The white space nearby

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January 10, 2015

Two Directions in AEC

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1. **Eventual Behavior** Assume there are arbitrarily large models (and often ap,jep and even tameness)
2. **Work from the bottom up**

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1. **Frames**: Place very strong (superstability) conditions in a fixed cardinal and bootstrap your way up. So ap and jep are assumed (with more) in a single cardinal.
2. **Explore** Can we fill in the white spaces on the map that are nearby?

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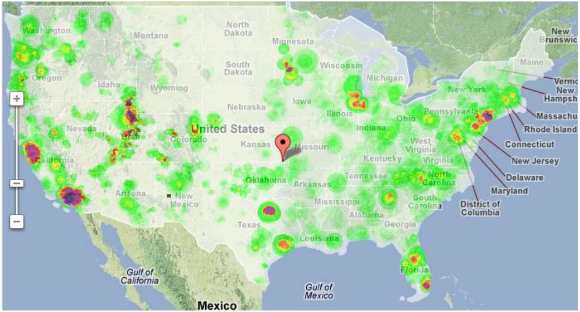

Methods

1. Set theoretic methods as in Larson (Friday) and Laskowski or Kolesnikov talks
2. extending Fraissé style arguments
   1. looking for atomic models
   2. the importance of (strong) disjoint amalgamation
3. excellence
4. combinatorics

Three lines of research

1. Understand the models in the Löwenheim number.
2. What are the spectra of existence, jep, ap, tameness ?

   Need to parameterize notions: e.g. (κ, λ)-tame
3. Are syntactic hypotheses such as 'complete sentence in $L_{\omega_1,\omega}$' significantly stronger than abstract AEC hypotheses?

Models in the Löwenheim number

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An AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is completely determined by its restriction up to the Lowenheim number.

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Theorem. B-Boney

$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has a witnessing sequence (a specified directed system of countable structures) in $LS(\boldsymbol{K}) = \aleph_0$ if and only if there are arbitrarily large models.

Analytically Presented AEC

Definition

An abstract elementary class **K** with Löwenheim number $\aleph_0$ is analytically presented if the set of countable models in **K**, and the corresponding strong submodel relation $\prec_{\mathbf{K}}$, are both analytic.

Theorem. (B-Larson)

Analytically presented ***K*** is the same as a $PC\Gamma(\aleph_0, \aleph_0)$ class:

reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types

crux: We recognize the type of presentation by looking only at countable models.

Almost Galois stability

Definition

1. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be Galois ω-stable if for each countable $M \in \mathbf{K}$, there are countably many Galois types over any countable model.
2. The abstract elementary class $(\mathbf{K}, \prec)$ is almost Galois ω-stable if for each countable $M \in \mathbf{K}$, no countable model has a perfect set of distinct Galois types.

Properties of Analytic AEc

A (B-Larson) ($2^{\aleph_0} < 2^{\aleph_1}$) Few models in $\aleph_1$ implies almost Galois ω-stability.

B (B-Larson-Shelah) Countably many models in $\aleph_1$ implies:

Almost Galois ω-stable implies Galois ω-stable.

C (B-Larson-Shelah/B-Larson) $\aleph_1$-categoricity absolute for Almost Galois ω-stable with amalgamation.

tools: forcing, stationary towers, descriptive set theory, Morley-Shelah trees for analyzing $L_{\omega_1,\omega}$

Examples

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I is a set of *absolute indiscernibles* in M if every permutation of I extends to an automorphism of M.

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The complete sentence ϕ with countable model M homogenously characterizes κ if

1. P^M is a set of absolute indiscernibles.
2. ϕ has no model of cardinality greater than κ.
3. There is a model N with $|P^N| = \kappa$.

Theorem (Gao)

If countable structure has a set of absolute indiscernibles, there is an $L_{\omega_1,\omega}$ equivalent model in $\aleph_1$.

Fraissé style arguments

Crucial idea: to build atomic models, require local finiteness but not uniform local finiteness.
The class $\boldsymbol{K}_0$ of finite models is not closed under substructure.
Laskowski-Shelah (1992); Hjorth (2002)

Theorem Hjorth

For every countable α, $\aleph_{\alpha+1}$ is homogenously characterizable.

Fraissé style arguments + excellence

Theorem: (B- Koerwien-Laskowski)

There are a family of complete sentences ϕ_r such that ϕ^r:

1. homogeneously characterizes $\aleph_r$.
2. ϕ_r
   1. has ap up to $\aleph_{r-1}$,
   2. fails ap in $\aleph_{r-1}$,
   3. trivially has ap in $\aleph_r$.

crux: ***K*** satisfies $(< \aleph_0, r+1)$ disjoint amalgamation – i.e. $r+1$-excellence in the finite.

Contrasts

Excellence is sufficient

If K is excellent then it has arbitrarily large models and the amalgamation property.

Excellence is not necessary

(B-Kolesnikov) Non-excellent classes with arbitrarily large models, ap (and much more).

B-Laskowski-Koerwien measures the strength of excellence as a sufficent condition for model existence (and ap).

Question

Is there an AEC that is categorical up to $\aleph_n$ and has no larger models?

Mergers

Mergers

1. Let θ be a complete sentence of $L_{\omega_1,\omega}$ and suppose M is the countable model of θ and $V(M)$ is a set of absolute indiscernibles in M such $M - V(M)$ projects onto $V(M)$. We will say θ is a *receptive* sentence.
2. For any sentence ψ of $L_{\omega_1,\omega}$, the *merger* of ψ and θ is the sentence $\chi = \chi_{\theta,\psi}$ obtained by conjoining with θ, $\psi \restriction N$.
3. For any model M_1 of θ and N_1 of ψ we write $(M_1, N_1) \models \chi$ if there is a model with such a reduct.

Fraissé style arguments: Applying merger

Theorem: (Hjorth, B-Friedman-Koerwien-Laskowski)

There is a receptive sentence that characterizes (has only maximal models) $\aleph_1$.

Corollary: (B-Friedman-Koerwien-Laskowski)

If there is a counterexample to Vaught's conjecture there is one that has only maximal models in $\aleph_1$.

crux: Disjoint amalgamation

Spectrum of disjoint amalgamation in AEC

Kolesnikov and Lambie-Hanson have given a family of AEC's (of coloring classes) in a countable vocabulary which satisfy the amalgamation property but have no models above $\beth_{\omega_1}$.

Specific classes fail dap for the first time arbitrarily close to $\beth_{\omega_1}$.

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Hidden fear: It is easy to make AEC examples by taking disjunctions of $L_{\omega_1,\omega}$.

B-Koerwien-Souldatos:
Define the notion of a pure AEC that avoids this problem.
Nevertheless, the disjoint embedding spectrum can be chaotic.

Spectrum of disjoint amalgamation in AEC

B-Koerwien-Souldatos

For any countable family of characterizable cardinals λ_i, there is an AEC that has $2^{\lambda_i^+}$ maximal models in λ_i, fails AP everywhere and has arbitrarily large models.

So maximal models can be arbitrarily close to $\beth_{\omega_1}$ and then no more maximal models.

Crux: combinatorics of bipartite graphs

Open Question

Is there an $L_{\omega_1,\omega}$-sentence that has maximal models in uncountably many cardinals but arbitrarily large models?

Theorems and Questions

Density

Theorem (Shelah)

If a sentence ϕ of $L_{\omega_1,\omega}$ is $\aleph_1$-categorical, then there is an $\aleph_1$-categorical *complete* ϕ' with $\phi' \rightarrow \phi$.

Question

If an AEC ***K*** is κ-categorical, must there be a κ-categorical ***K*** sub-AEC all of whose models are (∞, ω)-equivalent?

Model classes are wide or tall

A hyper-strong Shelah conjecture:

If a (complete) sentence of $L_{\omega_1,\omega}$ characterizes κ then it has 2^κ models in κ.

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Theorem: Baldwin-Laskowski-Shelah

If a complete sentence of $L_{\omega_1,\omega}$ characterizes a κ for $0 < \kappa < 2^{\aleph_0}$ then it has $2^{\aleph_1}$ models in $\aleph_1$.

Corollary to proof

The B-Koerwien-Laskowski sentences characterizing $\aleph_n$ have $2^{\aleph_1}$ models in $\aleph_1$.

Question

Is $\aleph_1$-categoricity absolute for complete sentences of $L_{\omega_1,\omega}$?

Hanf Numbers for JEP, AP etc

Lower bounds

The previous results show the Hanf number for JEP and DAP is at least $\beth_{\omega_1}$.

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Upper bounds: B-Boney

Let κ be strongly compact and ***K*** be an AEC with Löwenheim-Skolem number less than κ.

- If ***K*** satisfies *JEP*$(<\kappa)$ then $\boldsymbol{K}_{\geq\kappa}$ satisfies *JEP*.
- If ***K*** satisfies *AP*$(<\kappa)$ then ***K*** satisfies *AP*.

crux: strongly compact cardinals. Direct proof is by ultraproducts. Proof using modification of first order arguments and compactness of $L_{\kappa,\kappa}$ leads to interesting issues about the presentation theorem.

The big gap

Assuming a strongly compact cardinal κ, various Hanf numbers are that κ.

(tameness (Shelah 932), jep, dap, ap)

In ZFC, those Hanf numbers are at least $\beth_{\aleph_1}$.