Math 512: Finite Model Theory
Games

February 27, 1997

We work in a finite relational language.

For $m \leq \omega$ a game $G_m(A, B)$ played by two players, duplicator and spoiler, consists of m turns. On move k, Spoiler chooses an element from A or B (denoted by a_k or b_k respectively). Duplicator then chooses an element from the other structure (denoted by b_n or a_n respectively). Duplicator wins if the sequences $\langle a_i : i < m \rangle$ and $\langle b_i : i < m \rangle$ are partially isomorphic.

0.1 Definition.

1. $A \simeq_m B$ if duplicator has a winning strategy for the family of games: $G_m(A, B)$.

2. $A \equiv_m B$ if A and B satisfy the same formulas of quantifer rank m.

3. $A \cong_n B$ is defined by induction:

   (a) $A \cong_0 B$ iff $A \equiv_0 B$.

   (b) $A \cong_{n+1} B$ iff

      i. For every $a \in A$ there is $b \in B$ such that $(A, a) \cong_n (B, b)$ and

      ii. For every $b \in B$ there is $a \in A$ such that $(A, a) \cong_n (B, b)$.

Remark: $A \simeq_0 B$ iff $A \equiv_0 B$. This holds just if the quanifier free diagrams of the the constants are the same.

0.2 Lemma. *For all $n < \omega$, $A \simeq_n B$ iff $A \cong_n B$.*

Proof. Induct on n; 0 is clear. Moreover, $A \cong_n B$ implies $A \simeq_n B$ is clear. So suppose duplicator has a winning strategy in $G_{n+1}(A, B)$. Fix a_0 arbitrary in A. If duplicator chooses b_0 by his winning strategy then duplicator has a winning strategy in $G_n((A, a_0), (B, b_0))$ and we finish by induction.

0.3 Definition. The rank-m type of a k-tuple $\overline{a} \in A$,

$$\mathrm{tp_A^m(\overline{a}) = \{\phi(\overline{x}) : A \models \phi(\overline{a})\,\&\,qd(\phi) \leq m\}.}$$

Note that for a finite relational language and $m < \omega$, the existential quantification of this type is a sentence ϕ_m with quantifier rank $m + 1$;

0.4 Theorem. *For all A, B in all finite relational langauages and $m < \omega$, the following are equivalent:*

1. $A \equiv_m B$.
2. $A \cong_n B$.
3. $A \simeq_m B$.
4. *A and B realize the same rank-$(m-1)$-types of singletons.*

Proof. We have already noticed the equivalence of 2 and 3. Moreover, we have seen $m = 0$ in 0.1. Assume the equivalence at n; we prove $n + 1$. First, we show 1 implies 4. So $A \equiv_{n+1} B$. We must show A and B realize the same rank-n types of singletons. For arbitrary $a \in A$, we must find $b \in B$ such that $\mathrm{tp_A^n(a) = tp_B^n(b)}$. Note that the sentence ϕ_n determining $\mathrm{tp_A^n(a)}$ has quantifier rank $n + 1$. So it is satisfied by B and we can find a realization in B. Reversing the roles of A and B, we see B realizes exactly the rank-n types of singletons realized in A.

Now we show 4 implies 2. Suppose A and B realize the same rank-n types of singletons; we want $A \cong_n B$. Pick an arbitrary $a \in A$. Choose $b \in B$ with $\mathrm{tp_A^n(a) = tp_B^n(b)}$. So, $(A, a) \equiv_n (B, b)$. By induction, $(A, a) \cong_n (B, b)$, as required. The other direction is symmetric.

Finally, we show 2 implies 1. Assume $A \cong_{n+1} B$. Every formula of quantifier rank $n+1$ is a Boolean combination of formulas of the form: $(\exists x)\psi(x)$, where ψ has quantifier rank at most n. So it suffices to show that for each formula σ of that form.

$$A \models \sigma \text{ iff } B \models \sigma.$$

Fix such a σ and suppose witnessed by a in A. Since, $A \cong_{n+1} B$. we can choose $b \in B$ with $(A, a) \cong_n (B, b)$. By induction, $(A, a) \equiv_n (B, b)$, so $B \models \sigma$. The argument is symmetric for the case $B \models \sigma$.

This formalization of the argument was based largely on notes by Chris Laskowski.