EM Models and Downward Categoricity Transfer

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MAIN RESULT

Let $\mu(\kappa)$ be the Hanf number for omitting 2^κ types for a first order theory with vocabulary of size κ.

We write $\mu(\tau)$ for $\mu(|\tau|)$.

In [1] Shelah asserts the following result:

Theorem 1 *If the AEC* $\mathbf{K}$ *has*

1. *ap*
2. *jep*
3. *is categorical in a regular cardinal* λ *and* $\lambda > \mu(\mu(|\tau|))$

then $\mathbf{K}$ *is categorical in every* θ *with* $\mu_2 = \mu(\mu(|\tau|)) \leq \theta \leq \lambda$.

CONTEXT

Conjecture: The class of cardinals in which a *reasonably defined* class is categorical and its complement cannot both be cofinal.

(Note: So, PC-classes are not 'reasonable'.)

We know this conjecture for first order theories. I don't know it for any broader class at all. So it is reasonable to investigate if first with quite strong hypotheses.

Of course, it is only interesting when $\mathbf{K}$ has arbitrarily large models – EM methods are applicable.

Goals

1. Discuss weakenings of the hypotheses.
2. Outline this argument and expound some of the main techniques.
3. Explain the suspect point in the argument.

SOME REDUCTIONS

Theorem 2 *Suppose* $(\mathbf{K}, \preceq_{\mathbf{K}})$ *is an AEC that is categorical in* $\lambda > \Theta$*, has arbitrarily large models, and that* $\mathbf{K}_{<\lambda}$ *has the amalgamation property. There is an AEC* $\mathbf{K}'$ *such that* $\mathbf{K}_{\geq\lambda} = \mathbf{K}'_{\geq\lambda}$ *and* $\mathbf{K}'$ *has the jep and* $\mathbf{K}'_{<\lambda}$ *has the amalgamation property.*

Proof: Let $\mathbf{K}''$ be the submodels of the categoricity model. Let $\mathbf{K}'$ be the unique aec with $\mathbf{K}'_{\leq\lambda} = \mathbf{K}''$. Then $\mathbf{K}'$ is as required.

I will simplify the treatment of Galois types in this talk by assuming full ap. But, with somewhat more difficult arguments the results on Galois stability and saturation go through assuming only ap below the categoricity cardinal.

Some Notation

A class $\mathbf{K}$ of τ-structures is called PC if it is the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$.

A class $\mathbf{K}$ of τ-structures is called $PC\Gamma$ if it the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$ which omit all types in a specified collection Γ of types in finitely many variables over the empty set.

Theorem 3 *If K is an AEC with Lowenheim number* $\mathrm{LS}(\mathbf{K})$ *(in a vocabulary τ with $|\tau| \leq$* $\mathrm{LS}(\mathbf{K})$*), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:*

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then $M' \restriction L \preceq_{\mathbf{K}} N' \restriction L$.

EM models

Notation 4 *1. For any linearly ordered set $X \subseteq M$ where M is a τ-structure we write* $\mathrm{D}_\tau(\mathrm{X})$ *(diagram) for the set of τ-types of finite sequences (in the given order) from X.*

2. *Such a diagram of an order indiscernible set, $\mathrm{D}_\tau(\mathrm{X}) = \Phi$, is called 'proper for linear orders'.*

3. *If X is a sequence of τ-indiscernibles with diagram $\Phi = \mathrm{D}_\tau(\mathrm{X})$ and any τ model of Φ has built in Skolem functions, then for any linear ordering I, $EM(I, \Phi)$ denotes the τ-hull of a sequence of order indiscernibles realizing Φ.*

4. *If $\tau_0 \subset \tau$, the reduct of $EM(I, \Phi)$ to τ_0 is denoted $EM_{\tau_0}(I, \Phi)$.*

Morley's Omitting Types Theorem

Lemma 5 *If* $(X, <)$ *is a sufficiently long linearly ordered subset of a* τ*-structure* M*, for any* τ' *extending* τ *(the length needed for* X *depends on* $|\tau'|$*) there is a countable set* Y *of* τ'*-indiscernibles (and hence one of arbitrary order type) such that* $\mathrm{D}_\tau(\mathrm{Y}) \subseteq \mathrm{D}_\tau(\mathrm{X})$*. This implies that the only (first order)* τ*-types realized in* $EM(X, \mathrm{D}_{\tau'}(\mathrm{Y}))$ *were realized in* M*.*

The easiest formulation of 'sufficiently long' is:
$|X|$ greater than $\mu(|\tau|)$.

$\chi(\Phi)$ is essentially the smallest χ that can be taken as 'sufficiently long' for Φ in some application. $\chi(\Phi) < \mu(\tau)$. But since there are no tools for evaluating $\chi(\Phi)$, I stick with $\mu(\tau)$.

EM models for AEC

Theorem 6 *If* $\mathbf{K}$ *is an abstract elementary class in the vocabulary* τ*, which is represented as a* $PC\Gamma$ *class witnessed by* τ', T', Γ *that has arbitrarily large models, there is a* τ'*-diagram* Φ *such that for every linear order* $(I, <)$ *there is a* τ'*-structure* $M = EM(I, \Phi)$ *such that:*

1. $M \models T'$.
2. *The* τ'*-structure* $M = EM(I, \Phi)$ *is the Skolem hull of* I.
3. I *is a set of* τ'*-indiscernibles in* M.
4. $M \restriction \tau$ *is in* $\mathbf{K}$.
5. *If* $I' \subset I$ *then* $EM_\tau(I', \Phi) \preceq_{\mathbf{K}} EM_\tau(I, \Phi)$.

Model Homogeneity

Definition 7 *M is μ-model homogenous if for every $N \preceq_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \preceq_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.*

To emphasize, this differs from the homogenous context because the N must be *in* $\mathbf{K}$. It is easy to show:

Lemma 8 *(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then $M_1 \approx M_2$.*

Theorem 9 *If $\mathbf{K}$ has the amalgamation property and $\mu *^{<\mu *} = \mu *$ and $\mu * \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu *$ which is $\mu *$-model homogeneous.*

GALOIS TYPES

Definition 10 *Let $M \in \mathbf{K}$, $M \preceq_{\mathbf{K}} \mathcal{M}$ and $a \in \mathcal{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathcal{M}$ which fix M.*

Definition 11 *The set of Galois types over M is denoted* $\mathrm{ga-S(M)}$.

We say a Galois type p over M is realized in N with $M \preceq_{\mathbf{K}} N \preceq_{\mathbf{K}} \mathcal{M}$ if $p \cap N \neq \emptyset$.

Definition 12 *The model M is μ-Galois saturated if for every $N \preceq_{\mathbf{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.*

Again, *a priori* this notion depend on the embedding of M into $\mathcal{M}$; but with amalgamation it is well-defined.

MORLEY'S METHOD FOR GALOIS TYPES

Now we prove 'Morley's method' for Galois types.

Lemma 13 *[II.1.5 of Sh394] If $M_0 \leq M$ and M is substantial* with respect to $|M_0|$, *we can find an EM-set Φ such that the following hold.*

1. *The τ-reduct of the Skolem closure of the empty set is M_0.*
2. *For every I, $M_0 \leq EM(I, \Phi)$.*
3. *If I is finite, $EM_\tau(I, \Phi)$ can be embedded in M.*
4. *$EM_\tau(I, \Phi)$ omits every galois type over N which is omitted in M.*

Proof. Let τ_1 be the Skolem language given by the presentation theorem and consider M as the reduct of τ_1 structure M^1. Add constants for M_0 to form τ_1'. Now apply Lemma 5 to find an EM-diagram Φ (in τ_1') with all τ-types of finite subsets of the indiscernible sequence realized in M.

Now 1) and 2) and 3) are easy (using clause 5) of Theorem 6 since we chose Φ so all finite subsets of the indiscernible set (and so their Skolem closures) are realized in M.

The omission of Galois types is more tricky. Consider both M and $N = EM_\tau(I, \Phi)$ embedded in $\mathcal{M}$. Let N^1 denote the τ_1'-structure $EM(I, \Phi)$. We need to show that if $a \in N$, $p = \mathrm{ga-tp(a/M_0)}$ is realized in M. For some $\overline{e} \in I$, a is in the τ_1-Skolem hull $N_{\overline{e}}$ of $\overline{e}$. (Recall the notation from the presentation theorem.) By 3) there is an embedding α of $N_{\overline{e}}$ into M^1 over M_0. α is also an isomorphism of $N_{\overline{e}} \restriction \tau$ into M. Now, by the model homogeneity, α extends to an automorphism of $\mathcal{M}$ fixing M_0 and $\alpha(a) \in M$ realizes p. $\square_{13}$

Homogeneity and Saturation

Theorem 14 *For* $\lambda > \mathrm{LS}(\mathbf{K})$*, The model* M *is* λ*-Galois saturated if and only if it is* λ*-model homogeneous.*

We sketch the ap proof; it isn't that much harder with no amalgamation assumption whatsoever.

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. Let $M \preceq_{\mathbf{K}} \mathcal{M}$ be λ-saturated. We want to show M is λ-model homogeneous. So fix $M_0 \preceq_{\mathbf{K}} M$ and N with

$$M_0 \preceq_{\mathbf{K}} N \preceq_{\mathbf{K}} \mathcal{M}.$$

Say, $|N| = \mu < \lambda$. We must construct an embedding of N into M. Enumerate $N - M$ as $\langle a_i : i < \mu \rangle$. We will define f_i for $i < \mu$ an increasing continuous sequence of maps with domain N_i and range M_i so that $M_0 \preceq_{\mathbf{K}} N_i \preceq_{\mathbf{K}} \mathcal{M}$, $M_0 \preceq_{\mathbf{K}} M_i \preceq_{\mathbf{K}} M$ and $a_i \in N_{i+1}$.

The restriction of $\cup_{i<\mu} f_i$ to N is required embedding.

Let $N_0 = M_0$ and f_0 the identity. Suppose f_i has been defined. Choose the least j such that $a_j \in N - N_i$. By the model homogeneity of $\mathcal{M}$, f_i extends to an automorphism $\hat{f}_i$ of $\mathcal{M}$. Using the saturation, let $b_j \in M$ realize the Galois type of $\hat{f}_i(a_j)$ over M_i. So there is an $\alpha \in \operatorname{aut} \mathcal{M}$ which fixes M_i and takes b_j to $\hat{f}_i(a_j)$.

Choose $M_{i+1} \preceq_{\mathbf{K}} M$ with cardinality μ and containing $M_i b_j$. Now $\hat{f}_i^{-1} \circ \alpha$ maps M_i to N_i and b_j to a_j.

Let $N_{i+1} = \hat{f}_i^{-1} \circ \alpha(M_{i+1})$ and define f_{i+1} as the restriction of $\alpha^{-1} \circ \hat{f}_i$ to N_{i+1}. Then f_{i+1} is as required. $\square_{14}$

Functors Between AEC's

We really have three AEC's:

$(LO, \subset)$,

$(\mathbf{K}, \preceq_{\mathbf{K}})$

and the representing first order:
$\mathbf{K}'$ which is $Mod(T')$ with submodel as τ'-closed subset

We are describing the properties of the EM-functor between $(LO, \subset)$ and $\mathbf{K}'$ or $\mathbf{K}$.

UNIVERSALITY

Definition 15 *M_2 is σ-universal over M_1 in N if $M_1 \leq M_2 \leq N$ and whenever $M_1 \leq M_2' \leq N$, with $|M_1| \leq |M_2'| \leq \sigma$, there is a (partial isomorphism) fixing M_1 and taking M_2' into M_2.*

Definition 16 *M is* brimful *if for every $\sigma < |N|$, and every $M_1 \leq M$ with $|M_1| = \sigma$, there is an $M_2 \leq N$ with cardinality σ that is σ-universal over M_1 in M.*

Claim 17 (Lemma 3.7 of [2]) *The linear order $I = \lambda^{<\omega}$ is brimful.*

Since every L'-substructure of $EM(I, \Phi)$ has the form $EM(I_0, \Phi)$ for some subset I_0 of I, we have immediately:

Claim 18 *If I is brimful as a linear order, $EM(I, \Phi)$ is brimful as an L'-structure.*

Claim 19 *If I is brimful as a linear order, $EM_\tau(I, \Phi)$ is brimful as a member of* $\mathbf{K}$.

Proof. Let $M = EM(I, \Phi)$; we must show $M \restriction \tau$ is brimful as a member of $\mathbf{K}$. Suppose $M_1 \preceq_{\mathbf{K}} M \restriction \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I', \Phi)$ with $|I'| = \sigma$ and $M_1 \subseteq N_1 \leq M$. By Lemma 6.5, $N_1 \restriction \tau \preceq_{\mathbf{K}} M \restriction \tau$. So $M_1 \preceq_{\mathbf{K}} N_1 \restriction \tau$ by the coherence axiom. Let N_3 be the σ-universal τ'-structure containing N_1 which is guaranteed by Claim 18.

Let M_2 have cardinality σ and $M_1 \preceq_{\mathbf{K}} M_2 \preceq_{\mathbf{K}} M \restriction \tau$. Choose a τ'-substructure N_2 of M with cardinality σ containing N_1 and M_2. Now, N_2 can be embedded by a map f into N_3. But $f(N_2) \restriction \tau \preceq_{\mathbf{K}} N_3 \restriction \tau$ by the coherence axiom so $N_3 \restriction \tau$ is the required $\mathbf{K}$-universal extension of M_1. $\square_{19}$

GALOIS STABILITY

Definition 20 *1. Let $N \subset \mathcal{M}$. N is λ-Galois-stable if for every $M \subset N$ with cardinality λ, only λ Galois types over M are realized in N.*

2. $\mathbf{K}$ is λ-Galois-stable if $\mathcal{M}$ is. That is $\mathrm{aut}_M(\mathcal{M})$ has only λ orbits for every $M \subset \mathcal{M}$ with cardinality λ.

Since each Galois type over M_0 realized in M is represented by an M_1 with $M_0 \preceq_{\mathbf{K}} M_1 \preceq_{\mathbf{K}} M$, $M = EM(I, \phi)$ is brimful, and $|M_1| = |M_0|$, Claim 19 implies immediately:

Claim 21 *If $\mathbf{K}$ is λ-categorical, the model M with $|M| = \lambda$ is σ-Galois stable for every $\sigma < \lambda$.*

CATEGORICITY IMPLIES STABILITY

Theorem 22 *If* $\mathbf{K}$ *is categorical in* λ*, then* $\mathbf{K}$ *is* σ*-Galois-stable for every* $\sigma < \lambda$*.*

Proof. Suppose $\mathbf{K}$ is not σ-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model N of cardinality σ^+ which is not σ-stable. Let M be the σ-stable model with cardinality λ constructed in Claim 21. Categoricity and joint embedding imply N can be embedded in M. The resulting contradiction proves the result. $\square_{22}$

STABILITY YIELDS SATURATION

Corollary 23 *Suppose* $\mathbf{K}$ *is categorical in* λ *and* λ *is regular. The model of power* λ *is saturated and so model homogeneous.*

Proof. Choose in $M_i \preceq_{\mathbf{K}} \mathcal{M}$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each M_i has cardinality $< \lambda$ and M_{i+1} realizes all types over M_i. By regularity, it is easy to check that M_λ is saturated. $\square_{23}$

Corollary 24 *Suppose* $\mathbf{K}$ *is an AEC with vocabulary* τ *that is categorical in* λ *and* λ *is regular. Then for every* μ, $\mathrm{LS}(\mathbf{K}) < \mu < \lambda$ *there is a model* $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ *which is* μ*-model homogeneous.*

We haven't mentioned splitting. Saturation in singular cardinals follows (rather abstractly) from a reasonable notion of independence and calculations of $\kappa(T)$. But Shelah doesn't claim it in this context in [1].

Theorem 25 *Suppose* $M \in \mathbf{K}$ *omits a Galois type* p *over a submodel* M_0 *with* $|M| \geq \mu(|M_0|)$. *Then there is no regular cardinal* $\lambda \geq |M|$ *in which* $\mathbf{K}$ *is categorical.*

Proof. By Lemma 13, there is a model $N \in \mathbf{K}$ with cardinality λ which omits p. But by Lemma 23, the unique model of power λ is saturated. $\square_{25}$

TOWARDS DOWNWARD CATEGORICITY II

In [1] Shelah asserts the following result:

Theorem 26 *If* $\mathbf{K}$ *is categorical in a regular cardinal* λ *and* $\lambda > \mu(\mu(|\tau|))$ *then* $\mathbf{K}$ *is categorical in every* θ *with* $\mu_2 = \mu(\mu(|\tau|)) < \theta \leq \lambda$.

Here is a sketch of the argument. We have shown that there are saturated models of power θ for every $\theta < \lambda$. The obstacle to deducing downward categoricity is that Theorem 13 only allows us to transfer the omission of types when the model omitting the type is much bigger than the domain of the type.

Definition 27 *We say* $\mathbf{K}$ *is* (χ, μ)*-tame if for any saturated* $N \in \mathbf{K}$ *with* $|N| = \mu < \lambda$ *if* $p, q, \in \mathrm{ga-S(N)}$ *and for every* $N_0 \leq N$ *with* $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ *then* $q = p$.

Theorem 28 *Suppose* $\mathbf{K}$ *is* λ*-categorical for* $\lambda \geq \mu(\tau)$ *and* λ *is regular. Then* $\mathbf{K}$ *is* (χ, χ_1)*-tame for some* $\chi < \mu(\tau)$ *and any* χ_1 *with* $\chi < \chi_1 \leq \lambda$.

NEXT STEPS

A. Prove Theorem 28.

B. Assuming Theorem 28 prove and extend the categoricity theorem.

I will discuss A in the workshop sessions. Grossberg and VanDieren will discuss B.

References

[1] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394.

[2] S. Shelah and O. Kolman. Categoricity of theories in $L_{\kappa,\omega}$, when κ is a measurable cardinal. part 1. preprint 362.