Using Set theory in model theory ASL/AMS Annual Meeting Boston 2012

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January 8, 2012

Today's Topics

Using Extensions of ZFC in Model Theory

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2. Perhaps the hypothesis is eliminable
   - A. The cominatorial hypothesis might be replaced by a more subtle argument.
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   - B The conclusion might be absolute

     The elementary equivalence proved in the Ax-Kochen-Ershov theorem

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   - A. The cominatorial hypothesis might be replaced by a more subtle argument.
     E.G. Ultrapowers of elementarily equivalent models are isomorphic
   - B. The conclusion might be absolute

     The elementary equivalence proved in the Ax-Kochen-Ershov theorem
   - C. Consistency may imply truth.

Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

See also the Vaananen article in Model Theoretic Logic volume

Which ‘Central Notions’?

Chang’s two cardinal theorem (morasses)

‘Vaughtian pair is absolute’

saturation is not absolute

Aside: For aec, saturation is absolute below a categoricity cardinal.

Classification Theory

Crucial Observation

The stability classification is absolute.

Fundamental Consequence

Crucial properties are provable in ZFC for certain classes of theories.

1. All stable theories have full two cardinal transfer.
2. There are saturated models exactly in the cardinals where the theory is stable.

But this is for FIRST ORDER theories.

Geography

$$L_{\omega,\omega} \subset L_{\omega_1,\omega} \subset L_{\omega_1,\omega}(Q) \subset anal.pres.AEC \subset AEC.$$

In a central case explained below

Extensions of ZFC are used for $L_{\omega_1,\omega}$.

Extensions of ZFC are proved necessary for $L_{\omega_1,\omega}(Q)$.

Two notions of ‘use’

1. Some model theoretic results ‘use’ extensions of ZFC
2. Some model theoretic results are provable in ZFC, using models of set theory.

This Talk

1. A quick statement of some results of the first kind
2. Discussion of several examples of the second method.

One Completely General Result

Theorem: ($2^\lambda < 2^{\lambda^+}$) (Shelah)

Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. For any Abstract Elementary class, if amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality λ^+.

Is $2^\lambda < 2^{\lambda^+}$ needed?

Is $2^\lambda < 2^{\lambda^+}$ needed?

1. $\lambda = \aleph_0$:
   a. Definitely not provable in ZFC: There are $L(Q)$-axiomatizable examples
      i. Shelah: many models with CH, $\aleph_1$-categorical under MA
      ii. Koerwien-Todorcevic: many models under MA, $\aleph_1$-categorical from PFA.
   b. Independence Open for $L_{\omega_1,\omega}$
2. Grossberg and VanDieren have announced the AEC analog in larger λ using the generalized Martin's Axiom.

A simple Problem

Let ϕ be a sentence of $L_{\omega_1,\omega}$.

Question

Is the property ϕ has an uncountable model absolute?

False Start

Fact: Easy for complete sentences

If ϕ is a complete sentence in $L_{\omega_1,\omega}$,
ϕ has an uncountable model if and only if there exist countable $M \precneqq_{\omega_1,\omega} N$ which satisfy ϕ.

This property is Σ^1_1 and done by Shoenfield absoluteness.

Note: $L_{\omega_1,\omega}$ satisfies downward Löwenheim-Skolem for sentences but not for theories.

Fly in the ointment

There are uncountable models that have no $L_{\omega_1,\omega}$-elementary submodel.

E.g. any uncountable model of the first order theory of infinitely many independent unary predicates P_i.

So the sentence saying every finite Boolean combination of the P_i is non-empty has an uncountable model and our obvious criteria does not work.

Note that if we add the requirement that each type is realized at most once, then every model has cardinality $\leq 2^{\aleph_0}$.

L^*-submodel

Given a sentence ϕ. Let L^* be the minimal countable fragment of $L_{\omega_1,\omega}$ containing ϕ.
Suppose $M \prec_{L^*} N$, $M \neq N$.

Does there exist a proper extension N' of N with $N \prec_{L^*} N'$?
If so we have an absolute characterization of ϕ has a uncountable model.

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BUT NO! Asserted by Gregory; example found by Johnson, Knight, Ocasio, VanDenDriessche this Fall

Smallness

Definition

1. A τ-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many $L^*(\tau)$-types (i.e. only countably many $L^*(\tau)$-n-types for each $n < \omega$).
2. A τ-structure M is called small or $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.

Why Smallness matters

Fact

Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.

A Correct Characterization

Larson's characterization

Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})$,
the existence of a model of size $\aleph_1$ satisfying ϕ
is equivalent to
the existence of a countable model of ZFC° containing $\{\phi\} \cup \omega$ which thinks there is a model of size $\aleph_1$ with a member satisfying ϕ.

This property is Σ^1_1 and done by Shoenfield absoluteness.

Larger Cardinals

It is easy to see that there are sentences of $L_{\omega_1,\omega}$ such that the existence of a model in $\aleph_2$ depends on the continuum hypothesis.
S. Friedman and M. Koerwien have shown.

Assume GCH (and large cardinals for independence of the Kurepa hypothesis)

1. For any $\alpha \in \omega_1 - \{0, 1, \omega\}$ there is a sentence ϕ_α such that the existence of a model in $\aleph_\alpha$ is not absolute.
2. For $\aleph_3$, there is a complete such sentence.

Deja vu

The really basic proof

Karp (1964) had proved completeness theorems for $L_{\omega_1,\omega}$, and Keisler (late 60's/ early 70's) for $L_{\omega_1,\omega}(Q)$, $L_{\omega_1,\omega}(\mathrm{aa})$.

Deja vu

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The rest of the talk illustrates the advantages of missing the 'obvious' argument.

Method: ‘Consistency implies Truth’

Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let $\mathcal{A}$ be the countable model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct $\mathcal{B}$, an uncountable model of set theory, which is an elementary extension of $\mathcal{A}$
such that $\mathcal{B}$ is correct about uncountability. Then the model of ϕ in $\mathcal{B}$ is actually an uncountable model of ϕ.

How to build $\mathcal{B}$

- **MT** Iterate a theorem of Keisler and Morley (refined by Hutchinson).
- **ST** Iterations of 'special' ultrapowers.

ZFC° denotes a sufficient subtheory of ZFC for our purposes.

How to build $\mathcal{B}$

The main technical tool is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by NS_{ω_1}.

The ultrafilter

Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U).

The Ultrapower

Given such M and U, we can form the generic ultrapower $\mathrm{Ult}(M, U)$, which consists of all functions in M with domain ω_1^M,

where for any two such functions f, g, and any relation R in $\{=, \in\}$, fRg in $\mathrm{Ult}(M, U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha) R g(\alpha)\} \in U$.

Nota Bene

If M is countable, $\mathrm{Ult}(M, U)$ is countable.

By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M, U)$ with its Mostowski collapse.

The Ultrapower is useful

Fact

Suppose that M is a model of ZFC°, and that $j\colon M \to \mathrm{Ult}(M, U)$ is an elementary embedding derived from forcing over M with $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then for all $x \in M$, $j(x) = j[x]$ if and only if x is countable in M.

That is $\mathrm{Ult}(M, U)$ increases exactly the sets that M thinks are uncountable.

Iterations

Definition

Let M be a model of ZFC° and let γ be an ordinal less than or equal to ω_1.

An iteration of M of length γ consists of models

$$M_\alpha : \quad (\alpha \leq \gamma),$$

sets

$$G_\alpha : \quad (\alpha < \gamma),$$

and a commuting family of elementary embeddings

$$j_{\alpha\beta} : M_\alpha \to M_\beta : \quad (\alpha \leq \beta \leq \gamma)$$

such that the successor stages are the ultrapowers just discussed.

What is this good for?

Fact

Suppose that M is a model of ZFC°, and that M_{ω_1} is the final model of an iteration of M of length ω_1.
Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable.

So consistent sentences of $L_{\omega_1,\omega}(Q)$ are provable.

One can also make M_{ω_1} correct about stationarity, extending the absoluteness results to $L_{\omega_1,\omega}(aa)$.

Many Iterations

Remark

We emphasize that for any countable model M of ZFC° there are $2^{\aleph_0}$ many M-generic ultrafilters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$.

It follows that there are $2^{\aleph_1}$ many iterations of M of length ω_1.

Really distinct interations

Theorem (Larson)

If M is a countable model of ZFC° + $\mathrm{MA}_{\aleph_1}$ and

$$\langle M_\alpha, G_\alpha, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

and

$$\langle M'_\alpha, G'_\alpha, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\alpha,$$

where α is least such that $G_\alpha \neq G'_\alpha$.

G_α not defined for $\alpha = \omega_1$.

The Model Theory

Theorem: (Keisler, new proof Larson)

Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.

Corollary (Shelah using ch)

If a sentence in $L_{\omega_1,\omega}$ has less that $2^{\aleph_1}$ models in $\aleph_1$ then it is (syntactically) ω-stable.

CH used twice.

Sketching New Proof:

Constructing many models

Let N be a model of cardinality $\aleph_1$ realizing uncountably many F-types, let X be a countable elementary submodel of $H((2^{2^{\aleph_1}})^+)$ containing $\{N\}$ and the transitive closure of $\{F\}$. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse.

Build a tree of generic ultrapower iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M', each of length ω_1.

Since F-types can be coded by reals using an enumeration of F in M, the images of N_0 under these iterations will pairwise realize just countably many F-types in common.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an abstract elementary class: AEC if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

Analytically Presented AEC

Definition

An abstract elementary class **K** with Löwenheim number $\aleph_0$ is analytically presented if the set of countable models in **K**, and the corresponding strong submodel relation $\prec_{\mathbf{K}}$, are both analytic.

Context

Fact

Analytically presented $\boldsymbol{K}$ is the same as a $PC\Gamma(\aleph_0, \aleph_0)$ class:

reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types

AKA:

1. Keisler: PC_δ over $L_{\omega_1,\omega}$
2. Shelah: $PC(\aleph_0, \aleph_0)$, $\aleph_0$-presented

Example

Groupable partial orders (Jarden varying Shelah)

Let $(\boldsymbol{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order (equivalently admits a group structure)
with $M \prec N$ if $M \subseteq N$ and no component is extended.

This AEC is analytically presented.
Add a binary function and say it is a group on each component.
But it has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.

Recall: this 'is' the pseudo-elementary counterexample to Vaught' s conjecture.

Galois Types

Let $M \prec_{\mathbf{K}} N_0$, $M \prec_{\mathbf{K}} N_1$, $a_0 \in N_0$ and $a_1 \in N_1$ realize the same Galois Type over M iff
there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 : N_0 \to N$ and $f_1 : N_1 \to N$ such that $f_0|M = f_1|M$ and $f_0(a_0) = f_1(a_1)$.

Realizing the same Galois type (over countable models) is an equivalence relation

$$E_M$$

if $\boldsymbol{K}_{\aleph_0}$ satisfies the amalgamation property.

The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property for models of cardinality at most $\aleph_0$

and has at most $\aleph_1$-Galois types over models of cardinality $\leq \aleph_0$

then there is an $\aleph_1$-monster model $\mathbb{M}$ for $\boldsymbol{K}$
and Galois type of a over a countable M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.
So E_M is an equivalence relation on $\mathbb{M}$.

Some stability notions

Definition

1. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be Galois ω-stable if for each countable $M \in \mathbf{K}$, E_M has countably many equivalence classes.
2. The abstract elementary class $(\mathbf{K}, \prec)$ is almost Galois ω-stable if for each countable $M \in \mathbf{K}$, no E_M has a perfect set of equivalence classes.

Almost Galois Stable

Well-orders of type at most $\aleph_1$ under end-extension are an AEC where countable models have only $\aleph_1$ Galois types.

Galois equivalence is Σ_1^1

On an analytically presented AEC, having the same Galois type over M is an analytic equivalence relation, E_M. So by Burgess's theorem we have the following trichotomy.

Theorem

An analytically presented abstract elementary class $(\mathbf{K}, \prec)$ is

1. Galois ω-stable or
2. almost Galois ω-stable or
3. has a perfect set of Galois types over some countable model

Known to Shelah but independently rediscovered by Larson/Baldwin

Keisler for AEC

Theorem: (B/Larson)

Suppose that

1. $\mathbf{K}$ is an analytically presented abstract elementary class;
2. N is a $\mathbf{K}$-structure of cardinality $\aleph_1$, and N_0 is a countable structure with $N_0 \prec_{\mathbf{K}} N$;
3. P is a perfect set of E_{N_0}-inequivalent members of ω^ω;
4. N realizes the Galois types of uncountably many members of P over N_0.

Then there exists a family of $2^{\aleph_1}$ many $\mathbf{K}$-structures of cardinality $\aleph_1$, each containing N_0 and pairwise realizing just countably many P-classes in common.

$L_{\omega_1,\omega}$-case

Fact: Hyttinen-Kesala, Kueker

If a sentence in $L_{\omega_1,\omega}$, satisfying amalgamation and joint embedding, is almost Galois ω-stable then it is Galois ω-stable.

What about analytically presented?

Analytically presented Almost Galois ω-stable example

The 'groupable partial order' is almost Galois stable

Let $(\boldsymbol{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended.

Since there are only $\aleph_1$-isomorphism types of components this class is almost Galois ω-stable.

This AEC is analytically presented.

Getting small models I

Theorem: Shelah

If $\boldsymbol{K}$ is analytically presented and some model of cardinality $\aleph_1$ is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then $\boldsymbol{K}$ has an $L_{\omega_1,\omega}(\tau)$-small model M' of cardinality $\aleph_1$.

Getting small models II

Theorem: Baldwin/Shelah/Larson

If $\boldsymbol{K}$ has a model in $\aleph_1$ that is not $L_{\omega_1,\omega}(\tau)$-small then

1. there are at least $\aleph_1$ complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $\boldsymbol{K}$;
2. $\boldsymbol{K}$ has uncountably many models in $\aleph_1$;
3. $\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.

Proof: Iterate the previous theorem.

Corollary: Baldwin/Shelah/Larson

Vaught's conjecture is equivalent to Vaught's conjecture for extendible models.

A countable model is extendible if it has an $L_{\omega_1,\omega}$-elementary extension.

Absoluteness of (almost) ω-stability

1. first order syntactic: Π^1_1
2. $L_{\omega_1,\omega}$-syntactic: Π^1_1
3. analytically presented AEC: Galois ω-stable: perhaps Π^1_4
4. analytically presented AEC: almost Galois ω-stable: Π^1_2

Absoluteness of $\aleph_1$-categoricity

1. $\aleph_1$-categoricity of a class $\boldsymbol{K}$ defined in $L_{\omega_1,\omega}$ is absolute between models of set theory that satisfy any one of the following conditions.
    1. $\boldsymbol{K}$ is ω-stable;
    2. $\boldsymbol{K}$ has arbitrarily large members and $\boldsymbol{K}$ has amalgamation in $\aleph_0$;
    3. $2^{\aleph_0} < 2^{\aleph_1}$.

    `http://homepages.math.uic.edu/~jbaldwin/pub/singsep2010.pdf`
2. $\aleph_1$-categoricity of an analytically presented AEC $\boldsymbol{K}$ is absolute

    between models of set theory in which $\boldsymbol{K}$ is almost Galois ω-stable, satisfies amalgamation in $\aleph_0$, and has an uncountable model.

Why is this absoluteness of $\aleph_1$-categoricity true for AEC?

Fact

Suppose that **K** is an analytically presented AEC. Then the following statements are equivalent.

1. There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that:
   - $M \prec_{\mathbf{K}} N$;
   - the set of Galois types over M realized in N is countable;
   - some Galois type over M is not realized in N.
2. There is a countable model of ZFC° whose ω_1 is well-founded and which contains trees on ω giving rise to **K**, $\prec_{\mathbf{K}}$ and the associated relation $\sim_0$, and satisfies statement 1.

Summary

1. The set theoretic method provides a uniform method for studying models of various infinitary logic
2. We introduced analytically presented AEC and showed:
   - i Extended Keisler's few models implies ω-stability theorem to this class
   - ii Assuming countably many models in $\aleph_1$: Almost Galois ω-stable implies Galois ω-stable
   - iii $\aleph_1$-categoricity absolute for Almost Galois ω-stable with amalgamation.