SUPPLEMENT: Three Axiom Systems for Euclidean Geometry: How It Matters*

John T. Baldwin,† Andreas Mueller

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†University of Illinois at Chicago `jbaldwin@uic.edu`

Abstract

Three major axiom systems for organizing plane geometry are used in various GeT/high school textbooks. In the light of SLO 4, we describe the distinct challenges facing Euclid, Hilbert, and Birkhoff (SMSG: School Mathematics Study Group) in this project. Building on a workshop we gave to in-service high school teachers, a logician and a high school teacher describe an amalgam of the Euclid/Hilbert system designed to avoid the technical complexities of Hilbert's system while preserving his foundation for both analytic and synthetic geometry. The supplement expands substantially the treatment of independence, area, continuity, and logical background. It contains further proofs and student activities involving explorations and technology.

1 Introduction

This chapter is aimed primarily at (future) instructors of college courses in geometry for teachers. We stress that axioms are intended to organize the study of an area of mathematics by identifying fundamental assumptions needed to establish the results in that area and that different choices of fundamental notions (undefined terms) and axioms can provide different explanations. We describe in detail, compare, and contrast the three axiomatizations (Euclid, Hilbert, Birkhoff/SMSG), with attention to translations (e.g. Methodology 3.0.9) of geometry most used in GeT-textbooks to support any of them and assist in textbook choice.

The three systems agree on the results in Euclid. But Hilbert and Birkhoff have very different approaches to coordinate geometry. Hilbert's key insight into the axiomatic method is that one must leave a few basic concepts undefined. Euclid and Hilbert agree on these fundamental notions[1]: line, point, congruence ; their meanings are established by the axioms. The approaches of Birkhoff (Motivation 2.1.14) and Hilbert (Motivation 2.1.12) reflect different reactions to the 19th century definition of the real line. The axiomatizations of Euclid and Hilbert (in his first 4 axiom groups) are *synthetic*; theorems are proven from explicit geometric hypotheses in first order logic (quantify over elements). Birkhoff's basic notions include point, line, and angle. But his ruler and protractor axioms also invoke the real field because they measure distance (along lines and arcs). Thus they are *analytic* because they assume the real field with its metric and are second order (quantify over sets/set theoretic). In contrast, Hilbert defines a field from the geometry and uses it clarify the measure of area. He notes that an additional (second order) axiom can make that field the reals. Not distinguishing the synthetic and analytic approaches obscures the notion of axiomatization in a high school geometry course. Understanding the distinction is not required of high school students, but extremely desirable for their teachers.

The chapter began as a commentary on a course the authors (a logician and a high school teacher) gave to in-service mathematics teachers. The course starts with a guiding problem that illustrates the familiar distinction between construction and proof. It presents an easily understood (suitable for high school) algorithm for splitting a line into equal pieces whose use requires only Euclid's first three axioms. But, proving it works requires all the Euclidean axioms. The crucial ingredient in showing the algorithm works is the 'side-splitter' theorem asserting that in similar triangles, corresponding sides are proportional. Euclid proves this on geometrical grounds using the theory of area developed in his Books I and II (and some abstract assumptions about proportionality from Book V). Hilbert reverses the process, proving geometrically that a

[1]Euclid gives 'indicative' definitions (Methodology 3.0.4)

plane can be coordinatized by a field (defined in the process) and thus founding the theory of proportion geometrically, and then using the field to prove side-splitter and develop area.

As a contribution to this book, we integrated commentary on the learning standards and on the course notes. To organize this commentary we introduced labels of motivation, methodology, and pedagogy. Motivation seeks to explain why certain topics and problems arise and methodology the tools used to address them. Pedagogy concerns specific issues that arise in the geometry for teacher's course. We attempted by introducing concepts where they arise geometrically (independence proofs) to acquaint the reader with relevant notions of modern logic without writing an undergraduate logic text.

These many goals result in a text much longer than suitable for this volume. We coped with this by embedding the chapter in a *Supplement* which includes many more diagrams (especially in §6, 7), additional activities and remarks than the chapter. In particular, there is considerably more material[2] exploring the distinctions among the three approaches in Sections 5.4 9, 10, 11 and the appendix 13 on symbolic logic. The supplement is available at GeT: A Pencil site or at the link `https://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/ctti` takes one to a page with links to the supplement and group activities.

We agree with the advice in the narrative that the college instructor should scale up from the earlier levels of the Van Hiele hierarchy. We focus here on the development for college students of levels 3 and 4. Level 3 Deduction (Informal Deduction[3]): 'At this level students can give deductive geometric proofs. They are able to differentiate between necessary and sufficient conditions. They identify which properties are implied by others. They understand the role of definitions, theorems, axioms and proofs.' Level 4 Rigor: 'At this level students understand the way how mathematical systems are established. They are able to use all types of proofs. They comprehend Euclidean and non-Euclidean geometry. They are able to describe the effect of adding or removing an axiom on a given geometric system.'[4]. The common core demands level 3 of high school students but not level 4[5] As noted in SLO 2, 'there may be students in the college course who have not fully attained level 3, while there are a number of high school students that operate comfortably at level 4 and some who appreciate non-Euclidean geometries.'

Standard SLO 4 defines a theorem as 'a statement that can be proved from the axioms without regard to interpretation' (i.e., holds in every interpretation that satisfy the axioms (i.e., every model). More useful for students is 'can be deduced from the axioms by the rules of logic'. The equivalence of these two characterizations of theorem is precisely Gödel's completeness theorem for first order logic. In particular Gödel's theorem makes precise the meaning of *consistent*. All the axiom systems discussed here involve quantification (for all, there exists). First order logic allows quantification only over individuals, while second order logic allows quantifications over sets and functions[6]. In first order logic, a theory T is consistent if it satisfies one of the two equivalent conditions: i) One cannot derive a contradiction from T, ii) T has a model. We will examine such rules in Extension 5.1.5 of the supplement. The crucial point is that Books I-IV of Euclid and exaxiom groups I-IV of Hilbert are first order; Birkhoff is not. First order logic allows quantification only over individuals, while second order logic allows quantifications over sets and functions. This vastly

[2] Some references from chapter to Supplement may be a bit off; but only within a subsection.

[3] Van Hiele says formal but means what we call informal.

[4] This statement is from the easily accessible `https://physics.mff.cuni.cz/wds/proc/pdf12/WDS12_112_m8_Vojkuvkova.pdf`. the five levels are: 0 Visualization, 1 Analysis, 2 Abstraction, 3 Deduction, 4 Rigor. Many variants appear online; [Cro87] differs slightly on Level 4 and gives more background.

[5] 'During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.`https://www.thecorestandards.org/Math/Content/HSG/`

[6] A good introduction is `https://www.baeldung.com/cs/first-order-logic`.

greater strength[7] is embodied in Quine's dicta that 'second order logic is set theory in sheep's clothing'. The complexity of Hilbert continuity axioms (in particular, Dedekind completeness) is discussed in §10 of the supplement. Understanding them seems to require an extension of the Van Hiele hierarchy.

Motivation 1.1 (SLO 1, 3: Why axiomatics?)**.** A fundamental goal of K-12 education is to inculcate the ability to make and understand rational arguments. For over 2000 years Euclid's Elements performed this task more than any other single source. One of the standard goals for U.S. high school geometry is Common Core Standard 3 for mathematical practice: **Construct viable arguments and critique the reasoning of others**. A successful argument requires a clear statement of subject matter. The notion that reasoning skills learned in geometry transfer t, e.g., political discourse raises many distinct questions. However, [IA17, CS20] find studying mathematics develops general thinking skills. Our task here is not to defend that proposition. Rather, given that it is embedded in mathematics standards, the goal here is to provide a model of reasoning in a mathematical context which is accessible to high school students – geometry is everywhere. Moreover, via Euclid et. al., geometry is precise.

Synthetic geometry produces results from explicit geometric assumptions. In contrast, *analytic geometry* assumes the coordinization of the geometry by a field. If the field is required to be the real field the hypotheses are set theoretic rather than only about the geometric objects of Euclid.

We contrast three modes of persuasion: *argument:* reasoned persuasion in any subject: mathematics, law, politics, movies, *informal proof:* a typical argument in mathematics, the rules of inference are implicit and the global assumptions unstated although nominally reducible to formal set theory (e.g., Zermelo-Frankel with the axiom of choice), and *formal proof*: in a logic with strict rules for construction of sentences and deductions. This chapter concerns informal proof but clarifies the relation of proof in high school and college with formal proof, which in its most extreme form must be machine implementable [Hal08, FGH$^+$24].

Methodology 1.2 (Axiom Systems)**.** The introduction to [Hil62], published in 1899, heralds a new age in the foundations of mathematics.

> The following investigation is a new attempt to choose for geometry a simple and complete set of independent[8] axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms.

The aim is to determine fundamental, 'simple and complete' reasons for 'important geometrical theorems'. Hilbert's axioms did not enter the high school curriculum because of the complexity of their use. This complexity arises from the difficult construction of the linear ordering of a line from the abstract betweenness axioms and the tedious process of transcribing such important notions as circle (Hilbert omitted circles.) into his choice of basic concepts. By merging Hilbert's framework with Euclid's, we present a more accessible approach which embodies Hilbert's view what geometry is.

Old View: Until the 19th century it was thought that geometry demonstrated truths from *unassailable premises*. These premises were Euclid's axioms (common notions) and postulates (geometric assumptions).

New View: Geometry deduces conclusions from a specific set of geometric hypotheses. These hypotheses might be Euclidean, spherical, hyperbolic, etc. Whether these geometrical hypotheses are "true" is *not* a mathematical question. As the epigram of [HT20] puts it:

> Geometry doesn't contain the truth about how space is. Geometry is how you view space. Take charge of it – it's yours. Understand how you see things and how you imagine things. Geometry can say something about you and your universe. – David W. Henderson

[7] See [Vää10, §5.3] for a concrete expression of the continuum hypothesis in second order logic with empty vocabulary.

[8] [Hil62] checks the independence of the groups of axioms; however [Wyl44] showed dependence within the order-group.

But this new view leaves open the issue of how we are to understand these 'not known to be true' geometric hypotheses. What are the fundamental notions? What is true about them? What do they imply?

Motivation 1.3 (SLO 1vs SLO 4)**.** By contrasting axioms and models, SLO 4focuses on the roles of axiom systems for *organizing a topic* rather than particular proofs as in SLO 1and [SBM19]. We consider several alternative axiomatizations that each yield the propositions of Euclid. There is not a difference in most cases between the proofs of a particular theorem; the difference is in what statements are theorems rather than axioms, or provable or not. We examine how the different problems that motivated each author affects the actual development of the geometry and accessibility to students. Since the non-Euclidean geometries are rarely studied by deducing from their axioms but rather use tools from analysis to devise non-Euclidean metrics and reason about them, we concentrate on subsystems of Euclidean geometry and differing approaches, specifically those of Euclid, Hilbert, and Birkhoff, we concentrate on subsystems of Euclidean geometry and differing approaches, specifically those of Euclid, Hilbert, and Birkhoff (SMSG) to the Euclidean case.

Methodology 1.4 (SLO 1: Criteria for Choosing Axioms)**.** Natural criteria include that axioms should be intuitive and parsimonious. By intuitive, we mean the axioms can be easily illustrated for the students involved. An axiom system is independent if no axiom can be deduced from the others. Parsimony can be violated in two ways: i) including an axiom which is not needed for the intended collection of results or ii) failing to be independent. Mathematicians were convinced that the parallel postulate was not fundamental but should be provable from Euclid's other postulates; it took two thousand years to show the parallel postulate is independent.

A third natural criteria is that the axioms should be, as Hilbert said in 1.2, complete. But completeness turns out to be a rather complex notion that we will explore in Section 10. For now, we will say an axiom system is *descriptively complete*[9] if it implies all the propositions it was designed to axiomatize.

2 Interpretations, Models, and Axioms

Pedagogy 2.1.1 (SLO 1: Synthetic and Analytic proof / SLO 2: Critique reasoning)**.** Narrative SLO2 prescribes 'understanding different types of proof such as synthetic (from axioms), analytic (using coordinates), and proofs using transformations or symmetries.' This distinction between synthetic and analytic illustrates the difference between proof *from* axioms in the language of geometry and proof *about* interpretations (Example 2.1.9). A *synthetic* proof is an informal proof (Motivation 1.1) organized proofs as sequences of statements such that each statement is either an axiom, hypothesis, previously proved theorem, or follows from the earlier statements by a (perhaps vague) rule of inference. We call synthetic proof as taught in high school, 'semi-formal', reserving 'formal' for the stricter[10] notion of Motivation 1.1. An *analytic* proof is an algebraic proof about the coordinatized plane, which almost always uses symbols. As such, it is a proof *about* an interpretation of the axioms.

Notation 2.1.2. *[Syntax/semantics/interpretation]* The crucial divide between axioms and models is between *syntax* and *semantics*. Axioms are syntactic objects, sentences (English or symbolic). The sentences are in a regimented language with a fixed vocabulary of basic terms. Interpretations (models/structures) are semantic, mathematical objects. There is a clear method (either informally or by a technical definition) to determine when a particular sentence is true in a particular structure.

[9] More strongly it is *deductively negation complete* if every 'relevant sentence' is proved or refuted. See Definition 10.2.2 or [Det14].

[10] Increasingly the term formal is used only for computer proof (e.g.,`https://imsarchives.nus.edu.sg/files/CLThomasHales25Nov2009.pdf`.

More precisely, an *interpretation or structure* for a *vocabulary* (the basic terms) consists of a set (called e.g., world, domain, universe) and a meaning for each basic term on that domain. An interpretation is a *model* of a set of axioms if it satisfies each axiom.

We contrast in Example 2.1.4 the axiomatization of linear orders with the mathematical definition of a linear order thusly:

Definition 2.1.3 (Linear Order)**.** *A set X is linearly ordered by a relation $<$ if $<$ is asymmetric ($x < y$ implies $y \not< x$), irreflexive ($x \not< x$), transitive ($x < y$ and $y < z$ implies $x < z$), and satisfies trichotomy (for any x, y: $x < y$ or $x = y$ or $y < x$); it is dense if between any two points there is another.*

Example 2.1.4 (The theory of linear order)**.** *Fix a vocabulary with a single binary relation symbol $R^<$. The formal axioms of linear order are obtained by turning each item in Definition 2.1.3 into a formal sentence: e.g., asymmetry $(\forall x \forall y) x < y \rightarrow (y \not\leq x)$.*

A model of the theory of linear order is a pair $(X, <)$ with $<$ a binary relation on X such that each of the formal statements is true when $R^<$ is interpreted as $<$.

The fundamental distinction between $R^<$ (a formal symbol) and $<$ (a binary relation) can be discerned from the particular context and so *we follow below the common practice of using the same sign $<$ for both notions.*

The following basic mathematical structures (possible interpretations) should be known, but perhaps not so precisely as written here. A *structure* for the vocabulary of ordered fields (e.g., 'the rationals') is a set with a list of interpretations of basic terms. The ordered field of rational numbers $\langle \mathbb{Q}, +, \times, -, ^{-1}, 0, 1, =, < \rangle$ consists of the set of fractions with the specified constants, operations, and relations listed. The word field indicates that both addition and multiplication are groups (satisfy associativity, commutativity with identities $0, 1$ and inverses (unary functions $-, ^{-1}$), and that multiplication distributes over addition. 'Ordered' prescribes a linear order relation. Here is a particular *interpretation* of the vocabulary of fields (addition, multiplication, additive and multiplicative inverse and identities $0, 1$, equal, less than) on a particular set, the rational numbers $\mathbb{Q}$. Just use the usual meanings of the symbols. In particular, $\{\}^{-1}$ is the symbol for multiplicative inverse. Since all the field axioms are satisfied[11], this interpretation is a *model* of the theory of fields. Another model is the real field. But it, unlike the rationals, also satisfies the least upper bound principle, which cannot be expressed in the first order theory of fields. One point of these notes is that the least upper bound principle is largely irrelevant to high school geometry, but implicit in Birkhoff's treatment.

Definition 2.1.5. *The basic terms of an (incidence) geometry are points (P), lines (L) and a binary relation I between points and lines, 'A lies on ℓ'. The interpretation of the statement, 'the point A is on the line ℓ' is $\Pi(F) \models I(A, \ell)$.*

For any field F, the 'coordinate plane' over F is an interpretation for the incidence geometry vocabulary. By the coordinate plane $\Pi(F)$ over a field F we mean the interpretation $\langle P, L, I \rangle$ whose points are pairs $A = (u, v)$ in $F \times F$ and whose lines are the solutions of linear equations over F. That is, $A = (u, v)$ is on the line ℓ determined by $y = mx + b$, if $v = mu + b$. We say $\Pi(F)$ satisfies the statement 'A lies on ℓ' or formally $I(A, \ell)$.

In Theorem 7.13 we show the correspondence is invertible: the field is found in the geometry.

Exercise 2.1.6. *Here is a very different interpretation for the vocabulary of incidence geometries. Keep $P = F \times F$ but change L to the set of vertical lines: ℓ_a for $a \in F$. Then, I is defined by: $I(\langle x, y \rangle, \ell_a)$ if and only if $x = a$. Problem: Construct a plane with only horizontal lines.*

[11] Since addition does not distribute over multiplication, if we had perversely interpreted addition as $\times$ and multiplication as $+$, we would still have an interpretation; but not a model. Note $^{-1}$ denotes the multiplicative inverse.

Pedagogy 2.1.7 (The new view and student understanding)**.** We now consider axioms for projective planes, since they are much simpler than those for Euclidean geometry. [Har99] describes the distinction between the intuitive axiomatic (Greek) and structural conception (Hilbert) of axioms. Harel highlights that distinction as obstructing students' understanding of proofs and in particular of their understanding such exercises as 2.1.9. How can a plane be finite?

Definition 2.1.8 (Projective Plane)**.** *An incidence geometry is a* projective plane *if it satisfies the axioms: (P1) Any two distinct points lie on a unique line. (P2) Any two distinct lines meet in a unique point. (P3) There exist at least four points of which no three are collinear (i.e., are on the same line).*

Exercise 2.1.9.

1. **Fano Plane** *Draw a picture of the projective plane with* 3 *points on each line. (Hint: it has 7 points and 7 lines.)*

2. *Prove that in a projective plane there are four lines with no three sharing a common point.*

3. *Suppose* (P, L, I) *is a projective plane and there are* n *points on a given line* ℓ*. Prove each line has* n *points and there are* $n^2 - n + 1$ *points in the plane*[12].

Items 2) and 3) have very different nature; the first is a *theorem of projective geometry*; it is expressed in the vocabulary of geometry. The second is *a theorem about projective geometry*. Axiom P3 is expressed in English with the words two and four: The formal axiom expresses 'two' as $\exists x_1, \exists x_2$ such that .... In contrast, exercise 3 asks for a proof of a statement for each n about projective planes with n points on a line. This is an external statement about infinitely many projective planes, not a theorem that can be stated in terms of the axioms in Definition 2.1.8.

Deductions from Euclid's five axioms include some actual gaps and others that are questionable. Many of these gaps are more apparent than real; much of the difficulty came from later mathematicians ignoring the rigorous role diagrams played in Euclidean proof (Pedagogy 5.1.1). For example, Hilbert even postulates that if B lies between A and C then B lies between C and A. For high school this is unnecessary pedantry.

In the remainder of this section we describe the different challenges that motivated the organization of geometry by several authors. To situate Birkhoff's system with the others we need some definitions.

Definition 2.1.10.

1. *A metric on a set* X *is a function* d *from* $X \times X$ *into the positive elements of an ordered group (field for us) such that* $d(x, x) = 0$*,* $x \neq y \rightarrow d(x, y) > 0$*,* $d(x, y) = d(y, x)$*, and* $d(x, z) \leq d(x, y) + d(y, z)$ *(the triangle inequality).*

2. *If the vocabulary of an ordered field F is either included in the basic vocabulary (Birkhoff) or definable (Hilbert) the ruler postulate asserts: for each line* ℓ *in the plane there is a bijection* f_ℓ *from* ℓ *to* F *so that for* $A, B \in \ell$*,* $d(A, B) = |f_\ell(A) - f_\ell(B)|$*.*

3. *the* $\Re$*-ruler postulate (Birkhoff/SMSG) takes* F *to be the real numbers* $\Re$*.*

Motivation 2.1.11 (SLO2, 7: Euclid's Challenge)**.** Euclid aimed to provide a unified foundation for earlier geometry, specifically the side-splitter theorem of Thales around 600 BCE (Euclid VI.2: A line parallel to the base and intersecting both sides of a triangle creates two similar triangles) and the Pythagorean theorem.

[12] The notion of the projective line, as introduced in complex analysis or algebraic geometry is intimidating. But for incidence geometries, this exercise, generalizing from the construction of the Fano plane, is accessible to GeT students. Indeed, the first author took a college course in axiomatic projective geometry. His future wife, who had no college mathematics solved this problem.

The obstacle is incommensurability[13] in each case. He has five postulates. Using a theory of 'equal (area) figures' (but now called *equi-complementability or equal content*) of area, Euclid establishes the Pythagorean theorem as the culmination of Book I. By appealing to the Axiom of Archimedes, he establishes a theory of proportion that first yields: *VI.1 the area of a triangle is proportional to its base and altitude* and *VI.2 the side-splitter theorem*. While Eudoxus' method of exhaustion motivated Dedekind's construction [Ded63], the existence of continuum[14] many ($2^{\aleph_0}$) real numbers was completely foreign to Euclid.

Motivation 2.1.12 (Hilbert's Challenge)**.** 19th century mathematicians such as Cantor, Dedekind, and Frege revolutionized the foundations of mathematics by making the natural numbers rather than Euclidean geometry fundamental. Hilbert aimed for an independent development of geometry. He needed to develop notions of distance and proportion from geometric notions of point, line, between, and congruence (of angles or segments). He had to meet the new higher standards of rigor, in particular, avoiding any reliance on diagrams (Extension 5.1.5). He deduced VI.2, side-splitter, from a geometric foundation of the theory of proportion and then VI.1 from a new theory of 'measured area'. Hilbert presented his axioms as groups I-V. He proved his axiom groups I-IV are independent, although there are dependencies within the some of the groups. Definition 2.1.5 exhibits a model of the ruler postulate for $F = \mathbb{Q}$ (using the interpretation of the field in the plane); § 7 shows there is a such a field for any geometry satisfying Hilbert's first order (quantify only points) axioms (groups I-IV. Group is second order (quantify over sets of points).

Methodology 2.1.13 (SLO5: Congruence vs Distance)**.** This is partly a story of the chicken (congruence) and the egg (distance). A fundamental distinction between Hilbert and Birkhoff is that Hilbert takes the congruence relation as fundamental and *proves* that one can define a metric (with values in a field) and so the ruler postulate is satisfied. Birkhoff (and SMSG [SMS95] [Ced01, Appendix]) assume the $\Re$-*ruler postulate* and define congruence. A difficulty of these SMSG axioms for a high school course is that limits, which *Hilbert has shown are irrelevant to the geometry of lines*, are used implicitly, while basic observations are replaced by long proofs. E.g., common notion 3 (subtraction of line segments) is 'reduced' in some texts to using the ruler postulate twice and assuming the student knows the laws of algebra well[15]. In this chapter we take congruence of line segments and angles as fundamental, not some measure.

Motivation 2.1.14 (Birkhoff's Challenge)**.** Birkhoff differs from Hilbert by 'axiomatizing' analytic geometry rather than developing it from purely geometric hypotheses. He confronts the difficulty of using technical axioms about Hilbert's betweenness relation to the only slightly more intuitive concept of the *real* linear order. Raimi [Rai05] begins his discussion of geometry teaching before the 'new math' days (1960's) in US with the side-splitter theorem (2.1.11). Many texts and an influential mathematics educator [Rai05, p 9] propagated an incorrect proof of this theorem by implicitly assuming all line segments were commensurable. Birkhoff [Bir32, BB59] addressed this issue with 4 postulates: the ($\Re$)-ruler postulate (making the implicit assumption explicit) and an analogous $\Re$-protractor postulate, two points determine a line, and Postulate IV, a masterful amalgamation of SAS and the converse to side-splitter[16] (8.5) into a single statement of *analytic geometry*. From this he easily deduces the existence and uniqueness of parallels [Bir32, Theorem IX]. He writes [Bir32, p 344] 'On the basis of the preceding theorems, Euclidean arc length can be defined in the usual manner.' The existence of the real numbers and thus the $\Re$-ruler postulate can only be stated and proved in second order logic/set theory so do not provide a *geometric* foundation.

[13] Two line segments are commensurable if for some integers m and n, m copies of one are the same length as n copies of other.

[14] `https://en.wikipedia.org/wiki/Cardinal_number` for background.

[15] See [Bal06]

[16] [Moi90, §11.1-2] proves the forward direction of sidesplitter, (AAA) (If corresponding angles of a pair of triangle are congruent, the sides are proportional) via an implicit appeal to Archimedes.

Motivation 2.1.15 (Our Challenge)**.** A prime objection to Hilbert's axioms is that they are too abstract for high school. So the challenge in designing our course was pedagogical, to amalgamate the axioms of Hilbert and Euclid to provide a more accessible account of Hilbert's foundation of both synthetic and analytic geometry on purely geometric principles culminating in a proof of VI.I and VI.2. We vary from Hilbert primarily in accepting Euclid's careful use of diagrams and taking as an axiom (Pedagogy 5.3.3) that each line has a dense linear order based on betweenness. In addition, we integrated proofs of the independence of certain axioms (e.g. Exercise 5.2.8) to give future teachers an understanding of independence proofs. We expound Hilbert's bi-interpretation of Euclidean geometry and ordered fields because it not only is the key step in the bi-interpretation of hyperbolic and Euclidean geometry (Theorem 11.4) but because *it provides a synthetic basis for high school analytic geometry*. For simplicity and succinctness, *we axiomatize only plane geometry*.

Motivation 2.1.16. *[Why not Birkhoff?]* We began with Hilbert's admonition to seek simple, explanatory axioms. The ruler postulate is neither. It appeals to a 'magical' notion: 'the real numbers'. Similarly, assuming the side-splitter magically connects two radically different concepts of proportion (via fields or similarity) that in fact are provably (in Hilbert's system) equivalent. By magic, we mean that Hilbert's axioms identify the actual property that make the reals special, they are the largest Archimedean field. And he has proved his geometry is coordinatized by a field. There is a reason he avoids circles. A rigorous definition of angle measure involves the exponential and trigonometric functions, using either calculus or infinite series. All this is buried by the protractor postulate. Of course, this background is obvious to Birkhoff, one of the leading analysts of the 20th century. But it isn't obvious to a high school sophomore. Nor even to a college student who hasn't absorbed the least upper bound principle in Advanced Calculus. More practically, assuming the $\Re$-ruler postulate kills almost all examples of axiom independence in this chapter.

3 Common Notions vs Postulates

We now discuss Euclid's distinction between general and geometric premises and the 19th century quest for an autonomous basis for geometry.

Methodology 3.0.1. *[Common notions vs postulates]* Euclid's distinction between principles (common notions) that are true everywhere in mathematics and those that are true only of a particular topic remains important today. But it is answered in a different way. Euclid expounded only geometry and natural number (positive integers) arithmetic. His common notions essentially describe the properties of equality and order (among classes of 'comparable objects', i.e., magnitudes of various sorts). Length and area are incomparable magnitudes for Euclid. In modern mathematics (almost) all topics can be studied on a common basis in set theory.

Postulates describe the relations among the fundamental concepts of a particular subject. The best example for over 2000 years were Euclid's postulates for geometry. Nineteenth century geometers insisted that applicability of the common notions be explicitly based within geometry [Gio21]. Thus, the geometrical consequences of the common notions must be derived from the postulates; this required some additions (§5).

These are the common notions of Euclid. They apply equally well to geometry or numbers. Following modern usage, we call Euclid's postulates either 'axiom' or 'postulate'.

Common notion 1. Things which equal the same thing also equal one another.

Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

Methodology 3.0.2 (SLO5,7: Common Notion 1 (CN1))**.** Euclid used 'equal' in two ways: to describe congruence of segments/figures and to describe that figures have the same size (length, area, volume). While the only *numbers* for Euclid were the positive integers > 1, he did study the comparison of what we now interpret as lengths; numbers counted a unit length. See `http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/bookVII.html`: Definitions 1 and 7. Following Hilbert, in Section 7 we build an 'algebra of segments' (a semi-field) and explain how to consider the segments as 'numbers' that can measure areas, a concept totally foreign to Euclid.

CN1 asserts that equality is transitive[17]. For various notions (e.g., congruence) we may need to make this property (as well as symmetry) an explicit axiom. See Axiom 5.4.1.

Methodology 3.0.3 (SLO5,7: Common Notion 4)**.** What Euclid means by coincide and equal is unclear ([Euc56, p 224, 248],5.4.22). We adopt the view that X coincides with Y means 'one is mapped to the other by a rigid motion'; we follow the usual interpretation that in this context Euclid's equal means congruent. So, Euclid CN4 asserts any figure is congruent with itself. That is one of Hilbert's congruence axioms. We discuss the property of symmetry of congruence in Motivation 5.4.5.

Methodology 3.0.4 (SLO 1, 5, 7, 8 Definitions)**.** Euclid begins with a list of *definitions*. Some (e.g., 'A line is breadthless length') are really just an *indicative definition*; they point to intuitions. These indicative definitions become the basic terms (vocabulary) of Definition 2.1.2. Others (e.g., When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**.) are *stipulative definitions*. They precisely describe a new concept in terms of previous definitions. The geometric definitions in this chapter are stipulative.

Euclid and Hilbert take point, line (line segment for Euclid), incidence (a point is on a line), plane, and congruence of segments (and angles) as the most basic concepts. They regard triangles and other polygons as built from points and straight lines and facts about them follow from the axioms.

For Euclid, words in the proof refer to ideal geometric objects. But Hilbert's attitude is different. These basic concepts are named by words in the vocabulary. For him, the meaning of those words is given implicitly by the axioms [Dem94]. Blumenthal [dav11, **?**] reported, 'One must be able to say at all times–instead of points, straight lines, and planes – tables, chairs, and beer mugs'.

Before giving the postulates in §5 we clarify some of the stipulative definitions in Euclid. The activities in this chapter were all group activities for GeT-course or inservice.

Activity 3.0.5. *SLO5, CC Standard G-C0* 1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Why is distance along a circular arc given as an undefined notion? Can we define the length (congruence) of a circular arc in terms of the length (congruence of line segments)? Why is the length of the chord a less good measure than the length of the arc?

As noted [Har00, p 114], congruence of arcs can be defined by rigid motions. But in general, the length of an arc may not be the length of a straight line segment in a particular interpretation. E.g., when the

[17] A relation $R(x, y)$ is transitive if $R(a, b)$ and $R(b, c)$ implies $R(a, c)$. 'Descendent' is transitive; 'daughter of' is not.

interpretation is the plane over a field that does not contain π, the arc-length of a semicircle of radius 1. Specifically, F might be real algebraic numbers (i.e., the field of real solutions of polynomial equations in one variable with rational coefficients) as it does not contain π. The circle of radius 1 about the origin is the set of solutions of $x^2 + y^2 = 1$.

We give a stipulative definition of angle, one of the indicative definitions in Euclid.

Definition 3.0.6. *An angle $\angle ABC$ is a pair of distinct non-collinear rays from a point B. The rays BA (points C on the line BA such that such that B is not between C and A) and BC split the plane into two connected regions. The region such that any two points are connected by a segment entirely in the region is called the* interior *of the angle and the other the* exterior. *Two angles are* adjacent *if they share a ray but no interior points.*

Note that each interior angle (as defined) is less than a straight angle. The measurement of exterior angles is considered in Methodology 10.3.3 in the supplement.

Activity 3.0.7. *What are at least three different units for measuring the size of an angle? (Answers include, degree, radian, turn, grad, house (astrology), Furman[18].)*

Activity 3.0.8. *'Measure', don't 'calculate', the circumference of a convenient cylinder. Compare the result if you measure the radius or the diameter and then calculate the circumference. We have found this a useful exercise for college freshman; we urge future teachers to clarify this distinction for their students.*

We survey here some modern postulate systems for geometry that appear in textbooks for GeT. In line with SLO 4, we focus on those books that adopt an axiomatic approach and leave for other chapters those texts (e.g., [Ced01, HT20] among many) who treat other strands of geometry discussed in [Hen02]. Our categories reflect the intellectual needs of the system builders (Euclid, Hilbert, Birkhoff). We hope our discussion of the motivations of various results and argument can help the instructor respond to the intellectual needs of the students [Har13].

Methodology 3.0.9 (Postulate systems classified by basic notions[19])**.**

1. Hilbert [Hil62, Hil71, Har15, Har00, Ser93] makes points, lines, and *congruence* of segments and angles fundamental[20];
2. Birkhoff [Bir32, BB59], SMSG standards ([Ced01, SMS95]); *distance* is fundamental; all properties of the reals are implicitly assumed (via ruler and protractor postulates[21]);
3. Transformations are studied in two ways: i) within one of the Hilbert or Birkhoff systems [BH07, Cla12, Mar82] and ii) Viewing transformations as fundamental notions [Kin21, Wei97]. All use Birkhoff's axioms except Martin and Weinzweig[22].

[18] The last two were suggested by high school teachers and can be found online.

[20] [Man08] (historically) and [ADM09, Mil07] (mathematically) justify the Euclidean use of diagrams. [Tar59, Szm78] makes a logical but not pedagogical simplification reducing to one kind (sort) of object: a line is a set of collinear points (three points are collinear if they satisfy betweenness in some order).

[21] [Moi90, p 137] carefully distinguishes between what he calls *synthetic* and *metric* approaches. Roughly speaking, his synthetic corresponds to Hilbert (HP) and metric to Birkhoff. Hilbert with (HP5) establishes a metric, but the range is a field that depends on the model of HP5. It is only if Dedekind's axiom is assumed that this becomes a real-valued metric. From our standpoint, these are different synthetic approaches (different axioms in different logics – first vs second order).

[22] See Hartshorne's review [Har11] of [BH07] 'To begin with,the authors devote the first chapter to the axiomatic foundations of plane geometry. Here already, following a popular modern trend, they diverge from Euclid's purely synthetic geometry by presupposing the real numbers, and implicitly using some concepts of analysis.'

Some recent approaches to high school geometry (e.g., [Edu09a, Ill19]) adopt a local approach. Rather than positing a global axiom system, they carefully state and argue from premises for particular topics.

Notation 3.0.10 (Hilbert style axiom sets for plane geometry)**.** *[Bal18] extensively explores the relationship among the following important subsets of Hilbert's axioms for geometry.*

1. **Neutral Geometry (HP)** *The system HP denotes (our translation of) Hilbert's first three axiom groups (Euclid' s first four postulates.). A model is called a* Hilbert plane

2. **Circle free (HP5)** *The system HP5 is obtained by adding the parallel postulate to neutral geometry.*

3. **Euclidean geometry (EG)** *The system EG is HP5 plus circle-circle intersection; a model is called a Euclidean plane*

4. **Continuity axioms***: Axiom of Archimedes and Dedekind completeness (Section 10).*

4 A guiding problem

Pedagogy 4.1.1 (SLO2, SLO8: Role of this section)**.** We began our workshop with the following exercise, first used with future middle school teachers, to emphasize the importance of ruler (straight-edge) and compass constructions in basic geometry and with the hope that the questions in the activity would provoke a need for the proof in Sections 5-8. While a solution using analytic geometry is fairly straight forward, the process of creating a purely geometric proof gives a deep insight into '(a) recognize and communicate the distinction between axioms, definitions, and theorems, and describe how mathematical theories arise from them, (b) construct logical arguments within the constraints of an axiomatic system' (SLO 4).

Exercise 4.1.2. *Each group chooses an odd number n between 2 and 10. After the number is chosen, the group will be asked to fold a string to divide it into as many equal pieces as the number they chose. Other physical models will be used. Activity - Divide a line into n equal pieces.*

Construction 4.1.3. *SLO8: CCSS G-C0-12 For an arbitrary n, here is a procedure to divide a line segment into n equal segments.*

1. *Given a line segment AC.*

2. *Draw a line through A different from AC and lay off sequentially n equal segments on that line, with end points $A, A_1, A_2, \ldots$. Call the last point D.*

3. *Construct B on the opposite side of AC from D so that $AB \cong CD$ and $CB \cong AD$.*

4. *Starting at B, lay off n equal segments of length AA_1 and call the points so constructed on BC sequentially $B, B_1, B_2, ..., B_n - 1, C$.*

5. *Draw lines A_iB_i.*

6. *The points C_i where C_i is the intersection of A_iB_i with AC are the required points dividing AC into n equal segments.*

Exercise 4.1.4.

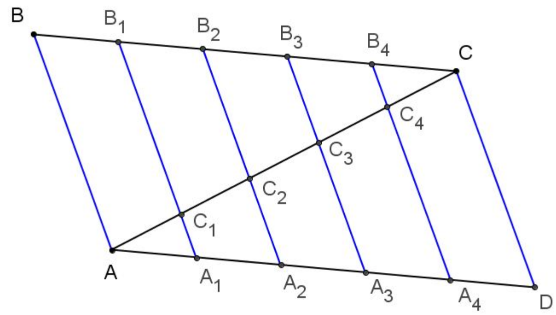


Figure 1: Dividing the line

1. *Use the algorithm described above to divide an arbitrary line segment into 5 equal segments. (Could be done in pairs. One person draws the line; the two have to divide it up.)*
2. *Show this construction used only Euclid's first 3 axioms, listed in Axiom 5.2.1 and 5.2.5 below.*

Pedagogy 4.1.5 (SLO2: Why is this assignment made?)**.** We are really asking, how and why does this construction work? Working in our system we see Euclid's first three postulates suffice to make the construction. See Exercise 5.2.7. We will need SAS and more to prove it works! We start with this exercise both to give the student a reason to prove (stimulate intellectual need [Har13]) and to emphasize this distinction between rule-based construction of geometric objects and a deductive verification of geometric propositions.

5 Book I: Propositions 1-34

The construction in the guiding problem, Exercise 4.1.3, is rather straightforward using only Euclid's first three axioms; the proof that the construction works involves much more. To prepare for this argument, we amalgamate the approaches of Euclid and Hilbert, trying to maximize both understanding and rigor. The material adapts some results from the first 34 proposition of Book I of Euclid to solving our guiding problem. In the remainder of Section 5, we develop material from Book I of Euclid that is used in Section 6 to almost prove the construction works. The remaining issue is the sticking point for each of Euclid, Hilbert, and Birkhoff: the side-splitter theorem. Adopting Hilbert's solution, in Section 7 we define a field of line segments and thus obtain an algebraic theory of proportion.

Pedagogy 5.1.1 (SLO5, 7: Reading a diagram)**. What diagrams meant classically.** Inexact properties can be read off from the diagram: slightly moving the elements of the diagram does not alter the property. Intersections, betweenness and side of a line, inclusion of segments are inexact.

What classical diagrams don't mean Anything about distance, congruence, size of angle (right angle!) may be deceptive. Since incidence is exact, you can't read off whether a point is on a line but you can read off that two lines intersect in a point and then name that point and then use the fact that it is on each line.

What high school diagrams mean Classical diagrams are enhanced in modern texts. Besides the inferences allowed above, SAT instructions say 'All figures in this test are drawn to scale unless otherwise indicated', e.g., 'Figure not drawn to scale'. Students are taught tick marks for congruent segments, angle marks for congruent angles, right angle marks, parallel marks. Figures on one side of a line are assumed to be in that half-plane. Points that appear on a line(s) can be assumed to be on that (those) line(s).

We now fulfill our promise to give more detail on rules of inference.

Definition 5.1.2 (Contraposition)**.** *Let A and B be mathematical statements. The* contrapositive *of 'A implies B' is '$\neg B$ implies $\neg A$'*

Fact 5.1.3 (Logical fact)**.** *Any implication is* equivalent *to its contrapositive.*

Pedagogy 5.1.4. *(*SLO 1) Fact 5.1.3 is easily checked to be valid by truth tables. High school geometry texts sometimes ask students to memorize the names of the four variants on a conditional (if-then) statement. One is the inverse that I know only from such books. This is counter-productive; only the conditional, converse and contrapositive are used frequently. A frequent difficulty is to understand why 'A implies B' is declared true when both A and B are false. The first author found it useful in undergraduate logic courses to emphasize that we are formalizing English. The ambiguity between inclusive or (either one or both) and exclusive or (but not both) or is easy to illustrate. Logicians decided use $\vee$ to mean inclusive or. A similar decision was made for implication $\rightarrow$. Of course if the instructor finds explanations that convince students that's even better.

Extension 5.1.5 (SLO 1, SLO 4, SLO 9: Supplemental Extension: Rules of Inference)**.** Late 19th century mathematicians banished the drawn diagram from semi-formal and even informal mathematics. The SLO 1narrative defines a theorem as 'a statement that can be proved from the axioms without regard to interpretation' (i.e., holds in every interpretation that satisfies the axioms). While correct in spirit, it misses an essential point; how is 'without regard to interpretation' guaranteed? The answer is to specify clear requirements on what statements are and rules for deducing one statement from earlier ones. These can be found in any introductory logic text and many discrete math books. [BE02] includes computer software that explains 'truth in a model' in a very basic way. [Lyn67] is old (My copy is stamped $3.25) but makes the distinctions immediately below very clearly.

Here is a short outline. *Propositional logic* has variables $p, q \ldots$ which stand for propositions (they are true or false). A sentence is a Boolean combination of propositions (combining by: and, or, not, implies).

($\#$) Every tautology is an axiom of propositional logic (check by truth tables). The only rule of inference is modus ponens: from ϕ and $\phi \rightarrow \psi$ infer ψ.

Sentential Logic replaces variables $p, q \ldots$ with atomic formulas[23] of a first order language (e.g., $I(A, \ell), B(C, A, E)$) and allows the same sorts of Boolean combinations (e.g., $I(A, \ell) \wedge B(C, A, E)$ means A lies on ℓ and is between C and E). This sentence does not choose between two contradictory extensions $I(A, \ell) \wedge B(C, A, E) \wedge I(C, \ell)$ and $I(A, \ell) \wedge B(C, A, E) \wedge \neg I(C, \ell)$. The first implies E is on ℓ and the second implies it is not. In order to continue the proof one may have to make case distinctions. See one of the many analyses online of fallacious proofs that 'all triangles are isosceles'.

We use the same rules of inference ($\#$) – translating a sentence into a Boolean combination of proposition by mapping each atomic formula with constants into a unique propositional variable. Then checking to see if it follows from the axioms by truth tables or ($\#$) .

The *logic of geometry* is slightly more complicated. The construction postulates below have the form 'Every set of point and lines satisfying a formula Δ_1 can be extended to a set satisfying Δ_2'. Theorems (and Euclid's 4th and 5th postulate) are even easier; they have the form 'Every set of elements and lines satisfy Δ'. That is, the most complicated results can be stated in the form: for every X satisfying ϕ there exists a

[23] Atomic formulas consist of single a predicate symbol with constants and variables inserted or an equation.

Y such that X and Y satisfy and ψ. E.g., For every pair of non-parallel lines ($X = \ell_1, \ell_2$ there is a point $Y = A$ such that $I(A, \ell_1) \wedge I(A, \ell_2)$.

Now there are two more rules:

1. **Existential instantiation:** Given a construction postulate and a sentence describing various points and lines some of which satisfy the hypothesis of a construction axiom. Choose a name for a witness to the construction postulate and deduce the conjunction of the given statement which the assert the conclusion of the postulate about the witness and the data which satisfies the hypothesis.

2. **Universal generalization:** From any statement ϕ about named points and lines $A, B, C, \ldots, \ell_1, \ell_2 \ldots$, we can deduce: '$\phi$ holds for all $X, Y, Z, \ldots, x_1, x_2 \ldots$'.

First order logic permits iterated use of both existential and universal quantifiers over elements. 'There is a line with seven points' is a permissible sentence. *Second order logic* permits iterated use of both existential and universal quantifiers *over sets*. The logical complexity of the continuity axioms is explored in Section 10.

Extension 5.1.6 (The fly in the ointment)**.** In more complicated arguments (unlikely to appear in high school), the location of the witness for a construction postulate in the existing diagram force a different proof[24].

Recent research clarifies and formalizes the ways that diagrams played an essential role in mathematical proof for 2000 years. [Man08] lays out the main issues and historical background. [ADM09] and [Mil07] provide formal systems with the diagram explicit and with methods to control the number of cases. [ADM09] show their diagram-based system is complete for a set of sentences that include the results of Euclid. See [Bal18, §9] for a summary.

Pedagogy 5.1.7. An excellent reference for grasping these connections is [BE02], which includes very helpful software (Tarski's world) to explore the connections between syntax and syntax. We discussed the importance of the equivalence of an implication with its contrapositive in Definition 5.1.2 through Pedagogy 5.1.4. Understanding this equivalence and fact that such an equivalence fails for an implication and its *converse* is very important. Spelling out the connection with *inverse* (inverse of $p \to q$ is $\neg p \to \neg q$) is known primarily because it was part of Aristotle's square of opposition.

5.2 Construction Postulates

Our vocabulary contains unary predicates P and **L**, binary I and ternary B , standing for point, line, incidence and between. We introduce further vocabulary such as predicates for congruence later. Here are Euclid's first three postulates. We don't list in detail Hilbert's betweenness axioms that imply Axioms I and II.

Axiom 5.2.1 (Euclid's first 3 axioms in modern language)**.**

- **Axiom I** *Given any two points there is a line segment*[25] *connecting them.*
- **Axiom II** *Any line segment can be extended indefinitely (in either direction).*

  The following is a translation of Euclid's Postulate II from a rule *for a construction into a Hilbertian* assertion *that for any witness to Euclid's 'given', there are further witnesses for his conclusion.*

[24] See the 'proof' that all triangles are isosceles [Gre93, p 48-50] and many explanations on the net. That is, one might apply Axiom 5.2.1.II to put a point on line where the given data X is sufficiently complicated that different cases arise in the proof.

[25] If Euclid is being used as a supplement, emphasize to students that a line for Euclid is a line segment for us.

For any points A and B there is a point C such that B is between C and A. (For the other direction interchange the roles of A and B. Implicitly we apply Axiom I, to know A and B are collinear.)

- **Axiom III** *Given a point and any segment there is a circle with that point as center whose radius is congruent to the segment.*

Axiom I varies from Euclid in two ways. Following Hilbert, we inserted 'unique straight'; Euclid's indicative definition of line, 'breadthless length' included curves. Thus, Hilbert replaces an indicative definition of straight, 'a line which lies evenly on itself, with the axiom, 'every two points determine a *unique* line[26]', which implicitly determines Euclid's insight.

Hilbert's first three axioms assert that two points determine a line and there are three non-collinear points. They follow from Euclid's first three, (Axiom 5.2.1).

Definition 5.2.2. *A circle C with center A and radius AB is the collection of points X such that $AX \cong AB$.*

Pedagogy 5.2.3. Circles Euclid gives a stipulative definition for a notion, circle, that does not appear in Hilbert. Hence, we include Axiom III which replaces [Hil71, Axiom III.1]. In addition to grounding the work students will do with circles, Axiom III is a much more tangible way to transfer distance than Hilbert's. [Har00, p 102-3] describes three of Hilbert's tools which, somewhat awkwardly, allow one to obtain the results of Euclid's constructions.

Fine historical point. Euclid does not explicitly mention that overlapping pairs of circles and circles overlapping a line actually intersect and Hilbert never mentions circles. Axiom 5.2.5 makes the assumption precise. In thinking about Exercise 5.2.4 consider why Euclid's notion of diagrams might have caused him to think no further Postulate was necessary to prove Proposition I.1.

Exercise 5.2.4. CCSS G-CO.13 *Prove Proposition I.1 of Euclid: To construct an equilateral triangle on a given finite straight line. Check with [Euc56].*

Following [Har00] we label this axiom E for Euclid as he treats circles while Hilbert doesn't.

Axiom 5.2.5 (Axiom E: Circle Intersections)**.** *If from points A and B, circles with radius AC and BD are drawn such that each circle contains points both in the interior (those points that are connected to the center of the circle by segments that don't cross the circle) and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

As Hartshorne notes, one can conclude from E a line circle axiom: If a line contains a point inside a circle, it intersects the circle (twice!). In many expositions (e.g., [Gre93, p. 80]), Axiom 5.2.5 is deduced from the continuity axiom and used to prove the circle propositions from Euclid's Books III and IV. But Hartshorne [Har00, p 114, 203] shows that only the theory EG (Notation 3.0.10) is needed for the circle theorems.

Lemma 5.2.6 (Euclid's Proposition 2: Rusty Compass)**.** *To place a straight line (segment) equal to a given straight line segment with one end at a given point. In modern language: Given any line segment AB and point C, one can construct a line segment of length AB and end point C.*

In straight-edge and compass constructions, we transfer segments by measuring with the compass, then copy that length to any other place on the paper (that is when we do the construction, our 'rusty compass' does not change the radius). See [Euc56, I.2] for his proof of Lemma 5.2.6 from the axioms I-III, which

[26] Viewing in 3 dimensions from the edge of the ambient plane, a *straight line* collapses to a point.

implicitly assumes our Axiom 5.2.5. Euclid's Propositions 2 and 3 essentially license the addition and subtraction of line segments.

Exercise 4.1.4.1 is now easy.

Exercise 5.2.7. *Using Axioms I-III and Lemma 5.2.6 show the algorithm in Section 4 can be carried out.*

The following exercise gives the student the chance to understand satisfaction in a model in a fairly familiar example and to look at independence where the models are straightforward. While the college students have seen analytic geometry over the reals, here we note that the construction can act on *any* field.

Exercise 5.2.8. *Prove the Cartesian plane over the rationals, defined as in Definition 2.1.5, models Axioms I and II from Axiom 5.2.1 but not Axiom 5.2.5 (Axiom E). Thus, Axiom E is independent from axioms I-III.*

Solution 5.2.9. *To verify Axioms I and II, use the two point form for the equation of a line. The line in point-slope form through (a_1, a_2) and (b_1, b_2), is $y = mx + b$ where $m = \frac{a_2 - a_1}{b_2 - b_1}$ and then set $b = b_2 - m(b_1)$. Note that this tells us how to find a line through two arbitrary points so Axiom I is the segment between if that line between the two points and the entire line verifies Axiom II. For a circle of radius r centered as a, bc take all solutions in Q to $(x - a)^2 + (y - b)^2 = r$.*

There is a close relation between these independence results and properties of fields.

Definition 5.2.10. *A field is Pythagorean if for every a, $\sqrt{}(1 + a^2)$ exists and Euclidean if for every a, $\sqrt{}a$ exists.*

The geometric context is in, e.g., [Har00, §12].

Fact 5.2.11.

1. *A field is Pythagorean iff it coordinatizes a Hilbert plane (model of HP5).*
2. *A field is Euclidean iff it coordinatizes a Euclidean plane (model of EG).*
3. *Characterizations of fields satisfying cubic equation and connections with origami can be found in [Alp05, Mak19].*

Studying such examples integrates the geometry with elementary field theory and gives very concrete examples of independent axioms.

Exercise 5.2.12. *Extend Exercise 5.2.8 to show that Axiom III is true $\Pi(F)$ if F is a Euclidean field.*

5.3 Betweenness, Order, and Planarity

Hilbert's 2nd group of axioms [Hil71, §I.3], labeled *Axioms of Order*, prescribe the behavior of the primitive concept: between. $B(x, y, z)$ means y is between x and z. His Theorem 6 roughly describes a linear order derived from the 'between' relation. Szmielew [Szm78, §7.1] gives ten axioms for betweenness (think of statements that are true of a symmetric relation (B $(A, B, C) \leftrightarrow$ B (C, B, A))) and then carefully derives the definition below of a relation $\leq$ that linearly orders (Definition 5.3.1) the line ℓ through ABC.

Recall Definition 2.1.3 of a linear order.

Definition 5.3.1.

1. *Fix $\ell = \overline{ABC}$, the line through the three points, and define $\leq$ for $P, Q \in \ell$ by*

$$P \leq Q \leftrightarrow (\boldsymbol{B}\ (P, Q, B) \wedge \boldsymbol{B}\ (P, B, C)) \vee (\boldsymbol{B}\ (P, B, C) \wedge \boldsymbol{B}\ (A, B, Q,)) \vee (\boldsymbol{B}\ (A, B, Q) \wedge \boldsymbol{B}\ (B, P, Q,)).$$

In fact, this definition can define a linear order in either direction. By a tricky argument, treating the rays in each direction separately, Szmielew proves:

Theorem 5.3.2 (Linear order and betweenness)**.** *[Szm78, §7.1] For any distinct A, B, C with B (A, B, C) the relation $\leq$ in Definition 5.3.1 is a linear order of ℓ. Assuming for all A, C there exists a B such that B (A, B, C) the order is dense.*

Pedagogy 5.3.3. The difficulty of the argument for Theorem 5.3.2 illustrates the intricacy of using the betweenness relation. Thus, Hilbert's axioms are not used in high school texts. However, *we will just use Theorem 5.3.2 in our development*. So an alternative axiomatization would be to replace Hilbert's order axioms with our Theorem 5.3.2 and certainly this would be a reasonable high school postulate.

Lemma 5.3.4. *Definition 5.3.1 asserts $B(A, B, C)$ yields is a collinearity; A, B, C lie on a line if and only if in some order they satisfy B. Thus, a line is a set of points in the plane.*

Definition 5.3.5. *Given a line ℓ and points A, B on ℓ and D, E not on ℓ.*

1. *the ray $\overrightarrow{AB}$ is all points C on ℓ the same side of A as B (i.e., B (A, C, B) or B (A, B, C).*
2. *A region is* connected *if any two points can be connected by a polygonal path (a sequence of segments such that successive segments share one endpoint).*
3. *D and E are in the same half-plane determined by ℓ if the line segment between D and E does not intersect ℓ.*

Like Euclid, Hilbert develops geometry of dimension 3 with plane as a fundamental notion and so a ternary predicate P for coplanar is in his formal vocabulary and the axiom holds when P (A, B, C). We guarantee the universe is plane by requiring Pasch's axiom to hold for *any* triplet of points; there is no predicate for plane in our system. Here are two equivalent formulations of Pasch [Har00, §7].

Axiom 5.3.6 (Planarity Axioms)**.** Pasch's Axiom: *Let A, B, C be three non-collinear points and let ℓ be any line which does not meet any of the points A, B, C. If ℓ passes through a point of the segment AB, it also passes through a point of segment AC, or through a point of segment BC.*

Separation Principle: *The points of a plane not on a line ℓ are divided into two disjoint connected regions. Two points are in different regions exactly if the line connecting them intersects ℓ.*

Exercise 5.3.7 (Betweenness and Pasch are consistent)**.** *Show the two planarity axioms are equivalent. Check that for any ordered field F, $\Pi(F)$ satisfies the betweenness and the Planarity axioms.*

5.4 Congruence Axioms

This section fills what is generally agreed to be a true gap in Euclid. In Proposition I.4, he purports to prove SAS. His argument implicitly relies on the superposition principle (Methodology 5.4.22). As in Euclid, we take the notions of segment congruence ($AB \cong A'B'$) and angle congruence ($\angle ABC \cong \angle A'B'C'$) as primitive. We follow Hilbert [Hil62, §6] and assert these axioms:

Axiom 5.4.1 (Congruence Axioms)**.** *Congruence* is an equivalence relation on undirected line segments (or angles) that is reflexive, symmetric, transitive and such that the sum (difference) of congruent (line segments, angles) is congruent.

Euclid uses 'equal' for our 'congruent' for undirected segments and angles.

Methodology 5.4.2 (On congruence axioms)**.** The symmetry of angle congruence arises because, following Euclid and Hilbert we are comparing angles not measuring rotation. We stated this axiom in English. Formally, for angles we would add a 6-ary predicate C for congruence (4-ary for segments) and write C (A, B, C, D, E, F) to translate the axiom for two angles ABC and DEF. Euclid uses 'equal' for our 'congruent' for undirected segments and angles.

Definition 5.4.3 (Triangle congruence)**. CCSS G-C0-7** *Two triangles are congruent if there is a correspondence (bijection between the angles and sides of one and the other) so that corresponding angles and corresponding sides are congruent.*

Rigid motions (5.4.21-5.4.23) clarify the notion of triangle congruence.

Definition 5.4.4. *A* rigid motion *is a bijection from points to points that preserves betweenness, collinearity (so it induces a bijection on lines), and congruence of segments and angles.*

A rigid motion is a reflection about ℓ *if it fixes* ℓ *pointwise and sends a point* A *not on* ℓ *to an* A' *such that* ℓ *is the perpendicular bisector of* AA'.

Methodology 5.4.5 (Labeled triangle congruence)**.** Some mathematicians and some high school texts treat congruence as a property of labeled triangles. But then a scalene triangle ABC is not congruent to ACB. By looking at the statement of I.4, it is clear this is not Euclid's intent. He specifies 'some correspondence'; in particular, reflected triangles are congruent. But requiring the labeling is to demand that the correspondence preserve orientation. Since rigid motions preserve congruence, under labeling reflections are no longer rigid motions. Hilbert also treats a weakening of SAS, [Hil71, Appendix II] to act only on oriented triangles (so rigid motions must preserve orientation).

While congruence is a property of triangles not of labeled triangles it is a useful convention to require that using the symbol $\cong$, writing $\triangle ABC \cong A'B'C'$ implies that the primes indicate the correspondence. Often, in describing a polygon ABCDE..., any consecutive letters in the name refer to consecutive (connected by a side) vertices in the polygon.

Methodology 5.4.6 (Axiom Choice)**.** Just as we had a choice of which concepts to specify as basic, we have choices to make for axioms. Euclid's Theorem I.4 (SAS) has been known since antiquity to rely on an implicit 'principle of superposition (Definition 5.4.4)'. In modern language we express this by saying the group of rigid motions (Definition 5.4.4) acts transitively [27] on each equivalence class of congruent angles. Hilbert chose to do this by simply making SAS an axiom. Euclid uses superposition (unnecessarily) again to prove I.8 SSS and proved without any hidden assumptions that SAS implies ASA and AAS. We chose SSS and prove SSS implies SAS. Here are two reasons for choosing SSS. 1) It is very practical: any three sticks that can form a triangle will always form the same triangle.

It is minimalistic: SSS only uses segments in its statement, all others use segments and angles, and defining angles is not trivial.

Pedagogy 5.4.7 (Too many axioms)**.** A major weakness of many high school texts is to think the equivalence proofs of the congruence propositions are too hard for high school. Some high school geometry texts list many of the congruence theorems (SSS, SAS, ASA, HL etc.) as separate axioms. This destroys one of the main features of axiomatics: the search for a small number of (ideally independent) assumptions from which the theory can be deduced. The cost is that students think mathematics is about memorization. This objection is not mere pedantry; calling a known theorem a postulate destroys the concept of axiom system. If to cover certain material (for reasons of time or perceived difficulty) one has to skip proofs, announce that. Don't pretend a new hypothesis has to be introduced.

[27] A group G acts transitively on a set X if for every $x_1, x_2 \in X$, there is a g with $g(x_1) = x_2$.

Axiom 5.4.8 (The triangle congruence postulate: SSS)**. CCSS G-C0-8** *Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$*

Methodology 5.4.9 (Failure of Protractor postulate)**.** *The protractor postulate fails for an F that doesn't contain π. The smallest model of the F-protractor postulate is $\Pi(K(\pi))$ where is K is the field of constructible numbers described at [Har00, 16.4.1]. But K satisfies $HP5$.*

Pedagogy 5.4.10. *We prove Theorem 5.4.11 twice to illustrate the close connections between two styles of presenting proofs. The paragraph style allows the use of English to smooth and emphasize the particular inferences. The 'two-column' style regiments giving a reason for each step.*

Theorem 5.4.11 (SSS implies SAS)**. CCSS G-C0-8, G-C0-10** *Assume HP^-. Let ABC and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle CAB \cong \angle C'A'B'$ then $\triangle ABC \cong \triangle A'B'C'$*

Proof. We must show $\triangle ABC \cong \triangle A'B'C'$. Draw circles with radius AC from A' and with radius BC from B' using Axiom 3. Let them intersect at a point D on the same side of $A'B'$ as C'. Note that triangle $A'DB' \cong ACB$ by SSS. ($AB \cong A'B'$, $BC \cong B'D$ and $AC \cong A'D$). So $\angle CAB \cong \angle DA'B'$. But then by transitivity of congruence, $\angle C'A'B' \cong \angle DA'B'$. But then D lies on $A'C'$ and in fact D must be C'. So we have proved the theorem. $\Box_{par}$

1	$AB \cong A'B', AC \cong A'C', \angle CAB \cong C'A'B'$	$given$
2	Draw circle with radius AC from A'	$Axiom\ 5.2.1.III$
3	Draw circle with radius BC from B'	$Axiom\ 5.2.1.III$
4	Choose the point of intersection D of the circles on the same side $A'B'$ as C'.	$Axiom\ 5.2.5$
5	$AD \cong AC$	$Axiom\ 5.4.8, 2, 3$
6	$\triangle A'DB' \cong \triangle ACB$	$Axiom\ 5.4.8, 5$
7	$\angle CAB \cong \angle DA'B'$	$Def\ 5.4.3$
8	$\angle C'A'B' \cong \angle DA'B'$	$Axiom\ 5.4.1$
9	D lies on A'C'	$Def\ 5.4.3$
10	$D = C'$	$DA' \cong CA'$
11	$C'B' \cong CB$	$6, 10$
12	$\triangle ABC \cong \triangle A'B'C'$	$SSS, 1, 10$

$\Box_{2-column}$

We have introduced a new axiom SSS and deduced the theorem SAS. This raises two questions. Are these propositions consistent with the earlier axioms? Are they needed? For ease of exposition, we name the collection of earlier axioms.

Definition 5.4.12 (HP^-)**.** *We denote the theory with axioms the construction postulates, the axioms of order, and the congruence axioms for angles and segments (Axiom 5.4.1) HP^-. HP adds SSS.*

Methodology 5.4.13 (relative consistency)**.** *We want to show that the new propositions are consistent relative to the earlier ones; that is, find a model of HP^- that also satisfies SSS. Then to show they are necessary, we must find a model of HP^- where SSS fails.*

Exercise 5.4.14 (Consistency of SAS/SSS)**.** *Refer to 2.1.5 and show that SAS is consistent with HP^- because all of them are true in $\Pi(\Re)$.) Modify the proof to show that SSS hold in $\Pi(F)$ for any Pythagorean ($c \in F \Rightarrow \sqrt{(1 + c^2)} \in F$) field F (Hint: [Har00, §16, §17]). Observe that by Theorem 5.4.11, SAS is also relatively consistent with HP^-.*

Methodology 5.4.15 (Consistency/Independence of SAS/SSS)**.** Exercise 5.4.14 shows SSS is consistent with the earlier axioms (HP^-). To show SSS is independent from the earlier axioms, we must show the negation of SSS is consistent. For this, following [Moi90, 112] we show the negation of SAS ($\neg SAS$) is consistent with HP^-. We explain why this suffices in Methodology 5.4.16.

Methodology 5.4.16 (Contraposition and independence)**.** In the next argument 5.4.17, we show that $\neg SAS$ was relatively consistent with HP^-. We now establish some notation of that argument and clarify why, since SSS implied SAS, we can deduce that SSS is independent from HP^-. Let $\phi_{SAS}(T, T')$ be the formulas saying the triangle T, T' have a correspondence between the points such that both T and T' have two sides and the included angle congruent.

Let $\phi_{SAS}(T, T')$ be analogous formula for SAS. And let $\psi(T, T')$ be the formula saying the triangle are congruent (by the same correspondence). Theorem 5.4.11 asserts that every model M of $HP^- \cup \{SSS\}$, $M \models (\forall T, T')\phi_{SAS}(T, T') \rightarrow \psi(T, T'))$. But Theorem 5.4.17 asserts there is a model M of HP^- such that $M \models (\exists T, T')\phi_{SAS}(T, T') \rightarrow \neg\psi(T, T'))$. Thus, there is a model of HP^- where SSS fails and so SSS is independent from HP^-.

We had trouble translating Moise's argument (below) from SAS to SSS. But once the consistency of not SAS by Hilbert's method was understood, the second author converted it to show SSS failed as well

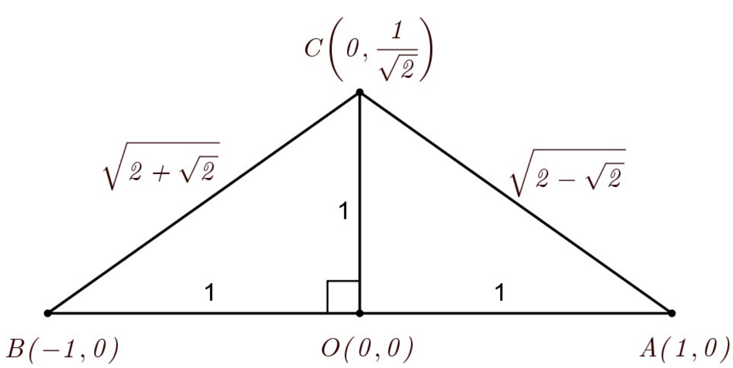
Theorem 5.4.17. *The negation of the postulates SAS and SSS are each consistent with* HP^-.

Proof. We will sketch two versions of the proof. The first is due to Hilbert in 1899, but we present a simplification due to Bernays[28].

Hilbert/Bernays: Hilbert presented his counterexample as a plane in 3-space (which may help in visualization) but we take Bernays' simpler planar version. Define a metric on the real plane by

$$d(\langle x_1, y_1\rangle), \langle x_2, y_2\rangle) = \sqrt{(x_1 - x_2 + y_1 - y_2)^2 + (y_1 - y_2)^2}$$

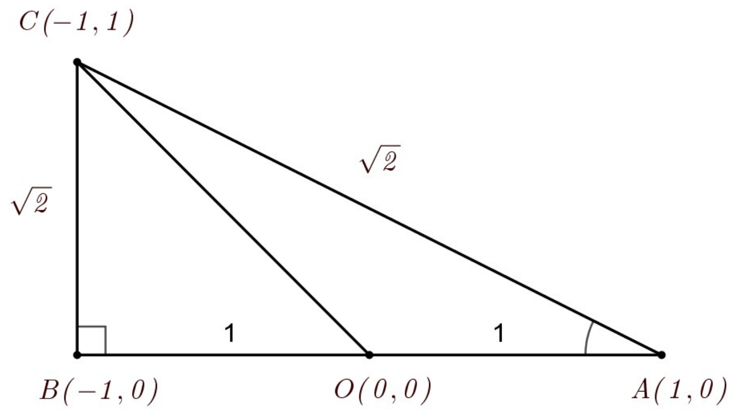
Now check that the axioms of HP^- are satisfied (not trivial). The triangle pictured below shows SAS fails.



Now the triangles $\triangle BOC$ and $\triangle AOC$ satisfy $\phi_{SAS}(BOC, AOC)$ with the right angles as the included angle. But $d(B, C) = \sqrt{2 + \sqrt{\frac{1}{2}}}$ and $d(A, C) = \sqrt{2 - \sqrt{\frac{1}{2}}}$ so SAS fails.

The following diagram shows that SSS fails in the same model.

[28]Bernays was 11 in 1899. However, he worked with Hilbert from about 1918. He cowrote various later editions of the Grundlagen with Hilbert and continued to revise after Hilbert died in 1943 with a final English edition in 1971.



Moise: [Moi90, 112] shows the independence of SAS from the postulates of metric geometry by varying the distance function along a single line. It is difficult to compare the metric geometries of Birkhoff with Hilbert or Euclid because the former are second order. □

We now discuss the group of rigid motions in order to illuminate the notion of superposition and set the stage for studying geometry by transformations. We will use right angles in this study. Activity 5.4.18 motivates Euclid's stipulative (using congruence) definition of *right angle*.

Activity 5.4.18. *Fold paper to make a right angle.*

Definition 5.4.19 (Right Angle)**.** *CCSS G-C0-1 [Euc56, Definition I.10] When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is* right*, and the straight line standing on the other is called a* perpendicular *to that on which it stands.*

Extension 5.4.20 (All right angles are equal)**.** *The 4th postulate of Euclid becomes a theorem of Hilbert ([Hil62, Theorem 15][Har00, 9.6].*

Definition 5.4.21 (ERM: Enough Rigid Motions[29])**.** *A plane* Π *has* enough rigid motions *if*

1. *For any* $A, A' \in \Pi$*, there is a rigid motion* ϕ *with* $\phi(A) = A'$.
2. *For any three points* $O, A, A' \in \Pi$*, there is a rigid motion* ϕ *that fixes O and sends the ray* $\overrightarrow{OA}$ *to* $\overrightarrow{OA'}$ *and*
3. *for any line* ℓ *there is a rigid motion* ϕ *that reflects* Π *over* ℓ.

Note that preserving the first three implies preserving congruence of angles by use of SSS.

Methodology 5.4.22. As we noted in Methodology 5.4.6, rigid motions are defined to clarify the concept of superposition: if a rigid motion takes one figure to another, then they are congruent. This makes Euclid's argument rigorous. [Har00, §17] shows 'enough rigid motions' (ERM) in any Hilbert plane and conversely that from the axioms for a Hilbert plane without SAS, ERM implies SAS. This is essentially Euclid's proof of Proposition I.4. Thus, the problem of superposition can be solved by adding any one of SAS, ERM, SSS to the system HP^-.

[29] [Har00, §17] calls it 'exists' rigid motions, but we say 'enough' to emphasize that transitivity properties are being assumed.

The most immediate formalization of rigid motions is to add second order quantifiers over arbitrary permutations of the set of points. But one use first order logic by adding a new sort $\mathbf{M}$ for motions and a ternary relation $\mathbf{R}$ on P $\times$ P $\times$ $\mathbf{M}$ that for each f in $\mathbf{M}$ the pairs $\langle a, b\rangle$ such that $\mathbf{R}(\mathbf{a}, \mathbf{b}, \mathbf{f})$ is the graph of a rigid motion.

Theorem 5.4.23. *Every rigid motion is a composition of reflections, translations and rotations.*

Proof. A rigid motion ϕ falls into one of four disjoint classes according to the number of points they fix.

1. Suppose ϕ fixes all points; then, $\phi = \psi^2$ for any reflection ψ.

2. Suppose ϕ fixes at least two points A, B but not all. In that case ϕ fixes the line ℓ through AB setwise. So under ϕ each X on ℓ remains the same distance from A and B; thus ℓ is pointwise fixed.

   Suppose $C \notin \ell$ and $\phi(C) = C''$ with $C'' \neq C$ is on the same side of ℓ as C. As ϕ takes the segment AC to AC''. But one is congruent to a proper subset of the other. So $C \notin \ell$ implies $\phi(C) = C'$ is on the opposite side of ℓ from C. Then for any $X \in \ell$, $XC \cong \phi(X)C'$ and $\phi(X) \in \ell$. In particular $AC \cong A\phi(C)$ and $BC \cong BC'$

   Let ℓ' be the line extending CC'. It is distinct from ℓ, so intersects ℓ only in one point D. But since ϕ fixes all lines setwise $\phi(D)$ is on $\ell \cap \ell'$, i.e, $\phi(D) = D$. So $DA \cong DB$ and $DC \cong DC'$. Thus $\triangle DBC \cong \triangle DBC'$ and $\triangle DAC \cong \triangle DAC'$. So $\angle CDB$ is a right angle and $\ell \perp \ell'$. Now we can see that ϕ is a reflection in ℓ.

   Let ℓ'' denote the image of ℓ under ϕ.

3. Suppose ϕ fixes a single point A. Then since ϕ preserves lines, it must be a rotation around A (not equal to a full turn).

4. ϕ fixes no point. Since ϕ sends lines to lines and no points are fixed; if for any ℓ, $\ell \parallel \phi(\ell)$; ϕ is a translation, if not it is a glide reflection [CK17, p 82].

□

Pedagogy 5.4.24 (SLO 1: Van Hiele level of transformational geometry)**.** Taking into account the necessity for a deep understanding of the notion of abstract function[30], one might posit a further 'Van Hiele' level (though not geometric): Ability to work with abstract functions. This may not be an issue for college students but additional work on functions might be helpful.

The HS teacher testifies against this, 'At the HS level we successfully work with transformations without using functions. Working in the coordinate system, given two possibly congruent shapes, visually draw a series of transformations of that shape to find out if the two coincide after the transformations.'

The method of proving the following important exercise is embedded in the proof of Theorem 5.4.11.

Exercise 5.4.25 (Move Angle)**.** *Prove: Let ABC be an angle. For any segment DE, choose a point F so that $\angle ABC \cong \angle DEF$.*

Construction 5.4.26 (Constructing perpendiculars)**. CCSS G-C0-12** *Given a line AD there is a line perpendicular to the line through AD at D.*

[30]See [Har14] for an argument against the use of transformation-based systems in high school; the unfamiliarity of sophomores with functions is a key point.

Proof. Let B be the intersection of the (infinite) line AD with the circle of radius AD centered at D. Now construct an equilateral triangle with base AB by using Axiom 5.2.1 twice to construct the vertex C. Draw CD. SSS implies $\triangle ACD \cong \triangle BCD$; so $\angle CDA \cong \angle CDB$ and therefore $CD \perp AB$. $\square_{5.4.26}$

Extension 5.4.27 (Independence of Congruence Axioms)**.** In the proof we constructed an equilateral triangle using only the first three postulates. We seem to need SSS to finish. [Hil71, p 39] shows by varying the distance formula in the real plane, that the congruence axioms are independent from his first two groups.

Definition 5.4.28 (Straight Angle)**.** *An angle $\angle ABC$ is called a* straight *angle if A, B, C lie on a straight line and B is between A and C.*

Since Euclid does not introduce a measure for angles, he explicitly defines 'right angle', and rough indications of size such as acute and obtuse.

Note a perpendicular creates two right angles on each side of a line. Constructing a perpendicular at the vertex of a straight angle and applying Euclid's fourth postulate (Extension 5.4.20) yields:

Theorem 5.4.29. *CCSS G-C0-9 All straight angles are equal (congruent).*

Proof. Let $\angle ABC$ and $\angle A'B'C'$ be straight angles. Construct lines BD and $B'D'$ perpendicular to AC and $A'C'$, respectively. Now $\angle ABD + \angle DBC = \angle ABC$ and $\angle A'B'D' + \angle D'B'C' = \angle A'BC'$. By Axiom 5.4.1, $\angle ABD = \angle A'B'D'$ and $\angle DBC = \angle D'B'C'$. $\square_{5.4.29}$

We differ from Euclid here in allowing straight angles. Thus, we avoid the awkward locution of 'two right angles' for 'straight angle'.

Theorem 5.4.29 is statement about the uniformity of the plane. In terms of transformations, it says any point and a line through it can be moved by a rigid motion to any other point and any line through it.

Definition 5.4.30. *If two distinct lines intersect, non-adjacent (Definition 3.0.6) angles that have only the vertex in common are called vertical angles.*

Exercise 5.4.31 (CCSS G-C0-9)**.** *Deduce from Theorem 5.4.29 that vertical angles are equal.*

Definition 5.4.32 (Isosceles)**.** *A triangle is* isosceles *if at least two sides have the same length. The angles opposite the equal sides are called the* base angles*. (Note some textbooks require exactly two sides have the same length).*

Activity 5.4.33. *[SLO8, 10: G-CO 11,12] Make two GeoGebra constructions using transformations so that a) one always yields an isosceles triangle but it may not be equilateral and b) the other also yields an equilateral triangle.*

Activity 5.4.34. G-CO 10 *Activity: Prove the isosceles triangle and exterior angle theorems. Compare 'paragraph' and 'two column' proof.*

Theorem 5.4.35. CCSS G-C0-10 *The base angles of an isosceles triangle are equal (congruent).*

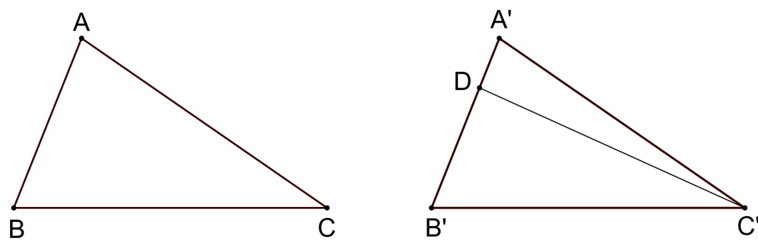
Proof. Let ABC be an isosceles triangle with $AC \cong BC$. We will prove $\angle CAB \cong \angle CBA$. The trick is to prove $\triangle ABC \cong \triangle BAC$. ($\triangle BAC$ is obtained from $\triangle ABC$ by flipping the triangle over its altitude.) We have two ways to prove the congruence. We know $BC \cong AC$ and $AC \cong BC$. We can also note $AB \cong BA$ and use SSS or $\angle ACB \cong \angle BCA$ and use SAS. In any case, since the triangles are congruent $\angle CAB \cong \angle CBA$. $\square_{5.4.35}$

Activity 5.4.36. *Prove the angles of an equilateral triangle are equal. (Note that there are two proofs, using either SSS or SAS, and they are distinguished by which correspondences are made in defining the congruence. Explain this by considering the theorem in terms of rotational or reflective symmetry.)*

The short proof of the following result is a typical use of *proof by contradiction* that emphasizes the close connection among the congruence theorems and the inanity of making ASA an axiom.

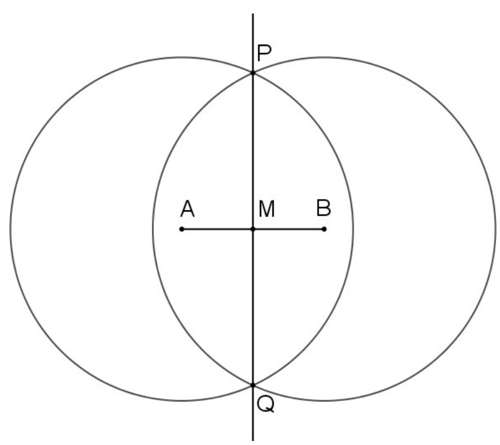
Theorem 5.4.37 (ASA)**. CCSS G-C0-8, G-C0-10** *If two triangles have two angles and the included side congruent, then the triangles are congruent.*

Proof. Suppose ABC and $A'B'C'$ satisfy $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle A'C'B'$ and $BC = B'C'$. We will show the triangles are congruent.



Choose D on $A'B'$ so that $AB \cong DB'$ (We'll assume D is between A' and B' for contradiction. If A' is between B' and D, there is a similar proof.) Now, $AB \cong DB'$, $BC \cong B'C'$ and $\angle ABC \cong \angle A'B'C'$ so by SAS, $\triangle ABC \cong \triangle DB'C'$. Since the angles correspond, $\angle DC'B' \cong \angle ACB$ and so by Common Notion 1, $\angle DC'B' \cong \angle A'C'B'$. But this is absurd since $\angle DC'B'$ is a proper subangle of $\angle A'C'B'$. $\square_{5.4.37}$

Theorem 5.4.38 (Constructing Perpendicular Bisectors)**. CCSS G-C0-12** *For any line segment AB there is a line PM perpendicular to AB such that M is the midpoint of AB.*



Proof. Set a compass at any length at least that of AB and draw two equal circles centered at A and B respectively. Let the two circles intersect at P and Q on opposite sides of AB and let M be the intersection of AB and PQ.

To show PQ perpendicular to AB, note first that $\triangle APQ \cong \triangle BPQ$ by SSS. So $\angle APM \cong \angle BPM$. Then by SAS, $\triangle APM \cong \triangle BPM$. Thus $\angle AMP \cong \angle BMP$. And, therefore these are each right angles by Definition 5.4.19. But $\triangle APM \cong \triangle BPM$ also implies $AM \cong BM$ so M bisects AB. $\square_{5.4.38}$

Note we could be more prescriptive and just as correct by requiring in the proof of Theorem 5.4.38 that the circle have radius AB. But this is an unnecessary additional requirement.

Definition 5.4.39. *If D is in the interior of angle $\angle ACB$, line CD bisects the angle $\angle ACB$ if $\angle ACD \cong \angle BCD$.*

5.5 The Parallel Postulate: SLO7

Of course, the change in viewpoint of what axioms mean (Methodology 1.2) stems from the proof of the independence of the parallel postulate. We do not rehearse here the well-known history but do discuss a subtle ambiguity of the phrase 'the parallel postulate'.

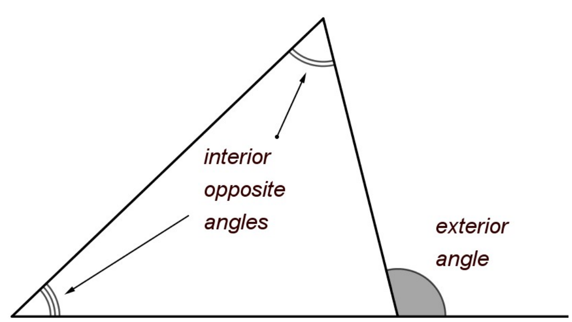
Definition 5.5.1. *Two lines are parallel if they do not intersect.*

The difference between several statements which are close to the parallel postulate provides interesting historical and pedagogical background. The most succinct statement is: For a line ℓ and point A not on ℓ, there is at most one line parallel to ℓ through A. Observe that Euclid proved the existence of parallel lines (Theorem 5.5.3). So spherical geometry, which was studied by the Greeks, could not have been seen as an example to show the independence since any two great circles intersect. Playfair and Hilbert rephrased the postulate as the existence of unique parallel lines; as [HT05] note, even prominent mathematicians were confused by this shift.

The definitions of corresponding, interior, and exterior angles can be found in any geometry text.

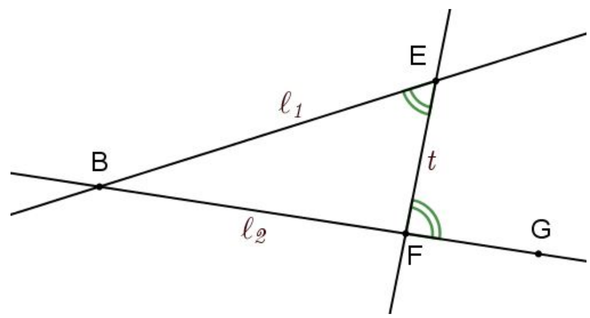
Theorem 5.5.2 (Exterior Angle Theorem, Euclid I.16)**.** *An exterior angle of a triangle is greater than either of the interior and opposite angles.*

Some modern texts write remote interior angles for interior opposite.



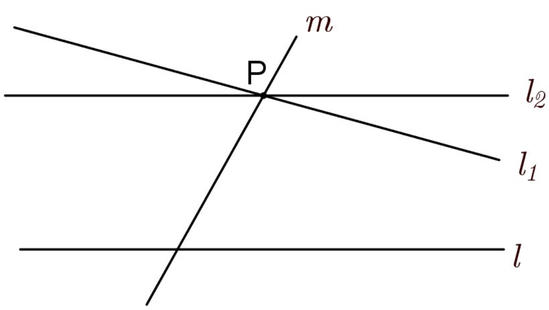
Euclid's proof: `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI16.html`. But there is a subtle dependence on betweenness. See the treatment in [Har00, p 36].

Theorem 5.5.3 (Euclid I.27)**.** *If two lines are crossed by a third and alternate interior angles are equal, the lines are parallel. Thus parallel lines exist.*



Proof. Suppose ℓ_1 and ℓ_2 are not parallel and intersect in point P. The hypothesis says the exterior angle EFG to triangle PFE is equal to the interior angle PEB. That contradicts the exterior angle theorem 5.5.2. So our assumption is wrong.

Axiom 5.5.4 (Heath's statement of Euclid's 5th postulate:)**.** *If a straight line crosses two straight lines in such a way that the interior angles of the same side are less than two right angles, then, if the two straight lines are extended, they will meet on the side on which the interior angles are less than two right angles.*



Theorem 5.5.5. *Axiom 5.5.4 implies there is at most one parallel to ℓ through P.*

Proof. Suppose two distinct lines ℓ_1, ℓ_2 through P are parallel to ℓ. Fix a transversal m that intersects ℓ with P on m. Since they are distinct the sum of the interior angles on the two sides of the transversal must differ. So, for one of ℓ_1, ℓ_2, say ℓ_1, and for one side of m, the sum of the interior angles must be less than a straight angle. Then by Axiom 5.5.4, ℓ is not parallel to ℓ_1, as required. □5.5.5

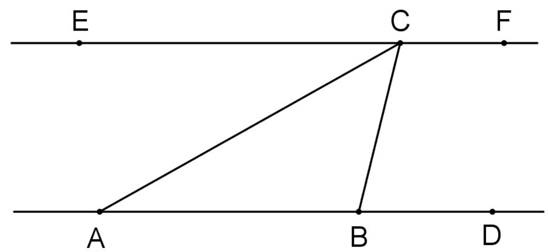
Theorem 5.5.3 and Axiom 5.5.5 establish the distinguishing feature of HP5. For any ℓ and P, there is a unique line parallel to ℓ through P.

A key equivalent to the parallel postulate is that the measures of the angles in a triangle sum to 180^o. In fact, the simplest definition of a degree is $\frac{1}{90}$ of a right angle. Non-Euclidean geometries can be classified by whether that sum is more (semi-elliptic) or less (semi-hyperbolic[31]) than a straight angle [Har00, p 311].

Theorem 5.5.6. CCSS G-C0-10 *HP5 proves the sum of the angles of a triangle is* 180^o.

Proof. We must show the sum of the angles of a triangle ABC is a straight angle.

[31] Elliptic is used when any two lines intersect and Hartshorne reserves hyperbolic for semi-hyperbolic satisfying the limiting parallels axiom.



Draw ECF so that $\angle BCE \cong \angle DBC$ (Exercise 5.4.25). Theorem 5.5.3 implies $EC \parallel AD$. The contrapositive to Axiom 5.5.4 implies that each pair of consecutive interior angles sum to a straight angle for any transveral, in particular BC, sum to a straight angle. So each of $\angle ACB + \angle ACB$ and $\angle ABC + \angle FCB$ is a straight angle. Subtracting $\angle ACB$, we have the equality of alternate interior angles: $\angle ABC = \angle BCF$. Now the sum of the angles of $\triangle ABC$ equals a straight angle as required. $\square_{5.5.6}$

6 Proof that the division of a line into n equal parts succeeds

We began this excursion into axiomatic geometry by trying to prove that for any n we could divide a line into n equal segments. The construction (Figure 1) used only Euclid's first 3 axioms. *We need to show the segments cut off by the C_i are actually equal.* We use the methods of Section 5 to *almost* prove the procedure in Exercise 4.1.3 works. We will discover that entirely different methods are needed for the last step in the proof – the side-splitter theorem 6.0.8.

Pedagogy 6.0.1 (SLO7)**.** The classification of quadrilaterals is a major topic in high school geometry. It is essential to first clarify the notion of 'classify'; it does not help to say 'a square is a rectangle just as a parallelogram is a quadrilateral' (heard from a high school teacher). The analogy the student needs is 'squares are rectangles just as dogs and cats are animals'.

Classifications may be 'exclusive' or 'inclusive'. Euclid requires an isosceles triangle to have *exactly* two equal sides while modern texts include classifications that are inclusive: equilateral triangles are isosceles.

Definition 6.0.2. *A parallelogram is a quadrilateral such that the opposite sides are parallel.*

Theorem 6.0.3. CCSS G-CO.11 *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Proof. Suppose $ABCD$ is the quadrilateral; draw diagonal AC. Then $\triangle ABC$ and $\triangle ACD$ are congruent by SSS. Therefore $\angle BAC \cong \angle ACD$. Now since alternate interior angles are equal, $AB \parallel DC$. Similarly, $BC \parallel AD$. $\square_{6.0.3}$

A similar argument shows:

Theorem 6.0.4 (Euclid I.34, **CCSS G-CO.11**)**.** *In any parallelogram the opposite sides and angles are equal. Moreover, each diagonal splits the parallelogram into two congruent triangles.*

We repeat the diagram from our guiding problem. Since a quadrilateral whose opposite sides are equal is a parallelogram (Theorem 6.0.3), we see $ABCD$ is a parallelogram. We DO NOT know that A_4B_4CD is a parallelogram. In order to establish that it is, we need some more information about parallelograms.

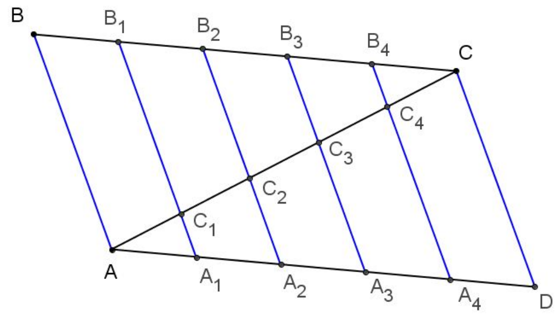


Figure 2: Dividing the line

Motivation 6.0.5 (SLO 1, SLO2)**.** We are giving the proof of our guiding problem in reverse to show how the abstract side-splitter theorem is needed to solve a concrete problem. The proof of it requires a new central idea - proportionality. The next two sections are devoted to providing a firm foundation for proportion. By using Hilbert's proof rather than Euclid's we are not relying on the Archimedean axiom.

Lemma 6.0.6. CCSS G-CO.11 *If one pair of opposite sides of a quadrilateral $ABCD$, labeled as in Figure 2, are equal and parallel, the figure is a parallelogram.*

Proof. Let $AD \| BC$ be congruent. Draw the diagonal AC. By alternate interior angles $\angle BCA \cong DAC$. The triangles ACB and ACD are congruent by SAS, using the hypothesis and that they share a side. So $\angle BAC \cong \angle ACD$. Now viewing AC as a transversal of BA and CD, they are parallel and we finish. $\square_{6.0.6}$

Lemma 6.0.7. *If $ABCD$ is a parallelogram, labeled as in Figure 1 (Section 4), and two points X, Y are chosen on the opposite sides BC and AD respectively so that $XC \cong YD$ then $XCDY$ is a parallelogram.*

Proof. Apply Lemma 6.0.6 with XC and YD equal and parallel.

To finish the proof, we need a very strong result:

Theorem 6.0.8 (Euclid VI.2: Side-splitter, CCSS G-SRT.4)**.** *If a line is drawn parallel to the base of a triangle the corresponding sides of the two resulting triangles are proportional and conversely.*

Proof of the guiding problem assuming sidesplitter: By repeating the argument for Lemma 6.0.7, we show all the lines $A_iC_iB_i$ are parallel. In particular the line C_4B_4 is parallel to the base B_3C_3 of triangle CC_4B_4. Applying Theorem 6.0.8 to $\triangle CAB$ with parallel CB_4, we complete our proof as follows:

$$\frac{CB_4}{CB_3} = \frac{CC_4}{CC_3}.$$

But we constructed $B_4C \cong B_3B_4$, so $C_4C \cong C_3C_4$, which is what we are trying to prove. Now move along AC, successively applying this argument to each triangle. $\square_{4.1.3}$

The analogous problem of trisecting an angle was open for 2000 years before being proved impossible in the 19th century using field theory.

7 Finding the underlying field

We reduced verifying our cutting the line algorithm to proving the side-splitter theorem VI.2. Hilbert defines a (semi)-field of segments (addition and multiplication on the positive elements of an ordered field). He thus has the modern algebraic theory of proportion and VI.2 follows easily (Section 8). Then (Section 9) he defines a measure of area function which recovers Euclid's theory of area and connects it with numerical measures of area.

Motivation 7.1. *[SLO 7 Irrationality: the Pythagorean scandal]* The geometry course is an excellent place to organize historically and conceptually the college students' understanding of irrational and transcendental numbers (Section 10). Two or more magnitudes are *commensurable* if they share a common measure[32]. Two feet and three feet are commensurable, each being a multiple of a foot; but the diagonal and side of a square are incommensurable. Thus, the irrationality of $\sqrt{2}$ is usually attributed to 5th century BCE Pythagoreans. Eudoxus found a way to define the ratio between incommensurables in the 4th century BCE and expounded in Euclid Book V on proportion, a couple of generations later. Crucially, this was a study of 'magnitudes' of various dimensions. The notion of ascribing a number to a measure of area was only adopted in geometry during the 19th century AD and put on a firm footing by Stolz and Pasch as expounded in [Hil62]. A beauty of Hilbert's approach is that he shows that (a suitable translation) of the (first order) axioms of Euclidean geometry allow the measure of area in any Euclidean plane (Notation 3.0.10) by interpreting a field into the plane. In Section 10, we will note how the real numbers provide the most commonly used example. For further background on Greek study of irrational numbers see [Smo08].

The proof of the side-splitter theorem (Theorem 6.0.8.) is difficult because the meaning of ratio between two incommensurable sides is obscure at best. To solve this problem, Hilbert defines *geometrically* a multiplication of line segments by line segments to give a line segment. Identify the collection of all congruent line segments and choose a representative segment OA for this class. There are three distinct historical steps[33]. i) In Greek mathematics numbers (i.e., $1, 2, 3 \ldots$) and magnitudes of various kinds ('length', 'area', 'volume') are incomparable categories. Numbers simply count the number of some unit; the unit varies from situation to situation. For them the notion of assigning a number as the length of the diagonal of a unit square is incomprehensible. ii) Hilbert introduces an addition and multiplication on line segments and proves the geometric theorems to show that these operations (on line segments on a fixed line) satisfy the field axioms except for the existence of an additive inverse. Here the points on fixed ray represent the positive elements (numbers) in the field,) iii) Finally define the additive inverse. Then both positive and negative numbers are identified with points on a line.

We first introduce an addition and multiplication on line segments and then prove the geometric theorems to show that these operations satisfy the field axioms except for the existence of an additive inverse.

Notation 7.2. *Recall (Axiom 5.4.1) that congruence forms an equivalence relation on line segments. We fix* a ray $\overrightarrow{\ell}$ *with end point* 0*. We consider the segment* $0A$ *on* $\overrightarrow{\ell}$ *as the representative of its congruence class of segments. We will often denote the class (i.e., the segment* $0A$*) by* a*. We say a segment (on any line)* CD *has length* a *if* $CD \cong 0A$*.*

Similarly, Theorem 5.2.6 (Euclid's Proposition I.2) established the way to add segments.

[32]the reciprocal of the least common multiple in Footnote 13.

[33]For SLO7, see [GG09] and Heath's notes to Euclid VI.12 (http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html.)

Definition 7.3 (Segment Addition)**.** *Consider two segment classes a and b. Fix representatives of a and b as OA and OB in this manner: Extend OB to a straight line, and choose C on OB extended (on the other side of B from A) so that so that $BC \cong OA$. OC is the sum of OA and OB.*

Diagram for adding segments

Activity 7.4. *Prove that this addition is associative and commutative.*

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. We discuss finding an additive inverse after Definition 7.12. Following Hartshorne [Har00], here is our official definition of segment multiplication.

Definition 7.5 (Multiplication)**.** *Fix a unit segment class 1. Consider two segment classes a and b. To define their product, construct a right triangle*[34] *with legs of length 1 and a. Denote the angle between the hypotenuse and the side of length 1 by α.*

Now construct another right triangle with base of length b with the angle between the hypotenuse and the side of length b congruent to α. The length of the leg opposite α is ab.

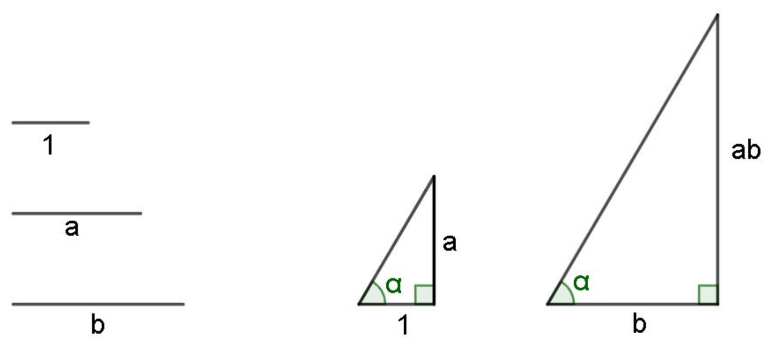


Figure 3: Multiplication

Commutativity of this operation must be shown.

Exercise 7.6. *We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length a end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e., 3 segments of length 1 laid end to end) by the segment of length a. Prove geometrically that these are the same.*

Before we can prove the field laws, in particular commutativity of multiplication, hold for these operations, we introduce a few more geometric facts. The crux of the argument is to prove that the multiplication

[34]The right triangle is just for simplicity; we only need to make the two triangles similar.

is associative and commutative. Hilbert and many successors give this argument as arising from the Desargues and Pappus theorems which hold in HP5 (neutral geometry plus the parallel postulate). We rely on the cyclic quadrilateral theorem, because the techniques of its proof are more similar to standard high school material.

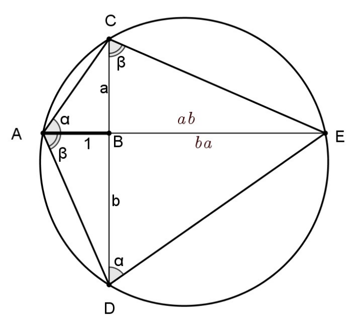
Theorem 7.7 (Euclid III.20)**. CCSS G-C.2** *In a circle, if a central angle and an inscribed angle intercept the same arc, the inscribed angle is congruent to half the central angle.*

Exercise 7.8. *Do the activity: Determining a curve (determinecircle.pdf).*

Activity 7.9. *Prove a central angle is twice an inscribed angle that intercepts the same arc. How many diagrams (cases) must you consider? This activity is on the website in both java and GeoGebra.*

We need (Corollary 7.10) of [Har00, Proposition 5.8] (Corollary 7.10), which is a routine (if sufficiently scaffolded) high school problem.

Corollary 7.10. *[***CCSS G-C.3: Cyclic Quadrilateral Theorem***] Let $ACED$ be a quadrilateral. The vertices of $ACED$ lie on a circle (the ordering of the name of the quadrilateral implies A and E are on the opposite sides of CD) if and only if $\angle EAC \cong \angle CDE$.*



Proof. Given the conditions on the angle draw the circle determined by A, C, E. Let D' be the intersection of DE with the circle. By Theorem 7.7, $\angle AD'E \cong \angle EAC$. But $\angle EAC \cong \angle AD'E$ so $D = D'$ as required. Conversely, given the circle, apply Theorem 7.7 to get the equality of angles. $\square_{7.10}$

Theorem 7.11. *The multiplication defined in Definition 7.5 satisfies:*

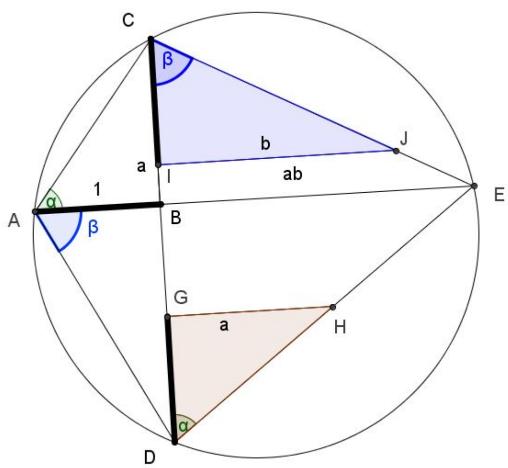
1. *For any a,*
$$a \cdot 1 = a$$
2. *For any a, b*
$$ab = ba.$$
3. *For any a, b, c*
$$(ab)c = a(bc).$$
4. *For any a there is a b with $ab = 1$.*
5. *$a(b + c) = ab + ac$.*

Proof. We prove item 2 (Figure below), since that requires some work. The slight variants for associativity and distributivity are in [Har00, 19.2].

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend CB to D on the other side of AB from C so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.

Since $\angle CAB \cong \angle EDB$ by Corollary 7.10, $ACED$ lie on a circle. Now apply the other direction of Corollary 7.10 to conclude $\angle DAE \cong \angle DCE$ called β. Now consider the multiplication beginning with triangle $\triangle DAB$ with one leg of length 1 and the other of length b. Then since $\angle DAB \cong \angle BCE$ and the leg adjacent to $\angle BCE$ has length a, the length of BE is ba. Thus, $ab = ba$.

The key point for proportionality is 4): the ability to find inverses. This is done by noting that in Figure 4, if multiplication by a is given by the angle α, multiplication by a^{-1} comes from β, the other acute angle in the right triangle.



$\square_{7.11}$

We have a *semi-field* because the addition does not form a group because there are no additive inverses (negative segments). This is important for Hilbert because he is giving an entirely geometric proof. We now show how to modify the construction to an additive group on each line. With this geometrically based field we give in the next section an algebraic basis for the theory of proportion which allows us to prove side-splitter.

Definition 7.12 (Adding points)**.** *Recall that a line is a set of points. Fix a line ℓ and a point 0 on ℓ (now all of ℓ instead of the ray $\overrightarrow{0A}$). We define an operation $+$ on ℓ. Now we identify a segment class a with the directed length of the segment $0A$. And write $-a$ for the segment class $A'0$ where $A'0 \cong 0A$ but on the opposite side of 0.*

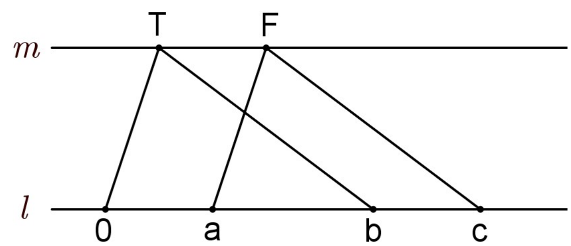
For any points a, b on ℓ, we define the operation $+$ on ℓ:

$$a + b = c$$

if c is constructed as follows.

1. *Choose T not on ℓ, and m parallel to ℓ through T.*
2. *Draw $0T$ and bT.*
3. *Draw a line parallel to $0T$ through a and let it intersect m in F.*
4. *Draw a line parallel to bT through F and let it intersect ℓ in c.*

Diagram for point addition



$$0b \cong ac$$

That is, $0a + ac = 0c$ which means $a + b = c$.

After extending multiplication to the whole line by requiring that multiplication by a negative reverses orientation we have proved:

Theorem 7.13. *If Π is a model of $HP5$, then, fixing any two points in Π as $0, 1$, there are first order formulas defining $<, +, \times$ such that $\langle \ell, <, +, \times \rangle$ is an ordered field.*

Methodology 7.14. Definition 2.1.5 showed we could define a coordinate plane over any field. Combined with Theorem 7.13, we have a bi-interpretation of fields and planes, described informally in Methodology 11.1 and formally in the Appendix to the supplement. This means that the algebraic proofs in high school analytic geometry can (but not easily) be converted to synthetic proofs in first order geometry. In particular, the following diagram illustrates how the equation of a line through the origin is immediate from the definition of multiplication.

Problem 7.15. *Add a and b (i.e., construct c) when a is to the left of 0 on ℓ.*

Algebraically, the additive inverse of a is a' provided that $a + a' = 0$. Construct the inverse of a.

8 Similarity, Proportion, and Side-splitter

This is one half of the culmination of Hilbert's program (Methodology 2.1.12). On a purely geometric basis (by Section 7) we define proportion and prove the side-splitter theorem. We need a couple of definitions. Recall that in Section 7 we defined a field whose elements were line segments on a fixed line $\overline{01}$. So we make the following definitions using a, b etc. to range over segments $(O, A), (O, B)$ etc. Most texts will have identified these segments with real numbers. We emphasize that the results are much more general than that.

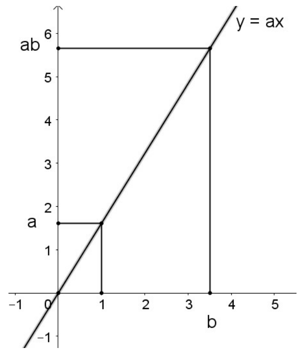


Figure 4: Multiplication

Definition 8.1. *Let a, b, a', b' be segments on a fixed line $\overleftrightarrow{01}$. Then we say the ratios $a : b$ and $a' : b'$* satisfy the proportion $a\colon b = a' : b'$ *(also written $a\colon b :: a' : b'$ or $\frac{a}{b} = \frac{a'}{b'}$) if $ab' = ba'$.*

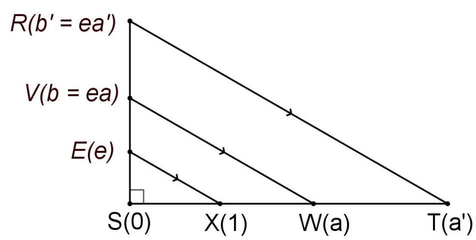
Definition 8.2. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if under some correspondence of angles, corresponding angles are congruent; e.g., $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.*

Activity 8.3. *Various texts define 'similar' as we did, or as corresponding sides are proportional or require both. Discuss the advantages of the different definitions. Why are all permissible?*

Theorem 8.4. *Similar triangles have proportional sides.*

Proof. Suppose SVW and SRT are similar triangles as displayed in the diagram below we show

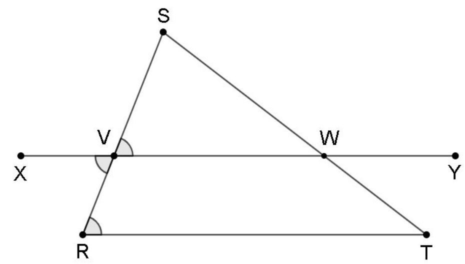
$$\frac{SV}{SR} = \frac{SW}{ST}.$$



Consider the special case that $\angle RST$ is a right angle. Label SW as a, ST as b, SV as a', SR as b', Then think of S as 0 and X as 1, using segment multiplication, the diagram shows $b' = eb$ and $a' = ea$. Now by multiplying algebraically and dividing by e we have picked a point X of ST with $SX \cong 01$. $ab' = ba'$. So by definition $a : b = a' : b'$ or $\frac{SW}{ST} = \frac{SV}{SR}$. [Hil71, p. 56] gives the half page argument that the restriction to a right angle is unnecessary. $\square_{6.0.8}$

Theorem 8.5. *Euclid VI.2: Side-splitter CCSS G-SRT.4 If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; conversely, if the sides of the triangle are cut proportionally, then the line joining the dividing points is parallel to the remaining side of the triangle.*

Proof. On $\triangle SRT$ draw VW parallel to RT. As in the following diagrams, extend VW to a line and pick points X and Y on VW on opposite sides of the triangle as shown.

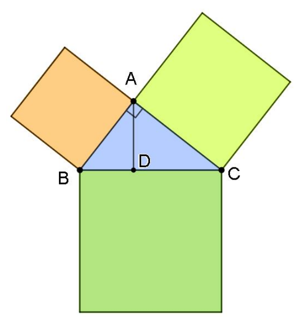


Now $\angle XVR$ and $\angle VRT$ are alternate interior angles for the transversal RS crossing the two lines XY and RT. So $\angle XVR \cong \angle VRT$ if and only if $VW \parallel RT$. But $\angle XVR \cong \angle SVW$ since they are vertical angles. So $\angle SVW \cong \angle VRT$ if and only if $VW \parallel RT$. So $\triangle SRT$ and $\triangle SVW$ are similar and we finish by Lemma 8.4.

Conversely, suppose we know VW cuts each side proportionally. By Theorem 5.5.3, choose W' on ST with $VW' \parallel RT$. The parallelism and our definition of multiplication imply VW' is to ST as SV is to ST. But we know SW' satisfies the same proportion so $W = W'$. Thus, $VW \parallel RT$ as required. $\square_{6.0.8}$

As we will sketch in Section 9, Euclid developed the notion of area (He says equal figure.) in I.35-I.48. *Commentators agree that this was specifically to avoid the use of proportion in the proof of Pythagoras.* In particular, Euclid needed the Archimedean axiom for his theory of proportion and so to prove the side-splitter. Hilbert grounds the theory of proportion *purely geometrically* without assuming Archimedes' axiom.

Exercise 8.6 (Euclid VI.31 CCSS G-SRT4)**.** *Prove the Pythagorean theorem using similarity.*



Activity 8.7. *Consider various proofs of the Pythagorean Theorem Activity: Pythagorean Theorem (pythag.pdf). Reconstruct President Garfield's diagram (Garfield.pdf has a copy of the original article.) and work out his proof of the Pythagorean theorem. ('On the hypoteneuse* cb *of the right angled triangle* abc*, draw the half-square* cbe*', means 'draw the right triangle* cbe *such that* be *is the diagonal of a square with side* cb*.)*

Activity 8.8. *The activity incenter.pdf contains some 'real-world' applications of incenter and Hartshorne's direct proof of the side-splitter theorem for segment arithmetic (Proposition 20.1 of [Har00]) without using area.*

9 Area of Polygons

Pedagogy 9.1 (SLO 1, 2, 5, 7)**.** Experience with college students in precalculus and calculus who react to min-max problems by saying ‘I know the formula is lw or $2l + 2w$ but I don’t know which’ motivates this section. The connection between (equi)-decomposition and area needs to be made in high (if not middle school).

We expound here the distinction between defining ‘equal area’ by one of several notions of ‘decomposition equivalence’ and by ‘equal measure’. In making this connection, Hilbert bridges one of *the most significant distinctions between Greek and 19th century geometry* and fulfills the challenge of Motivation 2.1.12.

Methodology 9.2 (SLO 4: What is area?)**.** This section expounds the differences among three methods of computing area that are frequently conflated in high school texts. Euclid begins by (implicitly) defining what it means for two figures to have same area (Euclid-equal, 9.9). By this means, he is able to prove the Pythagorean theorem without invoking the notion of proportion – showing it is a fully *geometric* result. In contrast, calculus based notions of measuring area rely fundamentally on approximating figures by infinitely many smaller figures and taking limits.

Using the field defined in Section 7, Hilbert defines ‘equal area’ by a slightly different notion (Hilbert-equal) and introduces a finite procedure to assign a numeric value as the area of a polygon. In fact, these three notions of equality are the same. However, they cannot be proved the same as equi-decomposable (scissors congruent) without the use of the Archimedean axiom.

Pedagogy 9.3 (equidecomposition in a high school text)**.** This notion (called area by dissection) is very nicely handled in [Edu09a, 197-205] with two caveats. The rectangle postulate [Edu09a, Postulate 3.3] (area of a rectangle is bh) should be labeled a definition and the Scissors-congruence postulate [Edu09a, Postulate 3.4] should be labeled as a theorem whose proof is beyond the scope of the course. This latter advice is reinforced by [Edu09a, §3.8] which exemplifies the method of proof. The authors are quite right not to discuss the inductive proof in a high school text.

In Section 5.3 we established a linear order on (congruence classes of) segments by $[AB] < [CD]$ if $AB \cong A'B'$ for some proper subsegment $A'B'$ of CD. This is not so easy in two dimensions; a long skinny rectangle might or might not ‘be bigger’ than a short fat one. There are even two incomparable triangles. In this section we discuss what sorts of objects we can assign area to and when two ‘figures’ have the same area?

Definition 9.4.

1. *A* (rectilineal) figure *is a subset of the plane that can be expressed as a finite union of disjoint triangles (Sides may overlap; interiors can’t.).*
2. *A* polygon *is a closed figure whose sides are line segments that intersect only at their endpoints and each endpoint is shared by exactly two segments. Closed means you can trace the outer edges and come back to where you started without any repetition.*

The term figure is introduced both to explicitly describe what configurations ‘have area’ and to allow for various types of decomposition.

Definition 9.5 (Two ways to measure)**.**

Method 1: ‘equal’ area Define an equivalence relation $E(P, P')$ on figures and define $[P_1] < [P_2]$ if some representative of $[P_1]$ is congruent to a proper subset of a representative of $[P_2]$.

We give three different equivalence relations of this sort in Definition 9.9, 9.10, 9.11. We see in Theorem 9.13 that the first and third are the same for HP5; Scissors congruence, 9.10, becomes equivalent in Archimedean geometries.

Method 2: equal numerical measure

Analytic measure Fix a unit of area; say, a square; tile the plane with congruent squares. Then to measure a figure, continually (perhaps infinitely often) refine the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure. This notion is well-defined only for Archimedean geometries; This hypothesis is used but never stated in [Bol78].

Geometric measure (Hilbert) Decompose the figure into finitely many disjoint triangles, which are each assigned area $\frac{bh}{2}$, and add those areas.

We call the last geometric area because the multiplication is the geometric multiplication of Section 7. We consider now the three ways to implement Method 1. Before giving the formal definition, we see how two of these methods are abstracted from the proof of Euclid I.35.

Theorem 9.6 (Proposition I.35)**.** *Parallelograms which are on the same base and in the same parallels equal one another.*

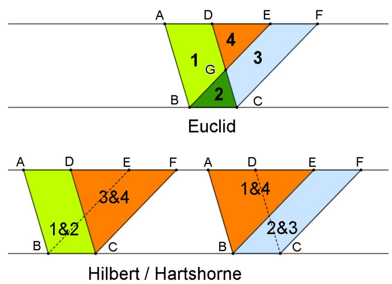


Figure 5: Euclid I.35

Proof. There are two ways of understanding this proof. The terms 'Euclid equal' and 'Hilbert equal' are defined below (generalized from this argument. Euclid says triangle 1 + 4 is congruent to triangle 3 + 4. Subtract 4 from the first to get trapezoid 1 and from the second to get trapezoid 3. So 1 and 3 have the same area. Add 2 to each to see the two parallelograms, 1 + 2 and 3 + 2, have the same area.

Hilbert says adding triangle 3&4 to parallelogram 1&2 gives the same as adding triangle 1&4 to parallelogram 2&3, and 1&4 and 3&4 are equidecomposable (in this case congruent) so we can conclude the two parallelograms have equal area. The distinction is that the weaker condition 'equidecomposablity' on the triangles 1&4 and 3&4 allows him to build scissors decomposition into his definition 9.11. $\square_{9.6}$

Both understandings of the proof required *both adding and subtracting area* rather than scissors congruent. One way of expressing the Archimedean postulate is to say 'every line segment is finite'. We show in Theorem 10.1.4 that there are non-Archimedean planes that satisfy HP5. Neither understanding of the proof of Theorem 9.6 requires finite line segments. But we now see that scissors congruence does.

Example 9.7. *Suppose the lines BE and CF are infinitely long while the lines AB and CD have finite length. The parallelograms $ABCD$ and $EBCF$ are not equidecomposable although they have the same area by Theorem 9.6.*

Proof. The sides of $ABCD$ are all finite and so a decomposition must be into finitely many triangles, each with all sides finite. But a decomposition into finite triangles of $EBCF$ requires infinitely many triangles because the entire line EB must be covered by edges of the decomposing triangles. However, $ABCD$ and $EBCF$ have the same altitude and same base so they have the same geometric measure. □9.7

Pedagogy 9.8. The distinction described in this section is not high school material. But it is background to avoid fallacious assertions. It is natural in K-12 education to describe equal area in terms of scissors congruence and certainly scissors congruent figures have the same area. But Example 9.7 shows that in some models of $HP5$ there are parallelograms of equal area that are not scissors congruent. Thus, their putative equivalence is still another example of an independent proposition. This is not a topic for high school. But teachers can remember to say 'scissors congruent figures have the same area' while not saying 'figures with the same area are scissors congruent'.

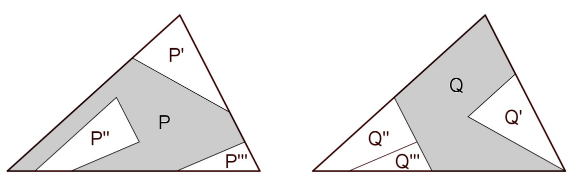
Definition 9.9. *[Euclid-equal polygons] For figures A and B:*

1. *A has 1-equal area with B if there is a figure C such that $A + C$ is congruent to $B + C$ or there is a C such that $A - C$ is congruent to $B - C$.*
2. Euclid-equal *(provably same as equi-complementable) is the transitive closure of the symmetric and reflexive relation* 1-*Equal content.*

Definition 9.10 (Scissors Congruence)**.** *Two polygons are* scissors congruent *or* equidecomposable *if one can be cut up into a finite number of triangles which can be rearranged to make the second.*

SLO7: It is a sign of Euclid's genius that he realized that a type of refinement of scissors congruent, dubbed *equal content* by Hilbert around 1900, allowed the proof of proportionality of area to base and height without appeal to Archimedes' axiom.

Definition 9.11 (Hilbert-equal: Equal content)**.** *Two figures P, Q have* equal content aka equicomplementable *or* Hilbert equal[35] *if there are figures $P'_1 \ldots P'_n$, $Q'_1 \ldots Q'_n$ such that none of the figures overlap, each P'_i and Q'_i are scissors congruent and $P \cup P'_1 \ldots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \ldots \cup Q'_n$.*



Here is Hilbert's schema for measuring area.

Lemma 9.12. *[Har00, §23] For any n (k) and any triangulation of a figure by n (k) triangles with base b_i and height h_i the sum $\Sigma_{i<n} \frac{b_i h_i}{2} = \Sigma_{i<k} \frac{b_i h_i}{2}$ is the same. That value is the geometric measure of the area of a polygon, so is represented by a line segment. Thus, the equivalence relation imposed by 'same geometric measure' is well-defined.*

[35]The diagram is taken from [Hil71].

While Euclid-equality is transitive by definition, it is considerably more difficult [Har00, p 199-201] to prove that Hilbert-equality is transitive.

Theorem 9.13. *[Har00, §23] In any plane satisfying HP5, figures have equal area under either Hilbert's or Euclid's notion of equal area if and only if they have the same geometric measure.*

However the analytic method of Definition 9.5 only joins the equivalence if the field is Archimedean.

Definition 9.14. *Two figures are* analytically equivalent *if they have the same analytic measure.*

Theorem 10.1.4 proves the existence of a non-Archimedean field and thus a non-Archimedean geometry (which will satisfy all the first order axioms).

Fact 9.15 (Wallace-Bolyai-Gerwien Theorem[36])**.** *Two polygons* in an Archimedean plane *are equidecomposable (scissors congruent) if and only if they have the same analytic measure.*

Note that the Archimedean hypothesis is essential. If the line BE in Figure 5 for Theorem 9.6 is infinite (Invert the segment $\overline{AB}$ created in Remark 13.2.), while all lines in $ABCD$ are finite then the parallelograms $ABCD$ and $EBCF$ are not equidecomposable even though they are Hilbert and Euclid equal. This equivalence often appears in high school text books without making it clear that it requires a vastly stronger hypothesis than any of the other results on polygons.

Interestingly, not all polyhedra (3D figures with plane polygonal surfaces) with the same volume are scissors congruent; for example, a regular tetrahedron cannot be cut up into polyhedra and rearranged into a cube [Bol78].

Fact 9.16 (Dehn-Sydler Theorem)**.** *Two polyhedra in $\Re^3$ are scissors congruent iff they have the same volume and the same Dehn invariant.*

Dehn [D] proved in 1901 that equality of the Dehn invariant is necessary for scissors congruence. Sydler proved the converse forty years later. We noted in Motivation 2.1.11 that Euclid first proved Theorerm 9.17 and then deduced sidesplitter. In contrast, Hilbert deduces the result via his theory of proportion.

Theorem 9.17 (Euclid VI.I)**.** *If two triangles have the same height, the ratio of their areas equals the ratio of the length of their corresponding bases.*

Proof. Definition 9.5 gave the geometric measure of a triangle to be $\frac{bh}{2}$ and Theorem 9.13 showed geometric measure is equivalent to Euclid equal. So the result follows from realizing that $A = \frac{bh}{2}$ can be read as 'the area is jointly proportional to the base and the height. $\square_{9.17}$

In Euclid this result holds for irrationals only by the method of Eudoxus, which is a precursor of the modern theory of limits, but did not envision the existence of arbitrary real numbers. Euclid implicitly relies on the Archimedean axiom in his definition of proportion. He deduces side-splitter from the proportionality while Hilbert goes in the other direction[37]. The development here shows that for any triangles which occur in a geometry satisfying the axioms here[38] the areas and their ratios are represented by line segments in the field.

[36]The forward direction was proven in the 19th century by William Wallace (not one of the progenitors of calculus), Farkas Bolyai (his son discovered non-Euclidean geometry) and P. Gerwien. See Wikipedia.

[37][Edu09b] shows the area of one of two similar figures is r^2 times the area of the other, where r is the constant of proportionality between lengths. They deduce this from side-splitter. It was Al Cuoco of the CME team who alerted the first author to Euclid going in the other direction.

[38]Crucially, neither Archimedean, nor Dedekind complete, is assumed.

10 Archimedes, Dedekind, and Completeness

We quoted in Methodology 1.2, Hilbert's desire 'to choose for geometry a simple and complete set of independent axioms'. In this section we first discuss Hilbert's continuity axioms in the context introduced in Methodology 1.4: T is *descriptively complete* [Det14] if T implies all the statements in our preexisting list of 'true geometrical statements'. Then we consider more formal notions of 'complete' which were developed in the first third of the 20th century.

A main theme of the preceding sections is that Hilbert (1899) established descriptive completeness of his first four groups of first order axioms (not only for Euclid's plane geometry but establishing Descartes' analytic geometry [Har00, §20-23]). Hilbert's second order continuity axioms (Group V) aimed at establishing

1. a *geometric* basis - set of geometric axioms - for what is variously called Cartesian/coordinate/analytic geometry over the real numbers (as understood by Hilbert, not Descartes, though Descartes was pointing the way).

2. These axioms are categorical. A theory T is categorical if it has a unique model (up to isomorphism).

In the late 19th century the only rigorous basis for the real plane was to construct the real numbers from the natural numbers by [Ded63] (1888) and then construct the Cartesian plane over the reals. But Hilbert in 1899 (Section 7) works from a plane satisfying geometric axioms and defines the field in it. By adding an axiom implying the plane and the field are unique both goals are reached. This gives descriptive completeness for the modern view of real geomety. The rather complicated story for completeness as conceived in modern logic is told in Methodology 10.2.3.

10.1 Continuity Axioms

Hilbert's Group V (continuity axioms) contains two axioms. The Archimedean axiom is usually taken as a property of an ordered group (or field). However, for geometry it says for any pair of line segments AB and CD there is a natural number n such that n copies of AB cover CD. Since the n is unbounded, this axiom is not first order but rather in a logic called[39] $L_{\omega_1,\omega}$. Note that the statement of the Archimedean axiom involves some notion of 'addition of lengths'.

Euclid uses the Archimedean axiom in Book V on proportion and then to prove VI.2, the side-splitter theorem. As we have seen (Theorem 8.5) Hilbert establishes VI.2 on the basis of axiom groups I-IV which are all first-order.

Although expressed in an unusual way[40], Hilbert's *completeness axiom* can be regarded as the *second order* statement asserting Dedekind completeness (equivalently, the least upper bound axiom) in the theory of ordered fields. The only use of these axioms is to [Hil62, §17] to prove the categoricity and unite synthetic and analytic geometry. Hilbert's other applications of these axioms are to proving metamathematical (independence/consistency) results.

A standard result in advanced calculus courses ([Spi80, p 124]) shows every Dedekind complete field is Archimedean. So the Archimedean axiom is redundant in Hilbert's system. He singles it out to show that the '(Dedekind) completeness' is not needed for such important results as Theorem 9.15 showing the equivalence between decomposition and measure for determining area.

The example of the plane over the real algebraic numbers defined in Activity 3.0.5 shows:

[39]Quantification is allowed only over individuals but infinite conjunctions and disjunctions are allowed. The Archimedean axiom asserts an infinite disjunction: $\bigvee_n \phi_n(A, B, C, D)$ where ϕ_n says n copies of AB cover CD.

[40]The axiom asserted 'a maximal Archimedean geometry', hence unique. Currently, 'categoricity' means uniqueness. And, complete means negation complete 10.2.2. Strictly speaking, Hilbert's 'maximality' axiom is only expressed in the arcane 'sort logic' [Vää14].

Theorem 10.1.1. *The ruler and protractor postulates and Hilbert's completeness axioms are independent from all the other axioms.*

Pedagogy 10.1.2 (Impact on other courses)**.** Theorem 10.1.1 is important for teaching pre-calculus and calculus as it emphasizes the gap between transcendental and algebraic numbers. That is, when teaching the hierarchy $\mathbb{Q} \subsetneq \Re \subsetneq \mathbb{C}$, the high school teacher should be aware that the first gap is biggest (in cardinality). In fact, there are only countably many algebraic numbers (real or complex), while the transcendentals have the same cardinality as the reals – although high school students are familiar only with π and expressions containing it as well values of trignometric functions.

Pedagogy 10.1.3 (Student background)**.** In particular, for Hilbert to show that his results do not depend on Archimedes, he must show that non-Archimedean fields exist. Hilbert gives a concrete proof (involving the study of rational function fields) of the existence of non-Archimedean fields, taking t to be infinite in an ordering of the rational function field $\Re(t)$. This is *not* usually taught in an undergraduate algebra course. We give now a proof using the 'compactness' theorem for first order logic – a standard topic in an undergraduate course in mathematical logic.

Theorem 10.1.4 (Proof of Existence of non-Archimedean fields)**.** *There exists a non-archimedean field.*

Proof. We note in Methodology 10.2.4 that Tarski's negation-complete extension of Euclidean geometry is the theory of $\Pi(M)$ for any M such that $M \models T_{rcf}$, the set of all first order sentences in the vocabulary of fields true in the real field. It has models of arbitrary cardinality and most are non-Archimedean (the Archimedean fields are imbedded in $\Re$. Consider the set Σ of sentences: $\{n \times \overline{AB} < \overline{01}\}$ for $n \in \mathbb{N}$. Clearly every finite subset of Σ is satisfiable. By the compactness Theorem[41], they are simultaneously satisfiable in some model M of T_{rcf}. Such an $\overline{AB} \in M$ is an infinitesimal. $\square_{10.1.4}$

Pedagogy 10.1.5 (The two uses of the continuity axiom)**.** There are two places where the continuity axioms are *necessary* for a topic that may occur in a high school geometry course:

1. formulas like $C = 2\pi r$ and $A = \pi r^2$ can be true only if π is in the coordinatizing field.
2. Theorem 9.15 (figures are equidecomposable (scissors congruent) iff they have equal analytic measure).

Since Hilbert showed the equivalence among equi-complementability and equal geometric area from HP5, the essential role of Archimedes is to show equal analytic measure implies scissors congruence.

Methodology 10.1.6 (Determining π)**.** A key question is whether it is the same π in each of the equations in Pedagogy 10.1.5.1 for area and circumference of a circle. Archimedes argues they give the same ratio, which is not a number for him. In [Bal19] we outline arguments of [Apo67, Spi80] using calculus and [Wei20] clarifying Archimedes' approximation.

10.2 Consistency, completeness, and categoricity

We return to the issue of making the notion of a complete theory rigorous.

[41] In first order logic, if every finite subset of a set Σ of first order sentences is satisfiable so is Σ.

Methodology 10.2.1 (LogicS)**.** Hilbert wrote [Hil62] in German, not in a formal language. So he had no precise way of expressing negation completeness. What makes a German sentence 'geometric'? Roughly 20 years after the publication of [Hil62], Hilbert developed his notion of formal logic. In his general formulation quantification is allowed over individuals (x), sets of individuals (X), sets of sets of individuals and so on (This corresponds to Russell-Whitehead's theory of types.) Hilbert later observed[42] that groups I-IV are what we now call first order (for him, the restricted predicate calculus): quantification is only over individuals and only finite conjunctions and disjunctions are allowed in combining statements. Now the key distinction arises from Gödel's completeness theorem: For first order logic, there is a system of inference rules so that θ can be derived from T if and only if θ is true for every model of T. So for first order logic, negation completeness implies the stronger *deductive negation completeness*: for $\phi \in \Phi$, either ϕ or $\neg\phi$ is provable from the axioms of T. Moreover, the set of true sentences is determined in Zermelo-Fraenkel set theory (with choice).

Given a collection of statements Φ about possible systems for geometry, there are several ways in which a subset Ψ of Φ can be thought complete for a collection of axioms T. Of course, each $\psi \in \Psi$ must be satisfied in each model of T. And the most natural notion of complete is:

Definition 10.2.2 (Categoricity and Completeness)**.**

1. *A consistent theory in a logic $\mathcal{L}$ is* negation complete *if for any $\mathcal{L}$-sentence ϕ, $T \vdash \phi$ or $T \vdash \neg\phi$.*
2. *A theory T is* categorical *if it has* exactly one model*.*

Methodology 10.2.3 (Completeness, categoricity and the choice of logic)**.** Categoricity was confused with negation completeness (Method 10.2.1, (Notation 10.2.2) until the late 1920's. It seems obvious that categoricity implies completeness. If each ϕ and $\neg\phi$ are consistent with T they both hold in the unique model of T, which is clearly impossible. For first order theories, this argument is almost correct[43]. For second order logic, categoricity is possible and the result holds. But the collection of true sentences is not *determined*. As in Methodology 10.2.5, whether ϕ or $\neg\phi$ holds may depend on the ambient model of set theory. For first order sentences this problem disappears since the truth value of ϕ in M is determined by the axioms of ZFC.

But the truth of a second order sentence about the real field may depend on the set theory in which you work (Methodology 10.2.5).

Methodology 10.2.4 (First order completeness)**.** The first order theory T_{rcf} of the Cartesian plane over real numbers is negation complete; one adds to EG the infinitely many axioms that say of the coordinatizing field that every odd degree polynomial has a root [Tar59]. Alternatively, analogously to the Peano axioms for arithmetic, Dedekind cuts are formalized to hold only for first order definable cuts [TG99, p 185].

Independence of the parallel postulate shows the axioms HP for neutral geometry are not complete. Even the descriptively complete theory EG is far from negation complete. In fact, [Bee, Zie82] (first in English) proves that if T is finitely axiomatized geometry consistent with T_{rcf} there is no algorithm to decide which sentences are consistent with T. There are uncountably many first order complete theories extending EG.

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[42] This was certainly known in the 20's; [HB34] is the earliest Hilbert reference for this fact that I could find, though likely it is in [HA38].

[43] Since the Löwenheim-Skolem theorem says that any first order sentence with an infinite model has a model of every infinite cardinality, a unique infinite model is impossible. The categoricity criteria can be weakened to a unique model in some infinite cardinality. But the first order theory geometry/real closed fields fails that criteria as well. Tarski establishes completeness of geometry by showing the quantifier elimination of ordered real closed fields. T has *quantifier elimination* if every first order formula is equivalent in T to one with no quantifiers.

Methodology 10.2.5 (Indeterminacy of 2nd order geometry despite categoricity)**.** Write the statement in pure second order logic expressing the continuum hypothesis[44]. By the celebrated work of Gödel and Cohen, the continuum hypothesis is independent from the Zermelo-Frankel axiom for set theory (even with the axiom of choice). Thus, while Hilbert's 2nd order axiomatization is negation complete, the truth value of the continuum hypothesis is not determined.

As we noted in Remark 3.0.9 Birkhoff's axioms are a description of the geometry over the real field ('Real field' is defined in set theory). With Hilbert's definition of the field, we can make this into a legitimate second order axiomatization of a theory that is categorical in any particular model M of ZFC. Just say the field has the least upper bound property. But the second order theory will depend not just on the given axioms but also on what set theoretic statements are true in M (as in 10.2.5).

Theorem 10.2.6. *Fix two points* $0, 1$ *on a Hilbert plane* M *and the line* ℓ *through them. Let* $<, +, \times$ *be the ordering relation and field operations defined on* ℓ *by Theorem 7.14. Add the least upper bound axiom:*

$$(\forall X)(\exists y)(\forall x \in X)[x < y \wedge [(\forall w)(\forall x \in X)x < w) \rightarrow y \leq w]].$$

The field on ℓ *is a complete ordered field and so is isomorphic to the reals.*

Proof. Clearly the ruler postulate holds on ℓ. But we know by Theorem 5.4.23 that the group of rigid motions acts transitively on lines so the ruler postulate holds on every line and so on M. □

10.3 Protractor Postulate and Radian Measure

We discuss in this section the subtleties introduced by Birkhoff's 'protractor postulate' and the connection with radian measure.

Axiom 10.3.1 (Birkhoff's 'protractor postulate')**.** There are several variants:

I. **Principle 3. ANGLE MEASURE.** [BB59, p 47]. 'All half-lines having the same end-point can be numbered[45] so that number differences measure angles.'

II. SMSG version [Ced01, SMS95]

1. **Postulate 11. Angle Measurement Postulate** To every angle there corresponds a real number between 0 and 180.
2. **Postulate 12. Angle Construction Postulate** Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m\angle PAB = r$.
3. **Postulate 13. Angle Addition Postulate** If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

III. First order[46] version: Protractor postulate If Π is a model of $HP5$ with associated ordered field F_Π, there exists a bijection α from (congruence classes of) angles ($\angle XAY$) to $(0, 180)$ in such a way that if D is a point in the interior of $\angle BAC$, then $\alpha(\angle BAC) = m\angle BAD + m\angle DAC$.

[44] Consider the second order sentence:

$(\exists X)(\exists f) f$ is an injective function from X onto a proper subset of $X \wedge (\exists Y)(\exists g)$is an injective function from Y onto X.

This says only there are at least two cardinals less than or equal the continuum. Adding a third assert the continuum hypotheis fails.

[45] By 'numbered', Birkhoff means that a real number can be assigned (to each half line).

The choice of 180 in II.2 and III stems from Babylonian times. But in a system based on 'grad', we would take 200. That, is the units of a protractor are 'arbitrary'. Fix a number of units for the half circle and then try to figure out how to choose points on the circle that measure many angles. There is a 13th century description of how to construct such a protractor as a part of an astrolabe [Fal20, §4].

Activity 10.3.2 (Construct a protractor)**.** *See the Quora article [Chi10] for instructions.*

Methodology 10.3.3 (Units)**.** Postulates 11 and 12 are from [Ced01]; the Wikipedia article, SMSG axioms, is identical, except the real numbers in Postulate 11 are said to be between 0° and 180°. As we now describe, this conflict illustrates one of two distinct uses of the coordinatizing field. If the field is just some abstract object, then we can choose the unit arbitrarily and proceed as in the Quora article [Chi10]. And devices for making this measurement are dated as early as 1400 BC. And we are free to call that unit a degree[47]. On the other hand, if we take that unit as measuring the distance along the arc from $(0, 1)$ to the point where the ray intersects the unit circle, and if we are in either Hilbert's segment geometry, where each number corresponds to a line segment, or assume Birkhoff's ruler postulate, we have established a definition of length for arcs of the circle. That way numbers are assigned to angles in Definition 10.3.1 suggests that he intends an abstract interpretation (every real number determines an angle). However, this intention is undermined by the ruler postulate -straight segments measure angles/arc length. And even more, [Bir32, fn 5, p 22] indicates he is thinking of radian measure by specifying angles are congruent whose measures (now any real number) are congruent mod 2π. As noted in [Bal19], this construction of radian measure requires calculus or at least infinite series.

Postulate 13 formalizes Birkhoff's 'number differences measure angle'. The exact statement is crucial; it specifies that only angles less than a straight angle are measured. Radian measure may not appear in grade 10 geometry.

Methodology 10.3.4 (Consistency of the $\Re$-Protractor postulate (Definition 2.1.10.2))**.** Recall Activity 3.0.5, specifically, 'how can one ensure that angle measure is additive?'. A natural idea is to use the tangent function (slope of the line). But it is not additive. The standard way to obtain an additive measure is to use the arc length. But, as noted in Methodology 10.1.6, this requires analysis (at least ϵ, δ arguments for summing series and at most fully integration). Thus the consistency of either Axiom 10.3.1 I or II requires advanced tools. Of course there are no countable models since the reals are explicit in the ruler and protractor postulates. On the other hand, Axiom 10.3.1.III with first order geometry, is shown consistent by a countable model M, where π^M is taken as some number realizing the same cut in M as $\pi \in \Re$, is shown in [Bal18, §10.3].

As noted in Methodology 10.3.3, radian measure is needed to extend the trigonometric functions from angles less than a straight angle (developed in Euclid) to functions on the entire real line.

Pedagogy 10.3.5 (Radian Measure)**.** Radians are not part of a standard high school geometry course. But (in 11th/12th grade when circle trigonometry and radians are introduced) teachers need to understand the connection between the 'conventional' method of degrees in Methodology 10.3.3 and the (implicit) use of arc length. The first author brought a bicycle wheel into a large lecture for precalculus and had students measure the length of the path as the wheel was rotated several times and then calculate the angle knowing the radius of the wheel.

[47] It is not clear whether SMSG per Wikipedia is assuming notion of degree is known.

11 Non-Euclidean Geometry

We showed in Section 7, specifically Methodology 7.14, that the theories of Euclidean geometry (EG) and fields were bi-interpretable. This depends on the notion of interpretation, Methodology 11.1 and formally defined in Definition 13.4 As we will see in this section Euclidean and hyperbolic geometry are also bi-interpretable. In particular, Poincaré showed that one could interpret hyperbolic geometry in a disc on the Euclidean plane. The converse is due to Hilbert.

The switch from the old to the new view of geometry (Comment 1.2) stemmed from the proof of the independence of the parallel postulate. Most of the modern work on non-Euclidean geometry assumes the existence of a real-valued metric (distance function) and is *not* done synthetically. However, [Har00, §34] elaborates on some axiomatic non-Euclidean geometry. In neutral geometry (Hilbert plane), he proves there is a rectangle if and only if the sum of the angles of a triangle is two right angles and introduces an axiomatic trichotomy of semi-hyperbolic, semi-Euclidean and semi-elliptic geometries depending on whether the sum of the angles of a triangle is $<, =, >$ (respectively) than 2 right angles. Further, he proves that a semi-hyperbolic plane satisfying Hilbert's 'limiting parallel axiom' (*hyperbolic geometry*) interprets a ordered field. [BP23] and [Har77, Ex. 18.4] give a nice examples of a semi-Euclidean planes where the parallel postulate fails (limited points in $\Re^\omega/D$ where D is a non-principal ultrafilter).

Methodology 11.1 (Informal description of Interpretation)**.** It is easy to confuse two meanings of interpretation: i) (somewhat archaic for logicians but used above for consistency with SLO 4) as a witness to truth: 'a model of ϕ or T' is called 'an interpretation of ϕ or T' and ii) a relation between two (languages or theories or models). We mean the second in this section.

Two theories are bi-interpretable if there are interpreting maps F, G from each to the other such that $F \circ G$ is the identity.

One way to prove the consistency of, say, hyperbolic geometry, is to interpret it in a Euclidean model; redefine the undefined terms of geometry (point, line, between, congruent, etc.) by formulas of Euclidean geometry and prove that *with this interpretation* the axioms of Hyperbolic geometry are satisfied in each model of Euclidean geometry. This yields that hyperbolic geometry is *relatively consistent* with Euclidean geometry. We give a full definition in Definition 13.4. A nice introduction to interpretation for those familiar with modern algebra is in [BN94, §3: Interpretability] and for the more logically oriented [**?**].

We define some theories of geometries and indicate interpretability relations.

Definition 11.2 (Limiting Parallels)**.**

1. *Two rays are* coterminal *if they eventually coincide.*
2. *A plane has* limiting parallels *if there are rays a through A and b through B that are either coterminal or they lie on distinct lines not equal to AB and every ray in the interior of the angle BAb meets the ray Bb [Har00, p 312].*

We conclude this section with a summary of the interpretability relations between the Euclidean and hyperbolic planes.

Theorem 11.3.

1. *The theory of ordered fields is bi-interpretable with HP5 (neutral geometry + parallel postulate). (Hilbert coordinatization and analysis of the cartesian plane over an ordered field.)*

   Whence, the theory of ordered fields is interpretable in EG (neutral geometry + Euclid's 5th + E (circle-circle intersection)
2. *The theory of Euclidean fields is bi-interpretable with EG. (See, e.g., [Har00, Mak19].*

3. *Call Hyperbolic geometry HL: neutral geometry + limiting parallels. The theory of Euclidean ordered fields is interpretable in HL [Har00, §41].*
4. *Call semihyperbolic geometry: neutral geometry + sum of the angles of a triangle* $< 2RA$.
   Exercise [Har00, 39.24] shows there is a semihyperbolic plane which is not hyperbolic[48].
5. *Clearly EG is not interpretable into HP5. If the coordinatizing field* $\Phi(M)$ *of a model of* $HP5$ *is not Euclidean (Some positive number doesn't have a square root.),* $\Pi(\Phi(M))$ *is not Euclidean (There are two overlapping circles that don't intersect.).*

While he doesn't state it quite this way, [Har00, §40] proves a converse to Theorem 11.3.3 and so

Theorem 11.4. *The theory of hyperbolic geometry with limiting parallels (HL) is bi-interpretable with EG.*

Thus we have shown, since both hyperbolic geometry and Euclidean geometry are bi-interpretable with the real field, they are themselves bi-interpretable. That is, it is possible to define a model of each in any model of the other. This emphasizes that interpretation preserves not meaning but consistency.

12 Conclusion

We have explained three 'axiomatic' approaches to resolving the tension between synthetic and analytic (coordinate) geometry: Euclid (no conception whatsoever of coordinates or even rational numbers[49]), Hilbert (a synthetic foundation for coordinate geometry) and Birkhoff (a set theoretic foundation for coordinate geometry – billed as synthetic). In the process we explore the methods to establish independence and consistency for first order theories of geometry[50]. We did this by sketching a system of synthetic geometry that proves the existence (from geometric axioms) of a coordinatizing field and thus grounds coordinatizing geometry. Rather than exploring in detail the independence of the parallel postulate (SLO 9), we illustrate crucial notions such as consistency and independence of propositions within the development of Euclidean geometry. We focused on the different paths of Euclid and Hilbert to establishing theories of proportion and area. Euclid relied on the $L_{\omega_1,\omega}$ axiom of Archimedes to develop a theory of proportionality for both arithmetic and geometry. From a more precise version of Euclid's other axioms, Hilbert constructed a field geometrically enabling an algebraic definition of proportionality.

13 Appendix: Formal Systems

This section is background for instructors who want more details on the logical notions that are sketched in the text. The aim is to give a precise notion of truth in a mathematical structure and give a more precise account of the interpretation of theories for the non-Euclidean case, which are much more complicated than the examples given in the chapter. Two accessible sources are [BE02] and [LK15]. This material is in any introductory course in mathematical logic – and much more fully explained.

Definition 13.1. *A formal system of first order logic consists of*

[48] It is unclear to me whether either semi-hyperbolic plane discussed in this exercise interprets a Euclidean ordered field.

[49] For Euclid numbers are $2, 3, 4, \dots$ full stop.

[50] This process is, at best, not well-understood for second order logic. So our analysis is restricted to Euclid and Hilbert's systems. But we do develop some of SMSG's approach since it is widely used.

1. *vocabulary*

   (a) *Logical vocabulary:* $(,), \wedge, \wedge, \neg, = \forall, \exists$ *variables* $v_1, v_2, \ldots$.

   (b) *non-logical vocabulary* τ[51]*: a list of relations, function, and constant symbols of prescribed arity.*

2. *Terms (expressions) are defined by induction.*

   (a) *A variable or a constant is a term.*

   (b) *If* f *is an* n*-ary function symbol and* $t_1, \ldots t_n$ *are terms then* $f(t_1, \ldots t_n)$ *is a term.*

3. *well-formed formulas (wff) are defined by induction.*

   (a) *Atomic formulas*

      i. *If* t_1, t_2 *are terms then* $(t_1 = t_2)$ *is an atomic formula.*

      ii. *If* $t_1, \ldots t_n$ *are terms and* R *is an* n*-ary relation symbol then* $R(t_1, \ldots t_n)$ *is an atomic formula.*
      For example, if $<$ *is a binary relation symbol,* 0 *a constant and* x *a variable,* $x < 0$ *is an atomic formula.*

   (b) *If* ϕ *and* ψ *are wffs*

      i. $\neg\phi$ *is a wff;*

      ii. $(\phi \wedge \psi)$ *is a wff;*

      iii. $(\phi \wedge \psi)$ *is a wff;*

      iv. $(\exists v_i)\phi$ *and* $(\forall v_i)\phi$ *are wffs.*

4. *A* τ*-structure*[52] *is a set* A *and for each*

   (a) *constant symbol* $c \in \tau$*, an element* c^A *of* A*;*

   (b) $n - ary$ *relation symbol* $R \in \tau$ *a relation* $R^A \subset A^n$*;*

   (c) $n - ary$ *function symbol* $f \in \tau$ *a function* $f : A^n \to A$.
   For example, the rational field $(\mathbb{Q}, +, \times, 0, 1)$ *is* τ *structure for the vocabulary* $+, \times, 0, 1$.

5. *To define truth of* τ*-sentences in an* τ*-structure* A*:*

   (a) *Expand* τ *to* τ^* *by adding a constant symbol* c_a *for each* $a \in A$*. (That is,* $c_a^A = a$*.)*

   (b) *The denotation* t^A *of terms* t *is defined by induction.*

      i. *The denotation of a constant* c *is* c^A.

      ii. *The denotation of a term* $t = f(t_1, \ldots t_n)$ *is* $t^A = f^A(t_1^A, \ldots t_n^A)$.

   (c) *Now truth of a formula* $\phi(t_1, \ldots, t_n)$ *(where the* t_i *are* τ^**-terms) is defined by induction:*

      i. *If* ϕ *is* $t_1 = t_2$*,* $A \models \phi$ *if* $t_1^A = t_2^A$.

      ii. *If* ϕ *is* $R(t_1, \ldots t_n)$*,* $A \models \phi$ *if* $\langle t_1^A, \ldots t_n^A \rangle \in R^A$.

      iii. *If* $\phi(t_1, \ldots, t_n)$ *is* $\psi(t_1, \ldots, t_n) \wedge \chi(t_1, \ldots, t_n)$ *then* $A \models \phi(t_1, \ldots, t_n)$ *if* $A \models \psi(t_1, \ldots, t_n)$ *and* $A \models \chi(t_1, \ldots, t_n)$.

[51] If there are no function symbols, the vocabulary is called *relational*; if there are no relations it is called *algebraic*.

[52] A structure for a relational vocabulary is called a *relational structure*; a structure for an algebraic vocabulary is called an *algebra*.

iv. *If* $\phi(t_1,\ldots,t_n)$ *is* $\neg\psi(t_1,\ldots,t_n)$ *then* $A \models \phi(t_1,\ldots,t_n)$ *if it is not the case that* $A \models \phi(t_1,\ldots,t_n)$.

v. *If* $\phi(t_1,\ldots,t_n)$ *is* $(\exists v_i)\psi(t_1,\ldots,t_n,v_i)$ *then* $A \models \phi(t_1,\ldots,t_n)$ *if for some* $a \in A$, $A \models \psi(t_1,\ldots,t_n,c_a)$.
For example, the sentence $(\exists x)(x^2 = 1+1)$ *is false in the structure* $(\mathbb{Q},+,\times,0,1)$ *and true in the structure* $\mathbb{C},+,\times,0,1)$ *(where the* QQ *and* $\mathbb{C}$ *indicate we are to interpret as the rational and complex field respectively.*

(d) *The sentence* ϕ *is valid if it is true in every structure. For every* M, $M \models \phi$.

(e) *The sentence* ϕ *is a* logical consequence *of the sentence* ψ *if for every* M*, if* $M \models \psi$ *then* $M \models \phi$.

If a sentence ϕ *is true in a structure* M*, we say* M *is a* model *of* ϕ*. If* M *satisfies all axioms of a theory* T*,* M *is a* model *of* ϕ.

Theorem 13.2 (Completeness and Compactness)**. Gödel's Completeness theorem** *For any sentence of first order logic and any* T*:*

$$T \models \phi \leftrightarrow T \vdash \phi.$$

Compactness theorem *For any and constants* $\mathbf{a}$ $\phi_n(\mathbf{a})$ *and collection of sentences* $\phi_n(\mathbf{a})$.

If there is a model M *for each* $N < \omega$*, there is an* $\mathbf{a}_N$*, and* M_N *such that* $M \models \bigwedge_{n<N} \phi_n(\mathbf{a}_N)$ *then there is a model* M_ω *and tuple* $\mathbf{a}_\omega$ *such that* $M_\omega \models \bigwedge_{n<\omega} \phi_n(\mathbf{a}_N)$.

Definition 13.3 (Proof system)**.** *We now specify a proof system for first order logic. However, we not recommend proofs in such a formal system in a GeT course.*

The key point is that the arguments in Euclid generally follow a simple form. A configuration of a finite number points is given and one must show that there exist further points satisfying a further configuration. That is the theorem can be expressed by formula $(\forall x_1,\ldots x_n)\theta(x_1,\ldots x_n) \rightarrow \exists(y_1,\ldots y_m)\psi(x_1,\ldots x_n,y_1,\ldots y_m)$*. For more detail see [ADM09, §2.4] or [Mue06, p 11-14].*

Logical Axioms

1. *Any sentence that is true in every* τ *structure (a tautology);*
2. *The equality axioms;*
3. $(\forall x)\phi \rightarrow \phi^x_t$ *(if t is substitutable for* x *in* ϕ*);*
4. $(\forall x)(\phi \rightarrow \psi) \rightarrow [(\forall x)\phi \rightarrow (\forall x)\psi]$*;*
5. $\phi \rightarrow (\forall x)\phi(x)$ *(if* x *not free in* ϕ*).*

Inference rule
(Modus Ponens): From ϕ *and* $\phi \rightarrow \psi$*, infer* ψ*.*

This is slightly altered version of the definition of interpretation in [Sho67]

Definition 13.4 (Formal definition of interpretation)**.** 1. *We say that* I *is an interpretation of* L *in* L'*, where* L *and* L' *are first-order languages, if* I *is a function such that* *i* *there is a universe for the image of* I, *represented by a unary predicate symbol* U_I *(or formula) of* L' *;*

ii *for each* n*-ary function symbol* f *of* L *, a corresponding symbol* f_I *of* L'*;*

iii *for each* n*-ary predicate symbol* P *of* L *(with the exception of =, which is generally taken to be a logical symbol), a corresponding symbol* P_I *of* L'.

iv *Moreover, we say that* I *is an interpretation of* L *in a theory* T' *if* I *is an interpretation of* L *in the language* L' *of* T' *and also:*

  i $T' \vdash (\exists x)U_I(x)$ *(it proves that the domain is non-empty);*

  ii *for each* $f \in L$ *,* $T' \vdash (U_I(x_1) \wedge \ldots \wedge U_I(x_n) \rightarrow U_I(f_I(I(x_1 \ldots, x_n)))$ *(it proves that the domain is closed under functions).*

v *Now, if* ϕ *is a formula of* L *and* I *an interpretation of* L *in* L'*, then we can define for* ϕ *its interpretation in* L' *,* ϕ_I *. We start by defining a formula* ϕ_I *of* L' *which is obtained by starting with* ϕ *and replacing each symbol of the original language by its interpretation in* L' *(e.g., if* ϕ *is* $f(x) = y$ *, then we replace f by* f_I *to obtain* $f_I(x) = y$*), and then relativizing the existential quantifiers to the domain (i.e., replace every* $(\exists x)\psi$ *by* $(\exists x)(U_I(x) \wedge \psi)$*. As the last step, if* $x_1 \ldots, x_n$ *are the free variables of* ϕ *, set* ϕ_I *to be* $(U_I(x_1) \wedge \ldots \wedge U_I(x_n) \rightarrow \phi_I$.

vi *Finally, an interpretation of a theory* T *in a theory* T' *is an interpretation* I *of the language of* T *in* T' *such that* $T' \vdash \phi_I$ *for every nonlogical axiom (i.e., an* L*-sentence* ϕ *that is not universally valid that has been taken as axiom of* T*).*

We have noticed that there are first order formulas defining the Cartesian plane over a field and, more surprisingly, conversely if the plane satisfies HP5. We say that the theory of fields and the theory of Hilbert planes satisfying the 5th postulate are *mutually interpretable*. As emphasized in [Mak19] a stronger connection is more useful: the theories are said to be *bi-interpretable* if the defining maps are inverses of each other. In particular, bi-interpretation preserves decidability while mutual interpretability may not. [Ena13] provides a basic exposition and the original definitions of interpretability [TMR68] is quite readable.

References

[ADM09] J. Avigad, Edward Dean, and John Mumma. A formal system for Euclid's elements. *Review of Symbolic Logic*, 2:700–768, 2009.

[Alp05] Roger C. Alperin. Trisections and totally real origami. *The American Mathematical Monthly*, 112:200–211, 2005.

[Apo67] T. Apostol. *Calculus*. Blaisdell, New York, 1967.

[Bal06] Baldwin, J. T. (2006) *Geometry and Proof* In: *Second International Conference on tools for teaching logic*, pages 5-8 University of Salamanca Press Online `https://homepages.math.uic.edu/~jbaldwin/pub/loggeomfor.pdf`

[Bal18] John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*. Cambridge University Press, 2018.

[Bal19] John T. Baldwin. Unifying the two definitions of π. 11, 2019. `https://homepages.math.uic.edu/~jbaldwin/CTTIgeometry/defpi.pdf`.

[BE02] J. Barwise and J. Etchemendy. *Language, proof, and logic*. University of Chicago Press, 2002.

[Bee] M. Beeson. Some undecidable field constructions. translation of [Zie82]http://www.michaelbeeson.com/research/papers/Ziegler.pdf.

[BH07] William Barker and Roger Howe. *Continuous Symmetry: From Euclid to Klein*. American Mathematical Society., 2007.

[Bir32] Birkhoff, George. A set of postulates for plane geometry. *Annals of Mathematics*, 33:329–343, 1932.

[BB59] George David Birkhoff and Ralph Beatley. *Basic Geometry*. 3rd ed. Chelsea Publishing Co., 1959. 1st edition 1941: Reprint: American Mathematical Society, 2000. ISBN 978-0-8218-2101-5; online at https://kupdf.net/download/birkhoff-amp-beatley-basic-geometry_58b4448e6454a79179b1e939_pdf.

[BP23] Błaszczyk and Petiurenko. New model of non-euclidean plane. https://arxiv.org/abs/2302.12768, 2023.

[BN94] A. Borovik and A. Nesin. *Groups of Finite Morley Rank*. Oxford University Press, 1994.

[Bol78] Vladimir Boltyanskii. *Hilbert's third problem*. V.H. Winston and Sons, 1978.

[Ced01] J. Cederberg. *A Course in Modern Geometries*. Springer, 2001.

[Chi10] Loring Chien. Construct your own protractor',, 2010. https://www.quora.com/How-were-the-first-protractors-made-How-can-I-make-a-fairly-accurate-one-withou

[Cla12] David M. Clark. *Euclidean Geometry: A guided inquiry approach*. Mathematical Circles Library. Mathematical Sciences Research Institute, 2012.

[CK17] V Climenhaga and A Katok. *From Groups to Geometry and Back*. AMS, Providence, 2017.

[CS20] Clio Cresswell and Craig P. Speelman. Does mathematics training lead to better logical thinking and reasoning? a cross-sectional assessment from students to professors. *Plus One*, 2020. open access: https://doi.org/10.1371/journal.pone.0236153.

[Cro87] M. Crowley. The van Hiele model of the development of geometric thought. In M.M. Lindquist, editor, *Learning and Teaching Geometry, K-12, 1987 Yearbook of the National Council of Teachers of Mathematics*, pages 1–16. National Council of Teachers of Mathematics, 1987. https://www.cns-eoc.colostate.edu/docs/math/mathactivities/june2007/The%20van%20Hiele%20Model%20of%20the%20Development%20of%20Geometric%20Thought.pdf.

[dav11] David Lowry Duda. Provenance of beer bottle quote, 2011. https://math.stackexchange.com/questions/56603/provenance-of-hilbert-quote-on-table-chair-beer-mug.

[Ded63] Richard Dedekind. *Essays on the Theory of Numbers*. Dover, Mineola, New York, 1888/1963. first published by Open Court publications 1901: first German edition 1888.

[Dem94] William Demopoulos. Frege, Hilbert, and the conceptual structure of model theory. *History and philosophy of logic*, 15:211–225, 1994.

[Det14] M. Detlefsen. Completeness and the ends of axiomatization. In J. Kennedy, editor, *Interpreting Gödel*, pages 59–77. Cambridge University Press, 2014.

[Edu09a] Educational Development Center: *CME Algebra I*. Pearson, 2009. EDC: Educational Development Center.

[Edu09b] Educational Development Center: Benson, Cuoco, et al. *CME Geometry*. Pearson, 2009.

[Ena13] A. Enayat. Interpretations and mathematical logic. 2013. `https://www.researchgate.net/publication/281936496_Interpretations_and_Mathematical_Logic`.

[Ena2x] A. Enayat. Interpretations and mathematical logic: A tutorial. 202x. accessed Dec. 2023:`https://gup.ub.gu.se/file/167690`.

[Euc56] Euclid. *Euclid's Elements*. Dover, New York, New York, 1956. In 3 volumes, translated by T.L. Heath; first edition 1908; online at `http://aleph0.clarku.edu/~djoyce/java/elements/`.

[Fal20] Seb Falk. *The Light Ages*. H.H. Norton, 2020.

[FGH+24] Maia Fraser, Andrew Granville, Michael H. Harris, Colin McLarty, Emily Riehl, and Akshay Venkatesh. Will machines change mathematics? *Bulletin of the AMS — Volume No. 2*, 61(2):197 – 212, 2024.

[Gio21] E. Giovannini. David Hilbert and the foundations of the theory of plane area. *Archive for History of Exact Sciences*, 2021. `https://doi.org/10.1007/s00407-021-00278-z`.

[GG09] I. Grattan-Guinness. Numbers, magnitudes, ratios, and proportions in Euclid's *elements*: How did he handle them? In *Routes of Learning*, pages 171–195. Johns Hopkins University Press, Baltimore, 2009. first appeared Historia Mathematica 1996.

[Gre93] M. Greenberg. *Euclidean and non-Euclidean geometries: development and History*. W. H. Freeman and Company, 1993.

[Hal08] T. Hales. Formal proof. *Notices of the AMS*, 55:1370–1380, 2008.

[Har99] G. Harel. Students understanding of proofs: A historical analysis and implications for the teaching of geometry and linear algebra. *Linear Algebra and its applications*, 302-303:601–613, 1999.

[Har13] Harel, G. Intellectual need. In K. Leatham, editor, *Vital Direction for Mathematics Education Research*, pages 119–151. Springer, 2013.

[Har14] G. Harel. Common core state standards for geometry: An alternative approach. *Notices of the AMS*, pages 24–35, 2014.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

[Har00] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.

[Har11] Robin Hartshorne. Review of Barker and Howe. *The American Mathematical Monthly*, pages 565–568, 2011. https://www.jstor.org/stable/10.4169/amer.math.monthly.118.06.565.

[Har15] Matthew Harvey. *Geometry Illuminated; An Illustrated Introduction to Euclidean and Hyperbolic Plane Geometry*. Mathematical Association of America, 2015.

[Hen02] D. Henderson. Geometry: Euclid and Beyond by Robin Hartshorne. *Bulletin of the A.M.S.*, 39:563–571, 2002. https://www.ams.org/journals/bull/2002-39-04/S0273-0979-02-00949-7/S0273-0979-02-00949-7.pdf.

[HT05] D. Henderson and D. Taimina. How to use history to clarify common confusions in geometry. In A. Shell-Gellasch and D. Jardine, editors, *From Calculus to Computers*, volume 68 of *MAA Math Notes*, pages 57–74. Mathematical Association of America, 2005.

[HT20] D. Henderson and D. Taimina. *Experiencing Geometry: Euclidean and Non-Euclidean with history*. Pearson, 2020. https://projecteuclid.org/ebooks/books-by-independent-authors/Experiencing-Geometry/toc/10.3792/euclid/9781429799850.

[HA38] D. Hilbert and W. Ackermann. *Grundzüge der Theoretischen Logik, 2nd edn*. Springer, Berlin, 1938. first edition, 1928.

[Hil62] David Hilbert. *Foundations of geometry*. Open Court Publishers, LaSalle, Illinois, 1962. original German publication 1899: reprinted with additions in E.J. Townsend translation (with additions) 1902: Gutenberg e-book #17384 http://www.gutenberg.org/ebooks/17384.

[Hil71] David Hilbert. *Foundations of Geometry*. Open Court Publishers, 1971. translation from 10th German edition by Harry Gosheen, edited by Bernays 1968.

[HB34] D. Hilbert and P. Bernays. *Foundations of Mathematics*. Bernays Project, 1934. translation Ian Mueller, 2003 http://www.phil.cmu.edu/projects/bernays/Pdf/bernays12-1_2003-06-25.pdf.

[IA17] M. Inglis and N. Attridge. *Does Mathematical Study Develop Logical Thinking*. World Scientific, 2017.

[Ill19] Illustrative Math. Illustrative mathematics: Geometry. https://im.kendallhunt.com/HS/index.html, 2019. Accessed: 2023-09-27.

[Kin21] J. King. *Geometry Transformed: Euclidean Plane Geometry Based on Rigid Motions*. Pure and Applied Undergraduate Texts. American Mathematical Society, 2021.

[LK15] C. Leary and L Kristiansen. *A Friendly Introduction to Mathematical Logic*. Milne Library Publishers, 2015. Geneseo Authors. 6. https://knightscholar.geneseo.edu/geneseo-authors/6.

[Lyn67] Roger Lyndon. *Notes on Logic*. Van Nostrand, New York, 1967. available on Abe books for less than $5.

[Mak19] J. A. Makowsky. Can one design a geometry engine? On the (un)decidability of affine Euclidean geometries. *Annals of Mathematics and Artificial Intelligence*, 85:259–291, 2019. online: https://doi.org/10.1007/s10472-018-9610-1.

[Man08] K. Manders. Diagram-based geometric practice. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 65–79. Oxford University Press, 2008.

[Mar82] George E. Martin. *Transformation geometry, an introduction to symmetry*. Springer-Verlag, 1982.

[Mil07] N. Miller. *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry*. CSLI Publications, 2007.

[Moi90] Edwin Moise. *Elementary Geometry from an Advanced Standpoint*. Addison-Wesley, 1990. 3rd edition.

[Mue06] Ian Mueller. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Dover Books in Mathematics. Dover Publications, Mineola N.Y., 2006. First published by MIT press in 1981.

[PSS07] Victor Pambuccian, Horst Struve, and Rolf Struve. Metric geometries in an axiomatic perspective. In L. et al. Ji, editor, *From Riemann to Differential Geometry and Relativity*, volume 48, pages 399–409. Springer International Publishing AG, 2007.

[Rai05] Ralph Raimi. Ignorance and innocence in the teaching of mathematics. `https://homepages.math.uic.edu/~jbaldwin/pub/Raimi.pdf`, 2005.

[SBM19] A.J. Stylianides, K.N. Bieda, and F. Morselli. Proof and argumentation in mathematics education research. In Á Gutiérrez, G Leder, and P. Boero, editors, *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues*, pages 315–351. Sense Publishers Rotterdam, Rotterdam, 2019.

[Ser93] Michael Serra. *Discovering Geometry: An inductive Approach*. Key Curriculum Press, 1993.

[Sho67] Joseph Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

[Smo08] C. Smorynski. *History of Mathematics: A supplement*. Springer-Verlag, 2008.

[SMS95] SMSG. The SMSG Postulates for Euclidean Geometry, 1995. Search for: Geometry/The SMSG Postulates for Euclidean Geometry, e.g. `https://faculty.winthrop.edu/pullanof/MATH%20393/The%20SMSG%20Postulates.pdf`

[Spi80] M. Spivak. *Calculus*. Publish or Perish Press, Houston, TX, 1980.

[Szm78] W. Szmielew. *From Affine to Euclidean Geometry: An Axiomatic Approach*. D. Reidel, Dordrecht, 1978. edited by Moszyńska, M.

[Tar59] A. Tarski. What is elementary geometry? In Henkin, Suppes, and Tarski, editors, *Symposium on the Axiomatic method*, pages 16–29. North Holland Publishing Co., Amsterdam, 1959.

[TG99] A. Tarski and S. Givant. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5:175–214, 1999.

[TMR68] Alfred Tarski, Andrezej Mostowski, and Raphael Robinson. *Undecidable Theories*. North Holland, 1968. First edition: Oxford 1953.

[Vää10] Jouko Väänänen. Second-order and higher-order logic, 2010. `https://plato.stanford.edu/entries/logic-higher-order/#PuttDistBetwSecoFirsOrdeLogi`.

[Vää14] Jouko Väänänen. Sort logic and foundations of mathematics. *Lecture Note Series, IMS, NUS*, pages 171–186, 2014.

[Wei97] A.I. Weinzweig. Geometry through transformations. privately published, 1997.

[Wei20] D. Weisbart. Modernizing Archimedes' construction of π. Σ *Mathematics*, page 23 pages, 2020. online open access: doi:10.3390/math8122204.

[Wu94] Wen-Tsü Wu. *Mechanical Theorem Proving in Geometry*. Texts and Monographs in Symbolic Computation. Springer-Verlag, New York, 1994. Chinese original 1984.

[Wyl44] C. R. Wylie. Hilbert's axioms of plane order. *The American Mathematical Monthly*, 51:371–376, 1944.

[Zie82] M. Ziegler. Einige unentscheidbare Körpertheorien. *L'Enseignement Mathmetique*, 28:269–280, 1982. Michael Beeson has an English translation.