Expansions of Models
Predicates and Automorphisms

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May 13, 2001

EXPANSIONS OF MODELS

Let M be a structure of a language L. Expand M by adding a name for some relation on M^n. When are 'good' properties of M preserved?

1. What are good properties
   (a) stability
   (b) o-minimality; NIP
2. Name what?
   (a) Unary predicate
      i. A model
      ii. A generic subset
      iii. A (small) set of indiscernibles
      iv. An arbitrary set
   (b) A generic automorphism

UNARY PREDICATES BACKGROUND

1. Shelah: Categoricity over a predicate
2. Poizat/Bouscaren: Belles Paires
3. Hrushovski/Buechler/Vassiliev: study geometry
4. Baizhanov: Expansions of (weakly) o-minimal
5. Baldwin-Benedikt: pseudo-small set of indiscernibles
6. Casanovas-Ziegler: relativized fcp
7. generic expansions: the Hrushovski construction

GENERIC AUTOMORPHISMS

T is assumed to admit *quantifier elimination.*

Expand L to L_σ by adding a unary function symbol σ.

T_σ extends T by asserting 'σ is an automorphism.

$\sigma \in \mathrm{Aut}(M)$ is a generic automorphism if (M, σ) is existentially complete.

T admits a *model companion* or 'TA exists' if the class of existentially complete models of T_σ is an elementary class.

When does TA exist?

What are its properties?

BACKGROUND

Chatzidakis-Hrushovski: TA exists for T as ACF.

Chatzidakis-Pillay: If T is stable and TA exists, then TA is simple.

Unstable Case;

Kikyo: If T is an unstable theory without the independence property, T_σ does not have a model-companion.

Kikyo and Shelah: If T has the strict order property, T_σ does not have a model-companion.

Kikyo: If T is an unstable model complete theory and T_σ has the amalgamation property then T_σ does not have a model-companion.

STABLE CASE

Wheeler/Khudaibergenov: If T is an stable theory with the finite cover property then T_σ does not have a model-companion.

Kikyo-Pillay: If T is strongly minimal with the definable multiplicity property then T_σ has a model-companion.

They conjecture the converse.

OBSTRUCTIONS

This work is due to Baldwin-Shelah.

T_σ has no obstructions if it satisfies the following condition.

For every finite set Δ_1 of L-formulas, there is a finite set $\Delta_2 \supset \Delta_1$ of L-formulas:

for any $(M, \sigma) \models T_\sigma$, and complete Δ_1 type $p(x, y)$ over M and complete Δ_2 type $q(x, y)$ over M extending $p(x, y)$,

if $q(x, y)$ implies that the Δ_2-type of y over M equals σ of the Δ_2 type of x over M,

then there is a complete type $q'(x, y)$ over M extending $p(x, y)$ such that $q'(x, y)$ implies $tp(y/M) = \sigma(tp(x/M))$.

(This formulation by Pillay simplifies the definition of Baldwin-Shelah.)

Lemma If T has the fcp then T has obstructions.

CHARACTERIZATION

Theorem Suppose T is stable. Then T_σ has a model companion iff and only if T_σ has no obstructions.

Proof Sketch: Assume no obstruction: For each formula $\psi(\overline{x}, \overline{y}, \overline{z})$ we add an axiom θ_ψ. Let $\Delta_1 = \{\psi\}$ and Δ_2 chosen from definition of obstruction.

θ_ψ asserts of a model (M, σ) of T_σ:

For any d, suppose $q(x, y)$ is a complete Δ_2 type (over M) such that $\psi(x, y, d) \in q(x, y)$ and such that $q(x, y)$ implies that $tp_{\Delta_2}(y/M) = \sigma(tp_{\Delta_2}(x/M))$ then there is (a, b) realizing $\psi(x, y, d)$ such that $b = \sigma(a)$.

Standard properties of stable theories without the fcp imply the hypothesis of the axiom is expressed by a single formula.

The lack of obstructions guarantees that the hypothesis implies the conclusion.

routine: (M, σ) is existentially closed iff it satisfies the conclusion for each ψ.

CONVERSE

It is well-known (Kudaibergenov/Winkler) that T has non-fcp.

Suppose there is a Δ_1 and for each finiteΔ_i an obstruction K_i. We take (in an expanded language) an ultraproduct of the obstructions. On the one hand, if T_σ has a model companion TA, this gives a set of formulas $\Gamma(\overline{x}, \overline{y})$ such that $TA \cup \Gamma(\overline{a}, \overline{b})$ implies there is an automorphism mapping $\overline{a}$ to $\overline{b}$. By compactness, this is implied by a finite piece of Γ and this contradicts the definition of obstruction.