FORKING AND MULTIPLICITY IN FIRST ORDER THEORIES

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

May 13, 2001

STABLE THEORIES

1. Origins of Stability theory
2. Forking and Multiplicity
   (a) Arbitrary theories
   (b) Simple theories
   (c) Stable theories
   (d) Multiplicity

DIVIDING

1. A formula $\phi(\overline{x}, \overline{b})$ *divides over* A if there exists an infinite sequence I and $k < \omega$ such that

   (a) $\mathrm{tp}(\overline{c}/A) = \mathrm{tp}(\overline{b}/A)$ for every $\overline{c} \in I$;

   (b) The set $\{ \phi(\overline{x}, \overline{c}) \mid \overline{c} \in I \}$ is k-contradictory.

Monotonicity. Let $A \subseteq B$ and suppose $p \in S(B)$ does not divide over A then p does not divide over C for every $A \subseteq C \subseteq B$.

Lemma. A formula $\phi(\overline{x}, \overline{b})$ divides over A if and only if there exist $k < \omega$ and an indiscernible sequence over A, $\langle \overline{b}_i \mid i < \omega \rangle$, such that $\overline{b}_0 = \overline{b}$ and $\{ \phi(\overline{x}, \overline{b}_i) \mid i < \omega \}$ is inconsistent.

CHARACTERIZING DIVIDING

[Basic Characterization of Dividing] Let A be a set, $\overline{b}$ be a finite sequence, and $\overline{a}$ be a (possibly infinite) sequence. The following conditions are equivalent.

1. $\mathrm{tp}(\overline{a}/A\overline{b})$ does not divide over A;
2. For every infinite sequence of indiscernibles I over A with $\overline{b} \in I$ there exists $\overline{a}'$ realizing $\mathrm{tp}(\overline{a}/A\overline{b})$ such that I is indiscernible over $A\overline{a}'$.

Pairs Lemma. If $\overline{b}$ is a finite sequence, $\mathrm{tp}(\overline{a}_0/A\overline{b})$ does not divide over A, and $\mathrm{tp}(\overline{a}_1/A\overline{b}\overline{a}_0)$ does not divide over $A\overline{a}_0$, then $\mathrm{tp}(\overline{a}_0\overline{a}_1/A\overline{b})$ does not divide over A.

MORLEY SEQUENCES

Notation. Fix a sequence $\langle \overline{e}_i : i \in I \rangle$. We write E_i for (the union of) $\langle \overline{e}_j : j < i \rangle$. The sequence is *coherent* if $i < j$ implies $\mathrm{tp}(\overline{e}_j / E_i) = \mathrm{tp}(\overline{e}_i / E_i)$.

Morley Sequence. Let $A \subseteq B$ and p be a type over B. Let $(X, <_X)$ be an infinite linearly ordered set. Then, $\langle\, \overline{a}_i \mid i \in X \,\rangle$ is a *Morley sequence for p over A* if

1. $\langle \overline{a}_i : i \in X \rangle$ is a coherent sequence of realizations of p
2. For every $i \in X$ the type $\mathrm{tp}(\overline{a}_i / BA_i)$ does not divide over A;
3. $\langle\, \overline{a}_i \mid i \in X \,\rangle$ is sequence of indiscernibles over B.

When $A = B$ we say that the sequence is a *Morley sequence for p*.

FORKING

1. A formula $\phi(\overline{x}, \overline{b})$ *forks over* A if there exist $n < \omega$ and $\{\phi_i(\overline{x}, \overline{b}^i) \mid i < n\}$, such that
   (a) $\phi_i(\overline{x}, \overline{b}^i)$ divides over A for every $i < n$;
   (b) $\phi(\overline{x}, \overline{b}) \vdash \bigvee_{i<n} \phi_i(\overline{x}, \overline{b}^i)$.
2. A type p *forks over* A if there exists a formula $\phi(\overline{x}, \overline{b})$ such that $p \vdash \phi(\overline{x}, \overline{b})$ and $\phi(\overline{x}, \overline{b})$ forks over A.

Extension Property. If p does not fork over A and $\mathrm{dom}(p) \subseteq B$, then there exists $q \in S(B)$ extending p such that q does not fork over A.

COMBINATORIAL INPUT

Notation. For any linearly ordered set $X \subseteq \mathcal{M}$ we write $\mathbf{D}(X)$ (diagram) for the set of types of finitely properly ordered sequences from X.

Now an application of the Erdos-Rado theorem originally due to Morley shows:

Lemma. If $(X, <)$ is a sufficiently long linearly ordered subset of $\mathcal{M}$ there is a countable set Y of indiscernibles (and hence one of arbitrary order type) such that $\mathbf{D}(Y) \subseteq \mathbf{D}(X)$.

Theorem. Let $A \subseteq B$. Suppose that $p \in S(B)$ does not fork over A. Then for every linearly ordered set $(X, <_X)$ there exists a Morley sequence $\langle b_i \mid i \in X \rangle$ in p over A.

COHEIRS

Let $M \models T$ and $M \subseteq A$, $p \in S(A)$ is a *coheir* of $p|M$ if every formula $\phi(\overline{x}, \overline{a}) \in p$ is satisfied in M.

Lemma. For any M any $p \in S(M)$ and any structure $A \supseteq M$, p has a coheir on A.

Lemma. Let M be a model of an arbitrary theory T, $M \subseteq B$ and $p \in S(B)$ the coheir of $p|M$.

1. p does not divide over M.
2. If $E = \langle \overline{e}_i : i \in I \rangle$ contained in MB is a coherent sequence such that $\mathrm{tp}(\overline{e}_i / ME_i)$ is $p|ME_i$, then E is a sequence of order indiscernibles. We call E a coheir sequence in p.

SIMPLE THEORIES

A complete first order theory T is *simple* if for some κ and each type p over a domain A there is an $A_0 \subseteq A$ such that p does not fork over A_0 and $|A_0| < \kappa$.

Lemma. Let T be simple. Then for every A and every p over A there is a Morley sequence in p over A.

Kim's Characterization of Dividing in Simple Theories Let T be simple. The following conditions are equivalent:

1. The formula $\phi(\overline{x}, \overline{b})$ divides over A;
2. For *every* Morley sequence $\langle \overline{b}_i \mid i \in X \rangle$ for $\mathrm{tp}(\overline{b}/A)$, the set $\{ \phi(\overline{x}, \overline{b}_i) \mid i \in X \}$ is inconsistent;
3. For *some* Morley sequence $\langle \overline{b}_i \mid i \in X \rangle$ for $\mathrm{tp}(\overline{b}/A)$, the set $\{ \phi(\overline{x}, \overline{b}_i) \mid i \in X \}$ is inconsistent.

CONSEQUENCES

Symmetry. Let T be simple. Then $\mathrm{tp}(\overline{a}/A\overline{b})$ divides over A if and only if $\mathrm{tp}(\overline{b}/A\overline{a})$ divides over A.

Divides = Forks. Let T be simple. Then, $\phi(\overline{x}, \overline{b})$ divides over A if and only if $\phi(\overline{x}, \overline{b})$ forks over A.

Transitivity. Let T be simple and $A \subseteq B \subseteq C$. If $\mathrm{tp}(\overline{a}/C)$ does not fork over B and $\mathrm{tp}(\overline{a}/B)$ does not fork over A, then $\mathrm{tp}(\overline{a}/C)$ does not fork over A.

INDEPENDENCE THEOREM

A Hard Lemma. Let T be simple. Let $p(\overline{x}, \overline{b})$ be a type over $A\overline{b}$ which does not fork over A. Let $\langle \overline{b}_i \mid i < \omega \rangle$ be a sequence of indiscernibles over A with $\overline{b} = \overline{b}_0$. Then there exists $\overline{a}$ realizing $p(\overline{x}, \overline{b})$ such that $\langle \overline{b}_i \mid i < \omega \rangle$ is $A\overline{a}$-indiscernible and $\mathrm{tp}(\overline{a}/A \cup \{\overline{b}_i \mid i < \omega\})$ does not fork over A.

The Independence Theorem. Let T be simple and M be a model of T. Let $\overline{a}$ and $\overline{b}$ be finite sequences such that $\mathrm{tp}(\overline{a}/M\overline{b})$ does not fork over M. Let $p(\overline{x}) \in S(M)$. Let $q(\overline{x}, \overline{a})$ be a nonforking extension of p over $M\overline{a}$ and $r(\overline{x}, \overline{b})$ be a nonforking extension of p over $M\overline{b}$. Then $q \cup r$ is a consistent, nonforking extension of p over $M\overline{a}\overline{b}$.

STABLE THEORIES

T is stable if for every infinite A and finite Δ, $|S_\Delta(A)| \leq |A|$.

Lemma. A theory T is stable if and only if for every λ with $\lambda^{|T|} = \lambda$, if $|A| = \lambda$, $|S(A)| = \lambda$.

The formula $\phi(\overline{x}, \overline{y})$ has the *order property* in T if there exist infinite sequences $\overline{a}_i, \overline{b}_i$ for $i < \omega$ such that:

$$\phi(\overline{a}_i, \overline{b}_j) \text{ iff } i < j.$$

T has the order property if some formula does.

T is stable if and only if no formula has the order property.

RANK

We say two Δ-types are explicitly contradictory if for some $\phi(\overline{x}, \overline{y}) \in \Delta$ and some $\overline{a}$, $\phi(\overline{x}, \overline{a})$ is in one and $\neg\phi(\overline{x}, \overline{a})$ is in the other.

Definition. Let p be a type and Δ a set of formulas. For λ a cardinal less than or equal to $\aleph_0$,

1. $R(p, \Delta, \lambda) \geq -1$ if p is consistent.
2. $R(p, \Delta, \lambda) \geq \gamma$ for limit γ if $R(p, \Delta, \lambda) \geq \beta$ for each $\beta < \gamma$.
3. $R(p, \Delta, \lambda) \geq \gamma + 1$ if for every finite $q \subseteq p$ there exist Δ-types q_i for $i < \lambda$ such that
   (a) the q_i are pairwise explicitly contradictory
   (b) and for each i,
   $$R(q \cup q_i, \Delta, \lambda) \geq \gamma.$$
4. $R(p, \Delta, \lambda) = \gamma$ if $R(p, \Delta, \lambda) \geq \gamma$ and
   $$R(p, \Delta, \lambda) \not\geq \gamma + 1.$$

Infinity rank

Lemma. [Ultrametric Property] For formulas $\{\, \psi_l(\overline{x}, \overline{b}_l) \mid l < n \,\}$ any type p, any finite set of formulas Δ, and $k, n < \omega$ we have

$$R[\, p \cup \bigvee_{l<n} \psi_l(\overline{x}, \overline{b}_l) \,, \Delta, \aleph_0] = \max_{l<n} R[\, p \cup \psi_l(\overline{x}, \overline{b}_l) \,, \Delta, \aleph_0].$$

Lemma. If $R(p, \Delta, \aleph_0) < \infty$ and A contains $\operatorname{dom} p$, there is a $q \in S(A)$ with $p \subseteq q$ and $R(p, \Delta, \aleph_0) = R(q, \Delta, \aleph_0)$.

Lemma. If p is a type over A, $R(p, \Delta, \aleph_0) < \infty$ and q is an extension of A which forks over A, then for some finite Δ, $R(p, \Delta, \aleph_0) > R(q, \Delta, \aleph_0)$.

Lemma. If T is stable, for every Δ and p

$$R(p, \Delta, \aleph_0) \leq R(p, \Delta, 2) < \omega.$$

Theorem. If T is stable then T is simple.

FROM INDEPENDENCE THEOREM

Let $p \in S(A)$; p is *stationary* if for any $B \supseteq A$, there is a unique extension of p in $S(B)$ which does not fork over A.

Theorem. If T is stable, complete types over models are stationary.

Corollary. If T is stable and I is an independent sequence of realizations of a type $p \in S(M)$ then I is a set of indiscernibles and thus a Morley sequence in p.

Corollary. If T is stable in $|A| = \lambda$ and $|B| > |A| \geq |T| + \kappa(T)$, B contains a set J with $|J| > |A|$ of indiscernibles.

TWO-RANK

Definition. The map d from $F(L)$ to $F(A)$, taking $\phi(\overline{x},\overline{y})$ to $d\phi(\overline{y})$ defines $p \in S(A)$ if

$$\phi(\overline{x},\overline{a}) \in p \text{ iff } d\phi(\overline{y}).$$

Let $M \models T$, $p \in S(M)$ and $M \subseteq A$. If d defines p, $d(p,A)$ denotes the type over A defined by d.

Lemma. If T is stable then for every formula $\phi(\overline{y},\overline{z})$, for every A and every $p \in S(A)$ there are formulas $d\phi(\overline{y},\overline{z})$ and $\theta(\overline{x},\overline{z})$ such that

1. For some $\overline{a} \in A$, $\theta(\overline{x},\overline{a}) \in p$.
2. For every $\overline{a},\overline{b} \in A$, if $\theta(\overline{x},\overline{a}) \in p$ then

$$\phi(\overline{x},\overline{b}) \in p \text{ iff } d\phi(\overline{b},\overline{a}).$$

NONFORKING=HEIR=COHEIR

Theorem. Let $M \models T$, $p \in S(M)$ and $M \subseteq A$.

1. If d defines p, $d(p, A)$ is the *unique* heir and coheir of p on A.
2. p has a unique coheir on A.

MULTIPLICITY

Let p be a type over A, the *multiplicity* $\mu(p)$ of p is the least μ such that there do not exist μ^+ pairwise contradictory nonforking extensions of p.

Definition.

1. A *finite equivalence relation* over a set A is an equivalence relation on k-tuples for some k which is definable over A and has only finitely many classes.
2. A set X (formula $\phi(\overline{x}, \overline{b})$) is *almost over* A if X (the solution set of ϕ) has only finitely many images under A-automorphisms of $\mathcal{M}$.
3. The type p is *definable almost over* A if it is defined over its domain by the map sending $\phi(\overline{x}, \overline{y})$ to $d\phi(\overline{y})$ and each $d\phi(\overline{y})$ is almost over A.

FORKING AND MULTIPLICITY

Theorem. If T is stable and $\hat{q}$ does not fork over A, then $\hat{q}$ has bounded number of conjugates over A. Moreover, $\hat{q}$ is definable almost over A. Thus, it has at most $2^{|T|}$ conjugates over A. Conversely, if $\hat{q}$ is definable almost over A, $\hat{q}$ does not fork over A.

Finite Equivalence Relation Theorem If T is stable and p, q are distinct nonforking extensions of $r \in S(A)$ to $S(B)$, there is a finite equivalence relation E over A such that:

$$p(\overline{x}) \cup q(\overline{y}) \vdash \neg E(\overline{x}, \overline{y}).$$

CONJUGACY

Conjugacy Theorem. If $\hat{p}_1$ and $\hat{p}_2$ are nonforking extensions of $p \in S(A)$ to types over the monster model $\mathcal{M}$, $\hat{p}_1$ and $\hat{p}_2$ are conjugate by an automorphism fixing A.

Open Mapping Theorem. If $q \in S(B)$ does not fork over A and q is isolated then $q|A$ is isolated.

Corollary. If T is stable, I is an indiscernible set over A and $\mathrm{tp}(\overline{b}/IA)$ does not fork over A then I is indiscernible over $A\overline{b}$.