Math 512: Finite Model Theory
0-1 Laws
Constant p

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References: Vaananen Chapter 5, E and F Chapter 3, E.M. Palmer *Graphical Evolution*, Wiley 1985, QA 166.17, Feller

0.1 Example. A (round robin) tournament is a directed graph with an edge between every pair of points.

Fix k. Is there a tournament such that for each set of k-players there is another who beats each of them?

0.2 Definition. Let Ω be a finite sample space with a probability measure P. (i.e. for each $G \in \Omega$, $0 \leq P(G) \leq 1$ and $\sum_{G \in \Omega} P(G) = 1$.)

1. An *event* is a subset of Ω.
2. A random variable is a map X from Ω to the reals.
3. $E(X)$, the expected value of X is $\sum_{G \in \Omega} X(G)P(G)$

Note that for any random variable X, Y, any reals a, b,

$$E(aX + bY) = aE(X) + bE(Y).$$

We will find the following inequality of Markov very useful. The proof is immediate from the definition.

0.3 Lemma. *If* $X \geq 0$ *and* $t > 0$*,*

$$P(X \geq t) \leq E(X)/t.$$

Returning to the original example, let S_n be the set of all tournaments with n players. $|S_n| = 2^{\binom{n}{2}}$. Each of these is equally likely.

Call a k-set X *bad* if no element dominates each member of X. Let $Y(T)$ be the number of bad k-sets in a tournament T and $P_n(Y \geq m)$ be the probability that a tournament of size n contains at least m bad k-sets. (The subscript n would usually be omitted; we write it once for emphasis.)

$$E_n(Y) = \binom{n}{k}(1-(1/2)^k)^{n-k}.$$

(Why? Let, for $i < \binom{n}{k}$, Y_i be a random variable which is 1 if the ith k-tuple is bad, and 0 otherwise. Then $E(Y) = \binom{n}{k}E(Y_i)$ and each $E(Y_i) = (1-(1/2)^k)^{n-k}$.

Then $E_n(Y) \to 0$ and by Markov's inequality $P_n(Y \geq 1) \to 0$. So a.a., there is such a tournament.

Now we move to a more general situation. Fix a finite relational vocabulary τ. Let $\mathbf{K_n}$ be the collection of all structures with universe n. These are the *labeled* τ structures of size n.

Simplest probability measure: for any $X \subseteq \mathbf{K_n}$: $P_n(X) = |X|/|K_n|$. For a global relation ϕ, $P_n(\phi) = P_n(\{A : A \in \mathbf{K_n} \wedge \mathbf{A} \models \phi\}$.

Consider graphs: What is $P_n(even)$, $P_n(connected)$, $P_n(planar)$

For simplicity we will restrict our consideration to graphs; the extension to arbitrary finite symmetric relational languages ($R(\overline{x}) \to R(\overline{x}')$ for any permutation of $\overline{x}$) is basically by complication of notation.

0.4 Definition. Uniform Distribution: For any structures A, B with size n, $P_n(A) = P_n(B)$. For a global relation ϕ, $P(\phi) = \lim_{n\to\infty} P_n(\phi)$ if this limit exists.

0.5 Lemma. *For graphs the uniform distribution is given locally by for any $a, b \in A$, the probability of an edge between a and b is 1/2.*

Proof. $P_n(A) = (1/2)^k \cdot (1-1/2)^{\binom{n}{2}-k} = (1/2)^{\binom{n}{2}}$ where k is the number of edges in A.

0.6 Definition. A class of global relations, e.g. the sentences in a logic $\mathcal{L}$, satisfy the 0-1 law if for every $\phi \in \mathcal{L}$ either $\lim_{n\to\infty\phi} = 1$ or $\lim_{n\to\infty\phi} = 0$.

0.7 Theorem. *The* 0-1-*law holds for first order logic under the uniform distribution.*

Let $\mathbf{K_0}$ be the set of all finite graphs. Write $A \leq B$ if A is a substructure of B. We will later abstract to a more general treatment of 'strong' submodels; we also say A is closed in B.

0.8 Definition. The class $\mathbf{K}$ of finite τ-structures with the relation $\leq$ satisfies the $\leq$-amalgamation property, if for any $A, B_1, B_2 \in \mathbf{K}$ such that $A \leq B_1, A \leq B_2$ there is a structure $D \in \mathbf{K}$ and an isomorphic embedding $f : B_2 \to D$ such that $B_1 \leq D$ and $f(B_2) \leq D,\ f \mid A = id_A$. The structure D is called the amalgamation of B_1 and B_2 over A.

0.9 Definition. The model M is called $(\mathbf{K}, \leq)$-*homogeneous-universal* (h.u.) or $(\mathbf{K}, \leq)$-*generic* if it satisfies following conditions:

1. If $A \leq M, A \leq B \in \mathbf{K}$, then there exists $B \leq M$ such that $B \cong_A B$,
2. If X is a finite subset of M, there is a finite $\bar{X}$ with $X \subseteq \bar{X} \leq M$.

0.10 Lemma. *If $\mathbf{K}$ satisfies the $\leq$-amalgamation property then there is a $(\mathbf{K}, \leq)$-generic model that is unique up to isomorphism.*

Proof. Construct a family of finite structures $\langle A_n : n < \omega \rangle$. To construct A_{n+1} consider all pairs $\langle B_i, C_i \rangle$ such that $B_i \subseteq A_n$, $B_i \leq A_n$, $B_i \leq C_i$ and $|C_i| \leq n$. Construct $A_{n,j+1}$ by amalgamating $A_{n,j}$ and C_j over B_j. Note that $\cup_{n<\omega} A_n$ is as required. For the isomorphism construct a back and forth.

0.11 Notation. Consider the special case of graphs. Let ϕ_k denote the sentence which asserts that there are two sets of size k and an element related to each member of the first set and none of the second. These are called the extension axioms. We call the consequences of the extension axioms T_{ran}.

Noting that in the case of graphs we can build the back-and-forth one element at a time (all substructures are strong substructures), we have:

0.12 Lemma. *The generic model G_{ran} for $\langle \mathbf{K_0}, \leq \rangle$, is the unique countable model for the extension axioms. Thus, the extension axioms axiomatize a complete ($\aleph_0$-categorical) decidable theory.*

0.13 Theorem. *Let the edge probability be p. For each extension axiom ϕ_k,* $\lim_{n\to\infty\phi_k} = 1$.

Proof. A pair of k-element sets (S, T) is said to be bad (in a graph G) if there is no $g \in G$ which is related to every element of S and not related to any element of T. Let $X(G)$ be the number of bad pairs in G ($|G| = n$).

$$E_n(X) = \binom{n}{2k}\binom{2k}{k}(1 - p^k(1-p)^k)^{n-2k}.$$

As before $E_n(X) \to 0$ so $P_n(X \geq 1) \to 0$. Thus $P_n(\phi_k) \to 1$.

0.14 Definition. A theory T is quantifier eliminable if for every formula $\psi(\overline{x})$ there is a quantifier free formula $\theta(\overline{x})$ such that

$$T \models (\forall \overline{x})[\psi(\overline{x}) \leftrightarrow \theta(\overline{x})].$$

Using the saturation and genericity of the random graph we have a very soft proof of:

0.15 Corollary. *T_{ran} is quantifier eliminable.*

Further corollaries:

0.16 Corollary.

1. *The collection of sentences ϕ such that $P(\phi) = 1$ is decidable.*
2. *The global relation '$|A|$ is a prime' is not first order axiomatizable.*
3. *Every sentence true in the random graph has a finite model, i.e. T_{ran} has the finite model property.*

0.17 Definition. The *spectrum* of sentence ϕ is $\{n : (\exists A) A \models \phi \wedge |A| = n\}$.
Problems: Fix ϕ with spectrum equal to all prime powers, to all primes?
Very open problem: Is the complement of a spectrum necessarily a spectrum?

Lemma 0.18 (Fagin). *For any first order ϕ either ϕ or $\neg\phi$ has co-finite spectrum.*

Proof. For some N, either for all $n > N$, $P_n(\phi) > 1/2$ or for all $n > N$, $P_n(\neg\phi) > 1/2$.