Necessity of the VWGCH ?

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April 5, 2008

Outline

The Weak Generalized Continuum Hypothesis

Setting

ZFC is the base theory throughout.

Axiom: WGCH Weak GCH

For every cardinal λ, $2^{\lambda} < 2^{\lambda^+}$.

Axiom: VWGCH Very Weak GCH

For every cardinal λ with $\lambda < \aleph_\omega$, $2^{\lambda} < 2^{\lambda^+}$.

Acknowledgements

This is primarily an exposition of work of Shelah followed by a series of problems.
Detailed proof of most of the results here are given in my monograph: Categoricity (available on line).

Definition: Devlin-Shelah Weak Diamond

Φ_λ is the proposition:

For any function $F : 2^{<\lambda} \rightarrow 2$ there exists $g \in 2^\lambda$ such that for every $f \in 2^\lambda$ the set

$$\{\delta < \lambda : F(f \restriction \delta) = g(\delta)\}$$

is stationary.

For every $X \subset \lambda$ and $\alpha < \lambda$,
Weak-$\diamond$ predicts whether $X \cap \alpha$ is in one side or another of a partition of $\mathcal{P}(\alpha)$.

Crucial Fact

Weak diamond is the operative form of WGCH.

$2^\lambda < 2^{\lambda^+}$ if and only if Weak-$\diamond$ on λ^+

Model Theoretic Context

In this talk, $\mathbf{K}$ is the class of models of a sentence ψ in $L_{\omega_1,\omega}$.

We write $M \prec_{\mathbf{K}} N$ where $\prec_{\mathbf{K}}$ is elementary submodel in the smallest fragment L^* containing ψ.

We will sketch how to study this situation as the class of atomic models of a first order theory.

More Background

A model is *small* if it realizes only countably many $L_{\omega_1,\omega}$-types over the empty set.

M is small if and only M is Karp-equivalent to a countable model.

ϕ is complete for $L_{\omega_1,\omega}$ if for every sentence ψ of $L_{\omega_1,\omega}$, either $\phi \rightarrow \psi$ or $\phi \rightarrow \neg\psi$.

Note that a sentence is complete if and only if it is a Scott sentence; so every model of a complete sentence is small.

Passing to Atomic

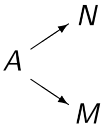
A model is atomic if every finite sequence realizes a principal type over $\emptyset$.

Theorem

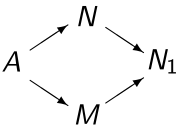
Let ψ be a complete sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ and a complete first order τ'-theory T such that reduct is a 1-1 map from the *atomic* models of T onto the models of ψ.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



Failure of amalgamation yields many models

Theorem (WGCH: Shelah)

If **K** is λ-categorical and amalgamation fails in λ there are 2^{λ^+} models in **K** of cardinality λ^+.

Upward Löwenheim Skolem

Definition

$\aleph_\alpha$ is characterized by ϕ_α if there is a model of ϕ_α with cardinality $\aleph_\alpha$ but no larger model.

Known

Morley: If ϕ has a model of cardinality at least $\beth_{\omega_1}$, ϕ has arbitrarily large models.

Hjorth: If α is countable $\aleph_\alpha$ is characterizable.

Conjecture

Shelah: If κ is characterized by ϕ, ϕ has 2^λ models in some $\lambda \leq \kappa$.

Few models and smallness

Theorem (Keisler)

If **K** has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then every model of **K** realizes only countably many types over the empty set in the countable fragment L^*.

Few models in $\aleph_1$ implies completeness

Theorem (Shelah)

If the $L_{\omega_1,\omega}$-τ-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then ψ has a small model of cardinality $\aleph_1$.

κ-Categoricity implies completeness ????

Thus, any $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class) But this result uses properties of $\aleph_1$ heavily.

Question

If the $L_{\omega_1,\omega}$-τ-sentence ψ has a model of cardinality κ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, must ψ have a τ-small model of cardinality κ?

Categoricity Transfer in $L_{\omega_1,\omega}$

An atomic class **K** is excellent if it is ω-stable and satisfies certain amalgamation properties for finite systems of models.

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

VWGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

Excellence gained: more precisley

VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and with $I(\mathbf{K}, \aleph_n) \leq 2^{\aleph_{n-1}}$ for each $n < \omega$ is excellent.

Context

K is the class of atomic models (realize only principal types) of a first order theory.
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

reprise: Few models and smallness

Theorem (Keisler)

If **K** has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then every model of **K** realizes only countably many types over the empty set in the countable fragment L^*.

ω-stability I

Definition

The atomic class **K** is λ-stable if for every $M \in$ **K** of cardinality λ, $|S_{\rm at}(M)| = \lambda$.

Corollary (Shelah) CH

If **K** is $\aleph_1$-categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then **K** is ω-stable.

ω-stability II

Consequences

1. This gets ω-stability without assuming arbitrarily large models.
2. We only demand few types over models, not arbitrary sets; this is crucial.
3. But, apparently uses CH twice! (for amalgamation and type counting)

ω-stability III

Getting ω-stability

1. Assume arbitrarily large models; use Ehrenfeucht-Mostowski models
2. Keisler-Shelah using CH.
3. Diverse classes (Shelah)

Fundamental question

Let ϕ be a sentence of $L_{\omega_1,\omega}$
Are the properties:

ϕ is $\aleph_1$-categorical, and

ϕ is ω-stable
absolute for cardinal-preserving forcing?

Is WCH is necessary?

Does MA $+ \neg$ CH imply there is a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$ categorical but

a is not ω-stable

b does not satisfy amalgamation even for countable models.

There is such an example in $L_{\omega_1,\omega}(Q)$ but Laskowski showed the example proposed for $L_{\omega_1,\omega}$ by Shelah (and me) fails.

Towards Counterexamples

For any model $M \in \mathbf{K}$,

1. P and Q partition M.
2. E is an equivalence relation on Q.
3. P and each equivalence class of E is denumerably infinite.
4. R is a relation on $P \times Q$ that is extensional on P. That is, thinking of R as the 'element' relation, each member of Q denotes a subset of P.
5. For every set X of n elements X from P and every subset X_0 of X and each equivalence class in Q, there is an element of that equivalence class that is R-related to every element of X_0 and not to any element of $X - X_0$.
6. Similarly, for every set of n elements Y from Q and every subset Y_0 of Y, there is an element of P that is R-related to every element of Y_0 and not to any element of $Y - Y_0$.

An AEC counterexample

Fix the class $\mathbf{K}$ as above and for $M, N \in \mathbf{K}$, define $M \prec_{\mathbf{K}} N$ if $P^M = P^N$ and for each $m \in Q^M$,
$\{n \in N : mEn\} = \{n \in M : mEn\}$ (equivalence classes don't expand).
This class does not have finite character (Trlifaj's talk).