UNIFYING THE TWO DEFINITIONS OF π

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Abstract. We expound modern rigorous arguments given by Apostol and Spivak for the equality of the proportionality constant π^a and π^c in the formulas $A = \pi^a r^2$ and $C = 2\pi^c r$ for the area and circumference of a circle. These proofs are deduced using the definition of area and arc length via integrals and require the development of trigonometric functions on the real line. We also point to the recent easily available work of David Weisbart [Wei20] who obtains the same identity more in the style of Archimedes, while avoiding the use of trigonomety.

For a circle C with radius r call the ratio of the area of a circle to r^2 π^a and the ratio of the circumference of C to r $2\pi^c$. Archimedes argues:

> Proposition 1. The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.
> Archimedes, Measurement of the circle [Arc97b, Arc97a]

Translating[1] to modern language with numbers rather than merely proportions, this is an assertion $\pi^a = \pi^c$. He assumes a circle can be deformed to a straight line of the same length by rolling a cylinder along a plane. In [Arc97b] he computes π^c to be between $3\frac{10}{71}$ and $3\frac{1}{7}$. 19th century analysis demanded more rigorous arguments for the equality of π^a and π^c. The background for this issue is the dissatisfaction with Euclid's general notion of magnitude resulting in the demand by e.g. Hilbert [Gio21], that separate *geometrical* notions of length and area must be developed. In particular, the relation of length of curves to the length of straight line segments must be clarified.

Textbooks such as [Apo67] and [Spi80] describe a rigorous approach to these results that maintain connection with the original geometry rather than proceeding by less-motivated power series for the trigonometric functions. One of their goals is to connect the definition of area under a curve as a definite integral with its geometric background. They use the standard definition in calculus of arc length of an arbitrary rectifiable[2] curve and calculate the circumference of a circle. In contrast, [Wei20] looks only at the case of a circle but objects that, *a priori, Archimedes construction depends on both the shape of the base polygon and the method of refinement.* After proving the result in [Wei20, §2] for $2^m n$-gons for fixed n, the remainder of the argument establishes in his §4 the equality for polygons whose

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[1] See Footnote 2 of [Arc97b].

[2] The notion of rectifiable was prefigured by Archimedes and is a necessary and sufficient condition for a curve to have a length.

vertices form a 'rational circuit.' For details see [Wei20]; this generalization is not used here.

Unfortunately, I am unable to find the calculation of the circumference via arc length in Apostol and it is relegated to problems in Spivak. We present here an outline, to be read in detail only with Spivak in hand[3], showing there is no circularity[4] in the argument identifying π^a with π^c. A crucial point is the use of the fundamental theorem of calculus.

Of course, this is not intended to be the Euclidean or even the Archimedean argument. The trigonometric functions are defined on the entire real line (by periodicity), not just for angles less than two right angles. Numbers rather than proportions are used to compare magnitudes.

In particular, the arguments of Apostol and Spivak, like most other modern treatments, require the use of trigonometric functions defined on the real line. However, [Wei20] establishes the result using modern notions of convergence but avoiding trigonometric functions.

(1) Using the definition of area under a curve given by a function f between a and b as the value of the definite integral of f from a to b, $\int_{-1}^{1} \surd(1-x^2)dx$ is the area of under the upper half of the circle so the area of the unit circle is $2\int_{-1}^{1} \surd(1-x^2)dx$. $\surd$ represents the positive square root [Spi80, p 12].

(2) We define π^a to be $2\int_{-1}^{1} \surd(1-x^2)dx$. We will justify this in terms of historical usage by proving that twice π^a is the circumference of the unit circle. This uses two lines of argument. A third line identifies the sin and cos defined here with those of right angle triangle trigonometry.
   (A) trigonometric functions
      (i) Define [Spi80, 289] the function

$$A(x) = \frac{x\surd(1-x^2)}{2} + \int_x^1 (1-x^2)dx$$

gives the area in the region bounded by the horizontal axis, the unit circle and the line $[(0,0),(x,\surd(1-x^2)]$. Note this function is continuous, decreasing, and onto the interval $[-1,1]$.

      (ii) For $0 \leqslant x \leqslant \pi^a$ define $\cos x = y$ where y is the unique number in $[-1,1]$ such that $A(y) = x/2$. Thus, cos is a function[5] with domain $[0,\pi^a]$ and $\sin x$ is $\surd(1-(\cos x)^2)$ on that interval. In particular[6], $A(-1) = \frac{\pi}{2}$ implies $\cos\pi = -1$ and $A(1) = 0$ implies $\cos(0) = 1$.

      (iii) [Spi80, 291-292] show $\sin' x = \cos x$ and $\cos' x = -\sin x$ on $(0,\pi)$. Spivak points out that this determination of the derivative of sin using the fundamental theorem of calculus avoids well-known difficulties with evaluating $\lim_{h\to 0}\frac{\sin h}{h} = 1$.

[3]Page numbers are different in later editions; the relevant section is entitled: the trigonometric functions.

[4]The paragraph on [Spi80, p 288] just before 'We can therefore define', is not a definition but a description of the groundwork for the definition. That groundwork is described below.

[5]There is a tacit appeal to the intermediate value to show cos is defined everywhere on $[-1,1]$.

[6]The evaluation A at ± 1 follows from noticing the first term in $A(\pm 1)$ is zero and the second is either 0 or $\pi/2$.

(iv) Note for use below that on the top half of page 291 Spivak calculates that $\cos(\frac{\pi^a}{2}) = 0$, and $\sin(\frac{\pi^a}{2}) = 1$.

(B) Arc Length and radian measure

(i) If f and f' are continuous on $[a, b]$, define the *arc length* of f on $[a, b]$:

$$L_{f,[a,b]} = \int_a^b \sqrt{(1 + f'(x)^2))}\, dx\,.$$

(ii) Calculate the arc length of $\sqrt{(1 - x^2)}$ from -1 to 1, which is denoted as $2\pi^c$.

(iii) This is done in `https://www.math.toronto.edu/jko/MAT186_week_12.pdf` which we copy as Problem 1.0.1. Read the π in the screenshot on the next page as π^a.

(iv) There are several fine points obscured in the screenshot. This is an improper integral of the second kind (cf. [Spi80, Apo67]). But [Spi80, Theorem 3 p. 294] establishes that the integrand[7] is $\arcsin'$ on the open interval $(-1, 1)$ and it is continuous on $[-1, 1]$ with $\arcsin(-1) = -\frac{\pi^a}{2}$ and $\arcsin(-1) = \frac{\pi^a}{2}$ so the integral converges as required and $\pi^a = \pi^c$.

(C) Now we justify *radian measure* as a measure of angles and conclude the agreement of Spivak's trig functions with those of right triangle geometry.

(i) Namely, using the definition of $\sin, \cos$ above, for $x \in [0, 1]$ the arc length of segment of the circle in the boundary of the region bounded by the horizontal axis, the unit circle and the line $[(0, 0), (\cos x, \sin x)]$ is x. For this, apply the same argument as in Problem 1.0.1 (below), but now integrating from x to 1. We see the answer[8] is $\arcsin 1 - \arcsin(\cos x)$. Now from the addition properties of $\sin x$ proved in [Spi80, Theorem 5], and the periodicity (and since $\cos x$ an even function) built into defining the trigonometric functions on the entire line, $\cos(x) = \cos(-x) = \sin(-x + \frac{\pi}{2})$. So

$$\arcsin 1 - \arcsin(\cos x)) = \frac{\pi}{2} - (\arcsin(-x + \frac{\pi}{2})) = \frac{\pi}{2} - (-x + \frac{\pi}{2}) = x$$

as required.

(ii) For angles less than two right angles, the functions sin and cos agree with the same-named functions on right triangles defined as ratios. That is, an angle in a right triangle with radian measure x has base $\cos x$ and height $\sin x$ (as defined by Spivak); since we are on a unit circle, that agrees with the traditional definition.

[7]The notation gets a bit messy here. In the first line of the proof f^{-1} is the inverse function $f = \sin$ as defined near the bottom of page 293. With that notation the passage from the first line of the proof to second is the general formula for the derivative of an inverse function [Spi80, Theorem 12.5 p 222].

[8]In the traditional definition it immediate that $\arcsin(\cos x) = \frac{\pi}{2} - x$, but we need to prove it for this definition of the trig functions.

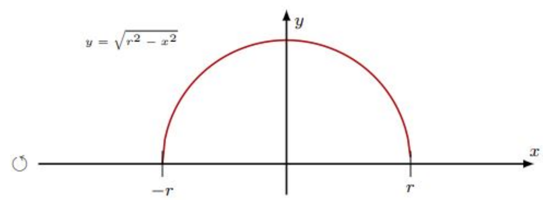
[Wei20] provides formulas for the area and perimeter of inscribed regular n-gons and gives a geometrical proof that as n increases they converge to the same value.

This note was stimulated by my description of a first order axiomatization of the Euclidean geometry along with π [Bal19] and [Bal18, Chapter 10]. The consistency of those axioms depends on arguments such as those described in this note.

Problem 1.0.1. *Calculate the arc length of* $\sqrt{(1-x^2)}$ *from* -1 *to* 1*, which is denoted as* $2\pi^c$.

Problem 1. ($\star\star$) Show that the circumference of a circle with radius r is $2\pi r$.

Solution 1. By symmetry, it suffices to compute the arc length of the semi-circle $y = \sqrt{r^2 - x^2}$ on the domain $[-r, r]$ and multiply our final answer by 2.



Finding the Integral: Since $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$, the arc length of the semicircle is given by

$$\int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx.$$

Computing the Integral: The integrand looks like the derivative of the $\sin^{-1}(x)$, but we need to do some algebraic manipulation first. We multiply the numerator and denominator by r^{-1} to conclude

$$\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx = \int_{-r}^{r} \frac{1}{\sqrt{1 - (\frac{x}{r})^2}}\, dx.$$

We will use the change of variables $u = \frac{x}{r}$,

$$\frac{du}{dx} = \frac{1}{r}, \Rightarrow r\,du = dx \qquad x = -r \to u = -1, \qquad x = r \to u = 1.$$

Under this change of variable, we have

$$\int_{-r}^{r} \frac{1}{\sqrt{1 - (\frac{x}{r})^2}}\, dx = \int_{-1}^{1} \frac{r}{\sqrt{1 - u^2}}\, du = r\sin^{-1}(u)\Big|_{u=-1}^{u=1} = r\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi r.$$

Therefore, the circumference of a semi-circle is πr and the circumference of the circle is $2\pi r$.

2. Weisbart

Here is a quick sketch of Weisbart's proof that approximations of π^a and π^c by regular $2^m n$-gons yield the same result. The major innovation for this argument is the use of Heron's formula in Step 2.0.2.2.a) below. Most of his paper (after §2) is devoted to a more general argument showing the result for arbitrary rectilinear approximations.

Notation 2.0.1. *(1) polygons*

   (a) *$g(m,n)$ and $G(m,n)$ are inscribed and circumscribed $2^m n$-gons.*
   (b) *$\ell_n(m)$ and $L_n(m)$ are side lengths of the respective polygons.*
   (c) *$p_n(m) = 2^m n\ell_n(m)$ and $P_n(m) = 2^m nL_n(m)$ are the perimeters of the respective polygons*
   (d) *$a_n(m)$ and $A_n(m)$ are the areas of the respective polygons.*

(2) *triangles*
   (a) *P and Q are adjacent vertices of $g(m,n)$ and P and Q are adjacent vertices of $G(m,n)$.*
   (b) *$\underline{\alpha_n(m)}$ and $\overline{\alpha_n(m)}$ are the areas of the respective triangles POQ and $\overline{AOB}$.*

Fact 2.0.2. (1) *He first argues in Proposition 3 that for each n each of the limits as $m \to \infty$ for p_n, P_n, a_n, A_n exists.*

(2) *He then argues in the first part of Proposition 4 on page 4, that the upper and lower area approximations converge to 1/2 the relevant perimeter approximation.*
   (a) *$a_n(m) = 2^m n\underline{\alpha_n(m)}$ and $A_n(m) = 2^m n\overline{\alpha_n(m)}$.*
   (b) *For fixed m, the perimeter of POQ is $2 + \ell_n$ so the semiperimeter of POQ is $1 + \frac{\ell_n}{2}$ and by Heron's formula*

$$A(POQ) = \sqrt{}[(1 + \frac{\ell_n}{2})(\frac{\ell_n}{2})^2)(1 - \frac{\ell_n}{2})).$$

   (c) *So in general $\underline{\alpha_n(m)} = 2^m n\frac{\ell_n(m)}{2}\sqrt{})1 - \frac{\ell_n}{2})^4)$ which tends to $\frac{p_n}{2}$.*
   (d) *On a unit circle, $\overline{\alpha_n(m)} = \frac{L_n(m)}{2}$, so $A_n(m) = 2^m\overline{\alpha_n(m)} = 2^{m-1}nL_n(m)$ which tends to $\frac{P_n}{2}$.*

(3) *It remains to show for each n that $P_n = p_n$; he argues this on page 5-6; a slightly shorter version appears at* [Smo22, 138-139].

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