Homogeneity and Saturation

John T. Baldwin
University of Illinios at Chicago
www.math.uic.edu/ jbaldwin

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HOMOGENEITY

The structure M is *"homogeneous"* if for every A, B, C contained in M
if $A \sim B$ then there is a D with $AC \sim BD$.

Three parameters:

1. constraints on A, B,
2. constraints on C, D,
3. What is $\sim$?

BACKGROUND EXAMPLES

1. **sequence homogeneity**
   (a) A, B are arbitrary subsequences of M with cardinality $< |M|$.
   (b) C, D are singletons.

   - If $\sim$ is elementary equivalence we have the usual model theoretic notion of homogeneity.

   - If $\sim$ is isomorphism and M is countable we have the notion of homogeneity studied in 'homogeneous structures': Fraissé, Jónsson, Lachlan, Cherlin, Hrushovsky, MacPherson, ...

2. **model homogeneity**: If A, B, C, are submodels of T with respect to a given notion $\prec$ of 'elementary submodel' and $\sim$ is the associated 'elementary equivalence', we have Shelah's notion of *model homogeneity*.

‘NEW’ EXAMPLES

1. L^n**-homogeneity**

   (a) A, B are arbitrary subsets of M with cardinality $< |M|$.

   (b) C, D are singletons.

   (c) $\sim$ is L^n-equivalence.

2. L^n**-**c**-homogeneity**:

   We fix a notion of closure: subalgebra generated by

   (a) A, B are closed subsets of M with cardinality $< |M|$.

   (b) C is an L^n-elementary submodel of M.

   (c) $\sim$ is L^n-equivalence.

3. **Generics**:

We fix a notion of closed substructure:

$$A \leq M.$$

(a) A, B are closed subsets of M with cardinality $< |M|$.

(b) C is a closed subset of M with cardinality $< |M|$.

(c) $\sim$ is isomorphism.

Examples of exotic strongly minimal sets, stable theories, 0-1 laws:

Hrushovski, Baldwin, Holland, Herwig, Shi, Shelah, Nesin, Tent, Wagner, Evans, Poizat, Pourmahdian, Kueker-Laskowsi. Work with $|M| = \aleph_0$.

FINITE VARIABLE LOGIC

Definition.

- An L^n-formula contains only n variables, free or bound
- An L^n-theory T is a consistent set of L^n-sentences, closed under logical consequence (in L^n).

We consider both infinite and finite models so we have compactness.

EXAMPLES

- Vector spaces over a finite field (L^3-complete)
- For any complete theory T, let T_n denote the L^n-sentences of T.
  - T_n is complete for L^n.
  - If T is totally categorical then for some n, T_n has arbitrarily large finite models.
- The n-extension axioms
- Dense linear order

L^n-THEORIES WITH FEW MODELS

Poizat characterized the complete L^n-theories which have models of size n or $n+1$ and larger finite models. He asked whether any complete L^n-theory with more than one finite model had infinitely many finite models.

Fact (Cherlin, Thomas)

- There are complete theories in L^n with models in finitely many finite cardinalities and *no* infinite model.
- There are complete theories in L^n with models in finitely many finite cardinalities and *with* infinite models.

L^n-TYPES OVER $\emptyset$

1. For $m \leq n$, an m-type in L^n, p, is the collection of L^n formulas satisfied by some $\mathbf{a} \in M \models T$.
2. $S_n^n(T)$ is the collection of such types.
3. The n-size of T is $|S_n^n(T)|$.

SCOTT SENTENCES

Lemma. For any finite structure M and any finite n there is a sentence of L^n which characterizes M up to L^n-equivalence. Moreover, the quantifier rank of this Scott sentence is less than $|S_n^n(T)|$ where $T = \mathrm{Th_{L^n}(M)}$.

Observe that the construction of the Scott sentence of a finite model, depends only on $S_n^n(T)$

Fact

1. If an L^n-theory T has a finite model, $S_n^n(T)$ is finite.
2. If $S_n^n(T)$ is finite, then
   - The L^n-theory T is axiomatized by a single sentence of L^n;
   - All models of T realize the same (all) members of $S_n^n(T)$.

L^n-STABILITY

A formula $\phi(\mathbf{x}, \mathbf{y})$ has the order property in a model M if there are sequences $\mathbf{a}_i, \mathbf{b}_j$ for $i, j < \omega$ such that

$$M \models \phi(\mathbf{a}_i, \mathbf{b}_j) \text{ iff } i < j.$$

A model M is *L^n-stable* if there is no L^n-formula $\phi(x, y)$ with the order property in M.

An L^n-theory T is stable if each model of T is.

The notions of stability and ω-stability in L^n were first worked out by Hyttinen and Djordjević.

SET AMALGAMATION PROPERTY

Definition. The L^n-theory T satisfies the *set amalgamation property* (**SAP**) if for any situation with $N, M \models T$:

$$\begin{tikzcd}[column sep=small, row sep=tiny] & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

with $\mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{M}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N})$ there exists an N_1

such that $\begin{tikzcd}[column sep=small, row sep=tiny] & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$

and $\mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N_1}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{M}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N})$.

FAILURE OF SAP

Let T be the L^3-theory of Abelian groups of exponent 2 in a language with one ternary relation symbol.

Let M be the Abelian group generated freely (in exponent 2) by b_1, b_2, b_3, b_4.

Let N be the Abelian group generated freely (in exponent 2) by c_1, c_2, c_3 and let

$$c_4 = c_1 + c_2 + c_3.$$

Now let f, g map $A = \{a_1, \ldots a_4\}$ into M and N by mapping a_i to b_i and c_i respectively.

- f and g are L^3 elementary.
- M and N can't be amalgamated over A.

WHAT IS AN L^n TYPE?

Two problems:

I. Formulas over parameters sneak in variables.

Solution: Each $\phi(\mathbf{x}, \mathbf{a})$ must be a substitution instance of an L^n-formula.

Drawback: Types are NOT closed under conjunction.

II. In the absence of amalgamation:

A. Types may be separately but not jointly consistent.

Hedman and Kueker built such an example where the countable model is ω-categorical.

B. $S(A)$ is ambiguous because we have no monster model.

$S(A)$?

Definition: T is ω-stable if for every $M \models T$ and every countable $A \subseteq M$, $S(A, M)$ is countable.

But, what is $S(A, M)$?

1. $S(A, M) = \{p : p \text{ is consistent}\}$.
2. $S(A, M) = \{p : p \text{ is realized in an } L^n\text{-extension of } M\}$.
3. $S(A, M) = \{p : p \text{ is realized in } M\}$.

Clearly ω-stability i implies ω-stability $i + 1$; but not conversely.

Amalgamation over models makes the last two equivalent; amalgamation over sets makes them all equivalent.

Fact:[Shelah] Let Δ be a finite set of formulas, closed under permutation of variables and negation. If M is a model which does not have the order property witnessed by a formula in Δ, then for each infinite cardinal λ, the model M realizes at most λ many Δ-types over each of its subsets of cardinality λ.

Corollary. If T is a complete stable L^n-theory and $S_n^n(T)$ is finite, then T is ω-stable in sense 3.

L^n-SATURATED MODELS

A structure M is L^n-κ-saturated if every $A \subset M$ with $|A| < \kappa$, every consistent type over A is realized in M.

Independently, Djordjević and Hyttinen showed:

Theorem. T has the L^n-set amalgamation property if and only if for any κ, every model of T can be L^n-embedded in an L^n-κ-saturated model.

Note that if T is the L^3 theory of $(Z, +, 1)$, there is no countable countably L^3-saturated model.

$\mathbf{T}^c$

Suppose the L^n theory T has the x-amalgamation property. (x = amalgamation over all sets, sufficiently closed subsets, or models) and has jep.

Then T has L^n x-homogeneous models; all such models are elementarily equivalent.
T^c is the first order theory of the L^n x-homogeneous models.

Question. (Djordjevic) Is '$S^n_n(T)$ finite' necessary for the following result?

GETTING FINITE MODELS

Theorem. (Djordjević) Suppose T satisfies SAP and $S_n^n(T)$ is finite. Let M be a model of T. If $p(\overline{x}) = \mathrm{tp_L(\overline{a}/\emptyset, M)}$ and $q(\overline{x}) = \mathrm{tp_{L^n}(\overline{a}/\emptyset, M)}$ then $T^c \cup q(\overline{x}) \vdash p(\overline{x})$.

Theorem. (Djordjević) If a stable L^k-theory T with the *set* amalgamation property has only finitely many L^k-types then T^c is ω-categorical and ω-stable. It follows that T has arbitrarily large finite models.

This requires Zil'ber/Cherlin-Harrington-Lachlan technology.

CAP AND FINITE MODELS

Hypothesis (H1). For any countable models M_ℓ for $\ell = 0,1,2$ with $M_0 \prec_{L^n} M_\ell$ for $\ell = 1,2$, and $\overline{a}_\ell \in M_\ell$ for $\ell = 1,2$ if

$$\mathrm{tp_{L^n}(cl(\overline{a}_1)/M_0, M_1) = tp_{L^n}(cl(\overline{a}_2)/M_0, M_2)},$$

then there exist N and L^n-elementary embeddings $f_\ell : M_\ell \to N$ such that $f_1(\overline{a}_1) = f_2(\overline{a}_2)$ and f_ℓ is the identity over M_0 for $\ell = 1,2$.

Let C denote cl($\emptyset$).

Theorem. (Baldwin-Lessmann) If a stable L^k-theory T with has (CAP, JEP) and satisfies $H1$ then $T^c(C)$ is ω-categorical and ω-stable. It follows that T has arbitrarily large finite models.

This extends Djordjević's result to theories with non-trivial forking.

Question (Lessmann) Can CAP and H1 be replaced by the AP?

Bicolored fields (Poizat)

L is $(+, 0, -, *, 1, B)$.

$\delta(\mathbf{a})$ is twice the transcendence degree of $\mathbf{a}$ minus the number of black points.

$(\mathbf{K_0}, \leq)$ has the amalgamation property. (Take linearly disjoint copies of the fields and paint all new points in the join white.)

Under what conditions is the generic ω-saturated? How do we make the rank of T finite?

Constraints on δ

Partial list:

1. If diag $(\overline{a})$ = diag $(\overline{b})$ then $\delta(\overline{a}) = \delta(\overline{b})$.
2. (monotonicity) $\delta(\overline{a}/\overline{b}) \geq \delta(\overline{a}/\overline{b}\overline{c})$ whenever $\overline{c} \cap \overline{a} = \overline{c} \cap \overline{b}$.
3. (definability) For any integer k and $\overline{a}, \overline{b} \subseteq C \in \mathbf{K}$ with $\delta(\overline{a}/\overline{b}) = k$, there is a *$\delta$-formula for* **a** *over* **b**, $\varphi(\overline{x}; \overline{y}) \in$ diag $(\overline{a}; \overline{b})$, such that the following hold for any $\overline{a}', \overline{b}' \subseteq B \in \mathbf{K}$: if $B \models \varphi(\overline{a}'; \overline{b}')$, then $\delta(\overline{a}'/\overline{b}') \leq k$.
4. (q.f. type determined) For any $N \in \mathbf{K}$ and $\overline{a}, \overline{a'}, \overline{b} \subseteq N$, if $N \models \phi(\overline{a}; \overline{b}) \wedge \phi(\overline{a'}; \overline{b})$ and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a'}/\overline{b}) = k$ and $\overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle$, $\overline{ba'} \leq \langle \overline{ba'} \rangle$, then $Diag(\overline{b}\overline{a}) = Diag(\overline{ba'})$.

Intrinsic Extensions

$\mathbf{a}$ is a *minimal intrinsic extension* of $\mathbf{b}$ if $\delta(\mathbf{a}/\mathbf{b}) < 0$ but for every $\mathbf{a}'$ properly contained in $\mathbf{a}$, $\delta(\mathbf{a}'/\mathbf{b}) \geq 0$,

L^+ adds a relation symbol $R_{AB}(\mathbf{x})$ for each minimal intrinsic extension.

T_{nat} asserts:

$$(\exists \mathbf{y} \Delta_A B(\mathbf{x}, \mathbf{y}) \leftrightarrow R_{AB}(\mathbf{x}).$$

T^+ is the universal theory of T_{nat}.

For any $\mathbf{b} \in A \in \mathbf{K}$, let $I^*(\mathbf{y})$ be a collection of formulas so that if $B \models I^*(\overline{b'})$, then $\overline{b'} \leq B$ and $\langle \mathbf{b} \rangle \cong \langle \mathbf{b}' \rangle$.

(I.e. the L^+ diagram of $\mathbf{b}$.)

SEPARATION OF QUANTIFIERS

Definition. If $A \subseteq B \in \mathbf{K}$ and $\overline{b} \subseteq B$, then $\overline{b}$ is *minimal strong* over A if $\overline{b} \not\subseteq A$, $A \leq A\overline{b}$ and for any nonnull, proper subtuple $\overline{c}$ from $\overline{b}$, $A\overline{c} \not\leq A\overline{b}$.

Definition. $(\mathbf{K}, \delta)$ *admits separation of quantifiers* if for any $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \mathbf{K}$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$, the following holds: For any formula $\tau(\overline{x}; \overline{y})$ in $I^*(\mathbf{a}, \mathbf{b})$ there is $\sigma(\overline{y}) \in I^*(\overline{\mathbf{b}})$ such that whenever $\overline{b'} \subseteq C \in \mathbf{K}$ and $C \models \sigma(\overline{b'})$, there is $D \in \mathbf{K}$ with $C \leq D$ and $\overline{a'} \in D$ such that $D \models \tau(\overline{a'}; \overline{b'})$.

Fact: (Baldwin-Holland) If $(\mathbf{K}, \delta)$ admits separation of quantifiers then the generic is ω-saturated.

Theorem: (Baldwin-Holland) For $\mathbf{K_0}$, the class of bicolored fields, $(\mathbf{K_0}, \delta)$ admits separation of quantifiers.

Proof sketch: Suppose that $\overline{a}$ is a black primitive over $\langle \overline{b} \rangle$. There are a primitive code $\mathbf{c}$ and $\overline{c} \subseteq \langle \overline{b} \rangle$ such that $\overline{a}$ is a generic solution of $\phi^f_{\mathbf{c}}(\overline{x}; \overline{c})$. Let $\rho(\overline{x}; \overline{y})$ be a complete δ-formula for $\overline{a}$ over $\overline{b}$.

By part 4 of the definition of δ-formula (and since $\delta(\mathbf{a}/\mathbf{b}) = 0$):

$$T \cup I^*(\overline{b}) \cup \{\rho(\overline{x}; \overline{y})\} \vdash I^*(\overline{a}; \overline{b}).$$

Thus, we may choose $\sigma(\overline{y}) \in I^*(\overline{\mathbf{b}})$ such that:

$$T \vdash \sigma(\overline{y} \wedge \rho(\overline{x}; \overline{y}) \rightarrow \tau(\overline{x}; \overline{y}).$$

Now let $\overline{b}' \in C$ with $C \models \sigma(\overline{b}')$.

Let $C[\overline{g}]$ be the white closure of the field generated by C and $\mathbf{g}$.

Then, $C[\overline{g}] \models \sigma(\overline{b}')$; it follows that

$$C[\overline{g}] \models \tau(\overline{g}; \overline{b}').$$

$\mathbf{K}^\mu$: Amalgamation

For any function μ from codes to natural numbers we define $\mathbf{K}^\mu$ to be the members of $\mathbf{K_0}$ which have less than $\mu(\mathbf{c})$ independent realizations over the same canonical parameter of each code $\mathbf{c}$.

Poizat proved for bicolored fields, $(\mathbf{K}^\mu, \leq)$ has the amalgamation property if

$$\mu(\mathbf{c}) \geq q(\mathbf{c})$$

(Use linear disjointness and identify maximal isomorphic strong substructures).

Lemma. Fix a primitive code $\mathbf{c}$ and suppose that $\overline{c} \subseteq C \models T_0^\mu$ with $\models \theta_{\mathbf{c}}(\overline{c})$. Suppose $\mathbf{d}$ is another primitive code, and that $\mathbf{e}_1, \ldots, \mathbf{e}_H \subseteq C[\mathbf{g}]$ with $E_{\mathbf{d}}(\mathbf{e}_1, \ldots, \mathbf{e}_H)$, and $H > \mu(\mathbf{d}) \geq q(\mathbf{d}) + m(\mathbf{d})$. Then, either $\mathbf{d}$ and $\mathbf{c}$ are equal, up to a possible reordering of variables, or

$$H \leq 3n(\mathbf{c}).$$

Lemma. $(\mathbf{K}_0^\mu, \delta)$ has separation of variables if μ is finite to one.

Proof sketch. Let $\mathbf{a} \leq \mathbf{ab} \leq \langle \mathbf{ab} \rangle$ and $\mathbf{b}$ be primitive over $\mathbf{a}$ with complete δ-formula $\phi_{\mathbf{c}}$. Fix any $\mathbf{a}'$ of the same length as $\mathbf{a}$ with $\mathbf{a}' \subset C$.

By our choice of μ finite to one, we may list all primitive codes $\mathbf{d}_1, \ldots \mathbf{d}_l$ with $\mu(\mathbf{d}_i) \leq 3n(\mathbf{c})$.

Note that for $i = 1, \ldots, l$, $T \cup I^*(\overline{a}; \overline{b})$ proves

$$\neg E_{\mathbf{d}_i}(\overline{w}_1, \ldots, \overline{w}_{\mu(\mathbf{d}_i)+1})$$

for all choices of $\overline{w}_1, \ldots, \overline{w}_{\mu(\mathbf{d}_i)+1}$ from among $\overline{xy}$, and also proves

$$\forall \overline{y}_1 \ldots \overline{y}_m (E_{\mathbf{d}_i}(\overline{y}_1, \ldots, \overline{y_m}) \wedge \ \overline{y}_1 \ldots \overline{y}_m \subseteq \overline{x} \rightarrow$$

$$\neg \exists \overline{z} (\overline{z} \text{ splits over } \overline{xy} \ \wedge \ E_{\mathbf{d}_i}(\overline{z}, \overline{y}_1, \ldots, \overline{y}_m)),$$

where $m = m(\mathbf{d}_i)$. Find $\tau_i(\overline{x}; \overline{y})$ in $\bigwedge I^*(\overline{a}; \overline{b})$ such that $T \cup \{\tau_i(\overline{x}; \overline{y})\}$ proves the displayed consequences of $T \cup I^*(\overline{a}; \overline{b})$.

By the previous paragraph, there is $\sigma_i(\overline{y}) \in I^*(\overline{y})$ such that

$$T \vdash \sigma_i(\overline{y}) \wedge \rho(\overline{x};\overline{y}) \to \tau_i(\overline{x};\overline{y}).$$

Replace σ by $\sigma \wedge \bigwedge_i \sigma_i$, so that if $\overline{b}' \subseteq C$, $C[\overline{g}] \models \sigma(\overline{b}')$, then $\mu(\mathbf{d}_i)$ is not violated in $\overline{b}'\overline{g}$ itself, and if for some i and some $\overline{e}_1, \ldots, \overline{e}_{m(\mathbf{d}_i)} \subseteq \overline{g}$, $E_{\mathbf{d}_i}(\overline{e}_1, \ldots, \overline{e}_{m(\mathbf{d}_i)})$, no $\overline{e}$ lying in $C[\overline{g}]$ with $E_{\mathbf{d}_i}(\overline{e}, \overline{e}_1, \ldots, \overline{e}_{m(\mathbf{d}_i)})$ can split over $\overline{b}'\overline{g}$.

Now fix $\overline{b}' \subseteq C$ with $C[\overline{g}] \models \sigma(\overline{b}') \wedge \alpha(\overline{b}')$, we then prove:

$C[\overline{g}] \models T^{\mu}$

to finish:

Theorem: (Baldwin-Holland) If μ is finite-one, the generic is ω-saturated (and its theory T^{μ} has rank 2).

Unsaturated generics

Lemma. For appropriate μ which is not finite to one,

1. The generic is not saturated.
2. In fact, T^μ is not ω-stable, nor even small.

Paradise Regained

Lemma. $Ex(T^{\mu}_{nat})$ is a Robinson theory.
Proof. (T^{μ} has amalgamation.)

Lemma. $Ex(T^{\mu}_{nat})$ is ω-stable.
Proof. Each quantifier free type over a countable set is determined by a complete δ-formula.

Corollary. $Ex(T^{\mu}_{nat})$ is not (in general) first order.
Proof. For appropriate μ the theory of the generic is not ω-stable.

Remark: In general, ω-stability in this sense may not even imply simplicity (since non-forking need not satisfy extension). It does in this case.

REPRISE

Homogeneity vrs L^n-saturation

L^n-saturation implies forking is trivial (Djordjevic); the finite model property is salvaged by dealing with a variant of model homogeneity.

Saturation vrs homogeneity

For bicolored fields T^μ, μ must be finite to one to prove the generic is saturated. But for arbitrary μ the finite diagram associated with T^μ is well-behaved.

In each case the 'universal domain' is first order homogeneous.

Question. If $Ex(T_{nat})$ is an ω-stable with finite rank Robinson theory of an (expansion) of a simple group, must the group be algebraic?