\nu+1}$ is defined.

The claim finishes the proof of the theorem by taking E to be the internal set of all a_i^j where $i, j \in \{1, \ldots, \nu\}$.

$L(G)$ as a topological vector space

Theorem

$L(G)$ becomes a real topological vector space with the above defined notions of scalar multiplication and vector addition.

It is a crucial part of the story that we prove that $L(G)$ is a *finite-dimensional* vector space. This will follow from the fact that $L(G)$ is *locally compact*, a fact we now indicate how to prove.

The Monad Structure of $L(G)$

In proving theorems about $L(G)$ as a topological space, the following description of its monad structure is helpful.

Lemma

Suppose $\mathbb{X} \in L(G)$ *and* $\mathbb{Y} \in L(G)^*$. *Then* $\mathbb{Y} \in \mu(\mathbb{X})$ *if and only if:*

- $\operatorname{domain}(\mathbb{X}) \cap (-2, 2) \subseteq \operatorname{domain}(\mathbb{Y})$;
- *for every* $t \in \operatorname{domain}(\mathbb{X})$ *and every* $t' \in \mu(t)$, *we have* $\mathbb{Y}(t) \in \mu(\mathbb{X}(t))$.

Proof that $L(G)$ is locally compact

Recall that $\mathcal{U}$ is our special neighborhood of 1 in G. Define

$$\mathcal{K} := \{\mathbb{X} \in L(G) \mid [-1,1] \subseteq \text{domain}(\mathbb{X}) \text{ and } \mathbb{X}([-1,1]) \subseteq \mathcal{U}\}.$$

- It is clear from the definition of the compact-open topology that $\mathcal{K}$ is a neighborhood of $\mathbb{O}$ in $L(G)$.
- To see that $\mathcal{K}$ is compact, suppose $Y : (-s, s) \to G^*$ is an internal local 1-ps such that $[Y] \in \mathcal{K}^*$. We must show that $[Y] \in L(G)^*_{ns}$.
- Define $X : [-1,1] \to G$ by $X(t) = \text{st}(Y(t))$.
- One can show that X can be extended to a local 1-ps of G and by the analysis of the monad structure of $L(G)$, one can see that $[Y] \in \mu([X])$.

The Local Exponential Map

Let $K := \{\mathbb{X}(1) \mid \mathbb{X} \in \mathcal{K}\}$. Let $E : \mathcal{K} \to K$ be defined by $E(\mathbb{X}) = \mathbb{X}(1)$. Then:

- E is continuous (use monad structure of $L(G)$);
- E is a bijection (use the fact that square roots are unique in $\mathcal{U}$);
- E is a homeomorphism (since $\mathcal{K}$ is compact);

We now show that K is a neighborhood of 1 in G. A consequence of this is a partial result to the problem, namely that G is locally euclidean.

A Countable Neighborhood Basis of the Identity

- For $x \in G \setminus \{1\}$, define $\mathrm{ord}(x)$ to be the largest n such that x^n is defined and $x^m \in \mathcal{U}$ for all $m \leq n$.
- If $x \in G^* \setminus \{1\}$, then $\mathrm{ord}(x)$ is the largest ν such that x^ν is defined and $x^\eta \in \mathcal{U}^*$ for all $\eta \leq \nu$.

Define $V_n := \{x \in G \mid \mathrm{ord}(x) \geq n\}$.

Lemma

$(V_n : n \geq 1)$ *is a decreasing sequence of compact symmetric neighborhoods of* 1 *in* G *and is a neighborhood basis of* 1 *in* G*.*

Proof.

For $\sigma > \mathbb{N}$, note that $V_\sigma \subseteq \mu$, so that $V_\sigma \subseteq U^*$ for any neighborhood U of 1 in G. By underspill, $V_n \subseteq U$ for sufficiently large n. $\square$

Proof that K is a neighborhood of 1 in G

We use the following lemma (whose proof uses the Gleason-Yamabe lemmas multiple times).

Lemma

Given $a \in \mu$ and $\nu > \mathbb{N}$, there is $b \in \mu$ such that $\mathrm{ord}(b) \geq 2^\nu$ and $a = b^{2^\nu}$.

We can prove that K is a neighborhood of 1 in G. Fix $\nu > \mathbb{N}$. Then the *internal* set of all η such that for each $x \in V_\eta$ there is $y \in \mathcal{U}^*$ such that $\mathrm{ord}(y) \geq 2^\nu$ and $x = y^{2^\nu}$ contains all infinite η. By underspill, there is some $n > 0$ in this set. Given $x \in V_n$, there is $a \in \mathcal{U}^*$ such that $\mathrm{ord}(a) \geq 2^\nu$ and $x = a^{2^\nu}$. Let $\sigma := 2^\nu$. Then for $t \in [-1, 1]$, $X_a(t) = \mathrm{st}(a^{[t\sigma]}) \in \mathcal{U}$, so $[X_a] \in \mathcal{K}$. But $x = X_a(1)$, so $x \in K$. Thus $V_n \subseteq K$ and K is a neighborhood of 1.

The Local Adjoint Representation Theorem

We are now ready to begin the proof that G NSS implies that G has a restriction that is a local Lie group. The first key ingredient is the Local Adjoint Representation Theorem.

Fix $g \in \mathcal{U}_6$ and $\mathbb{X} \in L(G)$. Pick $X \in \mathbb{X}$ with domain$(X) = (-r, r)$ such that image$(X) \subseteq \mathcal{U}_6$. Then define $gXg^{-1} : (-r, r) \to G$ by $(gXg^{-1})(t) := gX(t)g^{-1}$. One can check that gXg^{-1} is a local 1-ps of G.

Definition

For $g \in G$ and $\mathbb{X} \in L(G)$, defined $\mathrm{Ad}_g(\mathbb{X}) := [gXg^{-1}]$ where $X \in \mathbb{X}$ is as above.

Lemma

For $g \in \mathcal{U}_6$, $\mathrm{Ad}_g : L(G) \to L(G)$ is a vector space automorphism.

The Local Adjoint Representation Theorem (cont'd)

If $\dim_{\mathbb{R}}(L(G)) = n$, then give $\mathrm{Aut}(L(G))$ the topology such that the group isomorphism $\mathrm{Aut}(L(G)) \cong \mathrm{GL}_n(\mathbb{R})$ becomes a homeomorphism.

Definition

If G' is another local group, then a **morphism** from G to G' is a continuous function $f : G \to G'$ such that

- $f(1) = 1'$ and $(f \times f)(\Omega) \subseteq \Omega'$;
- $f(x^{-1}) = f(x)^{-1}$;
- $f(xy) = f(x)f(y)$ if $(x, y) \in \Omega$.

Theorem

Define the function $\mathrm{Ad} : \mathcal{U}_6 \to \mathrm{Aut}(L(G))$ *by* $\mathrm{Ad}(g) := \mathrm{Ad}_g$. *Then* Ad *gives rise to a morphism of local groups* $\mathrm{Ad} : G|\mathcal{U}_6 \to \mathrm{Aut}(L(G))$.

Some Facts About Local Lie Groups

Definition

G is **abelian** if there is a neighborhood $U \subseteq \mathcal{U}_2$ of 1 in G such that $xy = yx$ for all $x, y \in U$.

Lemma

If G is abelian, then G has a restriction which is a local Lie group.

Proof

Assume the special neighborhood $\mathcal{U}$ has been chosen so small that $\mathcal{U} \subseteq \mathcal{U}_6$ and elements of $\mathcal{U}$ commute with each other. Then one can show by induction that if $a, b \in \mathcal{U}$ and $(ab)^n$, a^n and b^n are defined and $a^i, b^i \in \mathcal{U}$ for all $i \leq n$, then $(ab)^n = a^n b^n$.

Some Facts About Local Lie Groups (cont'd)

But then, if $\mathcal{V} \subseteq \mathcal{K}$ is a symmetric open neighborhood of $\mathbb{O}$ in $L(G)$ such that $\mathbb{X} + \mathbb{Y} \in \mathcal{K}$ for all $\mathbb{X}, \mathbb{Y} \in V$, then:

$$\begin{aligned}(\mathbb{X} + \mathbb{Y})(1) &= \mathrm{st}((\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma}))^{\sigma}) \\ &= \mathrm{st}(\mathbb{X}(\frac{1}{\sigma})^{\sigma}\mathbb{Y}(\frac{1}{\sigma})^{\sigma}) \\ &= \mathbb{X}(1)\mathbb{Y}(1).\end{aligned}$$

This means that the local exponential map becomes an isomorphism of local groups after restriction to a smaller neighborhood of 1 in G. Since $L(G)$ is a Lie group, this completes the proof.

Some Facts About Local Lie Groups (cont'd)

Lemma

If $f : G \to \mathrm{GL}_n(\mathbb{R})$ is an injective morphism of local groups, then G is a local Lie group.

Theorem (Kuranishi)

Suppose H is a normal sublocal group of G and consider the local coset space G/H. Suppose:

1. *H is an abelian local Lie group;*
2. *G/H is a local Lie group;*
3. *there exists a "continuous local cross section."*

Then a restriction of G is a local Lie group.

NSS implies Lie

Let $H := \ker(\mathrm{Ad})$, a normal sublocal group of $G|\mathcal{U}_6$. Since H is also a locally compact NSS local group, H has an open (in H) neighborhood V of 1 "ruled by local 1-parameter subgroups", i.e. every element of V lies on a local 1-ps of H. Using that $L(H) \subseteq L(G)$ and interpreting what the kernel of Ad means, we see that elements of V commute and that H is an abelian local group. After restriction, we may as well assume that H is an abelian local Lie group.

Then the local adjoint representation theorem induces an injective morphism of local groups $G/H \to \mathrm{Aut}(L(G))$, whence we can conclude that G/H is a local Lie group. By Kuranishi, in order to show that a restriction of $G|\mathcal{U}_6$ is a local Lie group, it remains to show that G/H admits a continuous local cross section. This can be done using *canonical coordinates of the second kind.*

Let us just sketch how one proves that G locally euclidean $\Rightarrow$ G NSS.

- G locally euclidean $\Rightarrow$ G has **no small connected subgroups**: all that is used about the locally euclidean assumption is that G does not contain homeomorphic copies of $[0,1]^n$ for arbitrarily large n;
- G locally connected and no small connected subgroups $\Rightarrow$ G is NSS: the idea here is that there is a compact normal sublocal group H of G such that G/H is NSS and the quotient map $\pi : G \to G/H$ is injective on a neighborhood of 1.

There are numerous uses of the Gleason-Yamabe Lemmas in the proofs of these implications.