What is the right type-space? Humboldt University July 5, 2007

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Goals

The fundamental notion of a Stone space is delicate for infinitary logic.
I will describe several possibilities.

There will be a quiz.

Infinitary Logic and Omitting Types

Key Insight

(Chang, Lopez-Escobar) Any sentence ϕ of $L_{\omega_1,\omega}$ can be coded by omitting a set of types in a countable first order theory.

Reduction

(Shelah) Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory, T.

A model $A \subset M \models T$ is atomic if every finite sequence from A realizes a principal (isolated) type over the empty set.

Our Context

T is a countable first order theory that admits quantifier elimination.

$\mathbf{K}$ is the class of atomic models of T.

$\prec_{\mathbf{K}}$ is elementary submodel.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

$$\begin{tikzcd}
 & N \\
A \arrow[ur] \arrow[dr] & \\
 & M
\end{tikzcd}$$

there exists an N_1 such that

$$\begin{tikzcd}
 & N \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & N_1 \\
 & M \arrow[ur] &
\end{tikzcd}$$

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction.

Objective A

Expand $(\mathcal{C}, +, \cdot)$ by a unary function f which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ satisfied by $(\mathcal{C}, +, \cdot, f)$ is categorical and has quantifier elimination.

Objective B

Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Assumption

K has the amalgamation property over models and the joint embeddding property.

This is a highly nontrivial assumption; it follows from categoricity up to $\aleph_\omega$ and the WGCH.

Amalgamation over arbitrary subsets is a still stronger hypothesis which misses mathematically important examples.

The Monster Model

If $\mathbf{K}$ has the amalgamation property and the joint embedding property then we can work inside a monster model (universal domain) $\mathcal{M}$ that is $|\mathcal{M}|$-model homegeneous. That is,

If $N \prec_{\mathbf{K}} \mathcal{M}$ and $N \prec_{\mathbf{K}} M \in \mathbf{K}$ and $|M| < |\mathcal{M}|$ there is embedding of M into $\mathcal{M}$ over N.

Stone Space I

Ordinary $S(A)$

Let $A \subset M \in \mathbf{K}$. A syntactic 1-type is an ultrafilter in the Boolean algebra of 1-ary formulas with parameters from A.

$S(A)$ is the set of types over A.

Since T eliminates quantifiers there is no dependence on the choice of M.

Stone Space II

$S^*(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S^*(A)$ means there is an atomic model M of T with $A \subseteq M$ such that p is realized in M.

Stone Space III

$S_{at}(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

Simple Example

Wake up!

T is the theory of an infinite set under equality. $M \models T$. p asserts $x \neq m$ for every $m \in M$. Then $p \in S_{at}(M)$.

Complicated Example

Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

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What is the monster model for $\mathbf{K}$?
Every $p \in S_{at}(M)$ is realized in M.

Complicated Example

Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Consequences

What is the monster model for **K**?
Every $p \in S_{at}(M)$ is realized in M.

This does not mean:
For any $A \subset M$, every $p \in S_{at}(A)$ is realized in M.

Stone Space IV

$S_{\mathbb{M}}(A)$

Fix a monster model $\mathbb{M}$ of $\mathbf{K}$.
Let $p \in S(A)$ where $A \subset \mathbb{M}$.

Definition

$p \in S_{\mathbb{M}}(A)$ means that p is realized in $\mathbb{M}$.

Question

$S_{\mathbb{M}}(A)$

Fix a monster model $\mathbb{M}$ of **K**.

What are the relations among
$S_{\mathbb{M}}(A)$,
$S^*(A)$,
$S_{at}(A)$,
$S(A)$?

Does the cardinality of A matter?

Observations

$\subset$ means proper subset.

Always

$$S_{\mathbb{M}}(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$$

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Marcus

$S_{\mathbb{M}}(A) \subset S^*(A)$

***A*-countable**

$S^*(A) = S_{at}(A)$

Stone Space V: Galois Types

$\mathbb{S}(A)$

Fix a monster model $\mathbb{M}$ of $\mathbf{K}$.

Definition

Let $A \subseteq M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The *Galois type* of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of $\mathbb{M}$ which fix A.

The set of Galois types over A is denoted $\mathbb{S}(A)$.

Stone Space V: Galois Types: Subtleties

Over sets

Clearly, for fixed $\mathbb{M}$, there exists $\pi : \mathbb{S}(A) \twoheadrightarrow S_{\mathbb{M}}(A)$.
But there is a choice of $\mathbb{M}$.
In general the relation between $\mathbb{S}(A)$ and $S_{at}(A)$ is unclear.

Over models

But for models: $\pi : \mathbb{S}(M) \twoheadrightarrow S_{at}(M)$.
There is only one 'monster model'.
And Shelah's notion corresponds to the 'monster model' version.

GALOIS TYPES: General Form

Define:

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

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$$f(a) = f'(a').$$

'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N''
if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

locality Any increasing chain of types has at most one upper bound;

tameness two distinct types differ on a finite set;

compactness an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\mathbf{K}$ is (χ, μ)-*tame* if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathbb{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

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If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut_N}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut_M}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ^+-categorical and $(\lambda, < \infty)$-tame then $\mathbf{K}$ is categorical in all $\theta \geq \lambda^+$.

Theorem (Lessmann)

If $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\mathbf{K}$ is categorical in all uncountable cardinals

Two Examples that are not tame

1. 'Hiding the zero'
   For each $k < \omega$ a class which is $(\aleph_0, \aleph_{k-3})$-tame but not $(\aleph_{k-3}, \aleph_{k-2})$-tame. Baldwin-Kolesnikov (building on Hart-Shelah)
2. Coding EXT
   A class that is not $(\aleph_0, \aleph_1)$-tame.
   A class that is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame. (Baldwin-Shelah)

Very Complicated Example

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.

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2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
4. ϕ_k is categorical in μ if $\mu \le \aleph_{k-2}$;
5. ϕ_k is not $\aleph_{k-2}$-Galois stable;
6. But for $m \le k-3$, ϕ_k is $\aleph_m$-Galois stable;
7. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Some further consequences

In this example:

1. $\pi : \mathbb{S}(M) \twoheadrightarrow S_{\mathbb{M}}(A)$ is not 1-1.
2. There exists an independent pair of countable models M_1, M_2 over M_2 so that $|S_{at}(M_1 M_2)| = 2^{\aleph_0}$

The last example exhibits the failure of ‘automorphism stationarity’.
If $M_1 \approx M_1'$ over M_0 and both are independent from M_2 over M_0 then they are isomorphic over M_0.
But in 2) they need not be automorphic over M_0.

One more inequality

The Marcus example gives an uncountable atomic set A which cannot be embedded in an atomic model.
In fact, $S^*(A) \neq S_{\mathrm{at}}(A)$.

From the Baldwin-Kolesnikov analysis there is a model M with

$$S^*(M) \neq S_{\mathrm{at}}(M).$$

Locality and Tameness

Definition

K has (κ, λ)-*local galois types* if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of **K** with $|M| = \lambda$ and for any $p, q \in \mathbb{S}(M)$: if $p \upharpoonright M_i = q \upharpoonright M_i$ for every i then $p = q$.

Lemma

If $\lambda \geq \kappa$ and $\mathrm{cf}(\kappa) > \chi$, then (χ, λ)-tame implies (κ, λ)-local. If particular, $(\aleph_0, \aleph_1)$-tame implies $(\aleph_1, \aleph_1)$-local.

Whitehead Groups

Definition

We say A is a *Whitehead group* if $Ext(A, \mathcal{Z}) = 0$. That is, every short exact sequence

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow A \rightarrow 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathcal{Z}$.

Key Example

Shelah constructed (page 228 of Eklof-Mekler, first edition) of a group with the following properties.

Fact

There is an $\aleph_1$-free group G of cardinality $\aleph_1$ which is not Whitehead.
Moreover, there is a countable subgroup R of G such that G/R is p-divisible for each prime p.

THE AEC EXAMPLE

Let **K** be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\mathbf{G}, \mathbf{Z}, \mathbf{I}, \mathbf{H})$.
G is a torsion-free Abelian Group. Z is a copy of $(Z, +)$. I is an index set and H is a family of infinite groups.

Each model in **K** consists of

1. a torsion free group G,
2. a copy of $\mathcal{Z}$
3. and a family of extensions of Z by G.

Each of those extensions is coded by single element of the model so the Galois type of a point of this kind represents a specific extension. The projection and embedding maps from the short exact sequence are also there.

$M_0 \prec_{\mathbf{K}} M_1$ if

M_0 is a substructure of M,

but $\mathbf{Z}^{\mathbf{M_0}} = \mathbf{Z}^{\mathbf{M}}$

and $\mathbf{G}^{\mathbf{M_0}}$ is a pure subgroup of $\mathbf{G}^{\mathbf{M_1}}$.

NOT LOCAL

Lemma

$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_1, \aleph_1)$-local. That is, there is an $M^0 \in \mathbf{K}$ of cardinality $\aleph_1$ and a continuous increasing chain of models M_i^0 for $i < \aleph_1$ and two distinct types $p, q \in \mathbb{S}(M^0)$ with $p \restriction M_i^0 = q \restriction M_i$ for each i.

Let G be an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. There is an H such that,

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow G \rightarrow 0$$

is exact but does not split.

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

$M_1 = \langle G, \mathcal{Z}, \{a, t_1\}, \{G \oplus Z, H\} \rangle$

$M_2 = \langle G, \mathcal{Z}, \{a, t_2\}, \{G \oplus Z, G \oplus Z\} \rangle$

Let $p = \mathrm{tp}(t_1/M^0, M^1)$ and $q = \mathrm{tp}(t_2/M^0, M^2)$.
Since the exact sequence for $\mathbf{H}^{\mathbf{M}^2}$ splits and that for $\mathbf{H}^{\mathbf{M}^1}$ does not, $p \neq q$.

NOT $\aleph_1$-LOCAL

But for any countable $M_0' \prec_{\mathbf{K}} M_0$, $p \restriction M_0' = q \restriction M_0'$, as

$$0 \to Z \to H_i' \to G' \to 0. \tag{1}$$

splits.
$G' = \mathbf{G}(\mathbf{M_0'})$, $H' = \pi^{-1}(t_i, G')$.

NOT $\aleph_0$-TAME

It is easy to see that if $(\mathbf{K}, \prec_{\mathbf{K}})$ is $(\aleph_0, \aleph_0)$-tame then it is $(\aleph_1, \aleph_1)$-local, so
$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_0, \aleph_0)$-tame.
So in fact, $(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_0, \chi)$-tame for any χ.

Question

Could this example be formulated more naturally as
$\{Ext(G, Z) : G \text{ is } \text{torsion-free } \}$
(with projection and injection maps?

Incompactness

Theorem

Assume $2^{\aleph_0} = \aleph_1$, *and* $\diamondsuit_{\aleph_1}, \diamondsuit_{S_1^2}$ *where*

$$S_1^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}.$$

Then, the last example fails either $(\aleph_1, \aleph_1)$ *or* $(\aleph_2, \aleph_2)$*-compactness.*

BECOMING TAME ??

Grossberg and Van Dieren asked for $(\mathbf{K}, \prec_{\mathbf{K}})$, and $\mu_1 < \mu_2$ so that $(\mathbf{K}, \prec_{\mathbf{K}})$ is not (μ_1, ∞)-tame but is (μ_2, ∞)-tame.

Admits intersection

We say the AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ *admits intersections* if for every $X \subseteq M \in \mathbf{K}$, there is a minimal closure of X in M. That is, $M \restriction \bigcap\{N : X \subseteq N \prec_{\mathbf{K}} M\} = \mathrm{cl}_M(X) \prec_{\mathbf{K}} M$.

Tameness gained

Theorem

There is an AEC in a countable language that admits intersections with Löwenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Proof Sketch: Repeat the previous example but instead of letting the quotient be any torsion free group

1. insist that the quotient is an $\aleph_1$-free group;
2. add a predicate R for the group R G/R is divisible by every prime p where G is Shelah's example of a non-Whitehead group.

This forces $|G| \leq 2^{\aleph_0}$ and then we get $(2^{\aleph_0}, \infty)$-tame.
But $\aleph_1$-free groups fail amalgamation ??

Lemma

For any AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ which admits intersections there is an associated AEC $(\mathbf{K}', \prec_{\mathbf{K}})$ with the same (non) locality properties that has the amalgamation property.

Theorem

There is an AEC with the amalgamation property in a countable language with Löwenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Conclusion

- $\mathbb{S}(M)$ is the natural tool for Abstract elementary classes.
- $S_{\mathrm{at}}(M)$ is Shelah's tool for $L_{\omega_1,\omega}$.
  Excellence implies they agree.
- But Hytinnen and Kesala have interesting results with $S_{\mathbb{M}}(A)$ for finitary AEC.